Schrödinger, Heisenberg, and interaction pictures with time dependent Hamiltonians

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Since there was some discussion about the (in)equivalence of the three standard pictures in quantum mechanics in the fully time dependent case, I will show how to obtain the Heisenberg and interaction pictures from the Schrödinger one in such a situation. The equivalence then follows from the fact that the transformation from one picture to another is achieved via unitary transformations – these are bijective.

It will be useful to consider time-ordered exponentials first – they may not be familiar to everyone.

Time-ordered exponentials

Consider the initial-value problem for the real or complex function $y(t)$

$$\dot{y}(t) = a(t)\, y(t) \qquad\qquad [y(0) = 1]\,, \tag{1}$$

with $a(t)$ a given real or complex function. It is solved by

$$y(t) = \exp\left(\int_0^t a(t')\,\mathrm{d}t'\right)\,, \tag{2}$$

as can be easily verified by differentiation plus consideration of the initial value.

What about the same problem with Hilbert space operators instead of number-valued functions? Can we write down the solution of

$$\dot{Y}(t) = A(t)\, Y(t) \qquad\qquad [Y(0) = \mathbb{1}] \tag{3}$$

in a similar way as (2)? Let us define the operator

$$B(t) = \int_0^t A(t')\,\mathrm{d}t'\,. \tag{4}$$

This can be done by defining the integral as an appropriate limit of a sum of operators, essentially the same way a Riemannian integral is defined in the analysis of real functions. A practical way to compute the integral would be to expand the operator A in terms of its matrix elements with respect to a time independent orthonormal basis of the Hilbert space, according to

$$A(t) = \sum_{m,n} \langle m|A(t)|n\rangle\, |m\rangle\,\langle n| = \sum_{m,n} A_{m,n}(t)\, |m\rangle\,\langle n| \tag{5}$$

and to evaluate B via

$$B(t) = \sum_{m,n} |m\rangle\,\langle n| \int_0^t A_{m,n}(t')\,\mathrm{d}t'\,, \tag{6}$$

an expression that involves only integrals of ordinary (complex) functions.

The question then is whether $\exp[B(t)]$ solves the initial-value problem (3). It is easy to see, by taking the time derivative, that in general this is not the case:

$$\exp[B(t)] = \sum_{n=0}^{\infty} \frac{B(t)^n}{n!} = 1 + B(t) + \frac{1}{2}B(t)^2 + \frac{1}{3!}B(t)^3 + \dots \tag{7}$$

$$\begin{aligned}
\rightsquigarrow \qquad \frac{\mathrm{d}\exp[B(t)]}{\mathrm{d}t} &= \dot{B}(t) + \frac{1}{2}\left[\dot{B}(t)B(t) + B(t)\dot{B}(t)\right] \\
&\quad + \frac{1}{3!}\left[\dot{B}(t)B(t)^2 + B(t)\dot{B}(t)B(t) + B(t)^2\dot{B}(t)\right] + \dots \\
&= A(t) + \frac{1}{2}\left[A(t)\int_0^t A(t')\,\mathrm{d}t' + \int_0^t A(t')\,\mathrm{d}t'\,A(t)\right] \\
&\quad + \frac{1}{3!}\Bigg[\,A(t)\left(\int_0^t A(t')\,\mathrm{d}t'\right)^2 + \left(\int_0^t A(t')\,\mathrm{d}t'\right)A(t)\left(\int_0^t A(t')\,\mathrm{d}t'\right) \\
&\quad + \left(\int_0^t A(t')\,\mathrm{d}t'\right)^2 A(t)\,\Bigg] + \dots
\end{aligned} \tag{8}$$

and we cannot simplify this expression any further, unless $[A(t), A(t')] = 0$ for all t' in the interval $[0, t]$. If A "commutes with itself at all times", we can pull $A(t)$ to the left of the expression and it is then a factor of $\exp[B(t)]$, meaning that the differential equation (3) is satisfied. Since $B(0) = 0$, the initial-value problem is also solved.

So the condition $[A(t), A(t')] = 0$ is sufficient for $\exp[B(t)]$ to be a solution of (3), and it is also necessary, although I will not attempt to prove that.

However, suppose the condition is not satisfied. Can we still give a formal solution to the initial-value problem (3)? Yes, we can, and this is where time ordering comes into play.

Taking $t_1 \geq t_2 \geq t_3 \geq \dots t_n$ to be an ordered time sequence and $(t_{p_1}, t_{p_2}, \dots t_{p_n})$ an arbitrary permutation of these times, we define two time-ordering operators $\overleftarrow{T}$ and $\overrightarrow{T}$ by

$$\overleftarrow{T} A(t_{p_1})A(t_{p_2})\dots A(t_{p_n}) = A(t_1)A(t_2)\dots A(t_n)\,, \tag{9a}$$

$$\overrightarrow{T} A(t_{p_1})A(t_{p_2})\dots A(t_{p_n}) = A(t_n)A(t_{n-1})\dots A(t_1)\,, \tag{9b}$$

requiring this to be true for arbitrary $n \in \mathbb{N}$.

Two remarks are in order. First, the time-ordering operators are not operators on the Hilbert space of quantum states. Rather, they operate on linear operators on that space. Sometimes, operators of this type are referred to as superoperators. (Another example of a superoperator is the commutator operator $C^\times$, defined by $C^\times D = [C, D]$.) Second, the time-ordering operators are really only a convenient tool to indicate a sequence of application of operators different from the order in which they are written on the paper. So it allows one to form expressions that otherwise might be impossible or at least unwieldy.

Note that a rule in doing calculations with a time-ordering operator is that operators ordered by it may be treated as commuting quantities. Clearly, if we change the permutation on the left-hand sides of Eqs. (9), the right-hand sides remain unchanged. To make this even more obvious, consider

$$\overleftarrow{T}\,[A(t_1), A(t_2)] = \overleftarrow{T} A(t_1)A(t_2) - \overleftarrow{T} A(t_2)A(t_1) = A(t_1)A(t_2) - A(t_1)A(t_2) = 0\,. \tag{10}$$

Now I claim that the formal solution to the initial-value problem (3) is simply

$$\overleftarrow{T}\exp\left(\int_0^t A(t')\,\mathrm{d}t'\right) , \tag{11}$$

and the proof is easily obtained by first noting that this is unity for $t = 0$, i.e., satisfies the initial condition, and by then taking the derivative:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\overleftarrow{T}\exp\left(\int_0^t A(t')\,\mathrm{d}t'\right) &= \overleftarrow{T}\Bigg(A(t) + \frac{1}{2}\left[A(t)\int_0^t A(t')\,\mathrm{d}t' + \int_0^t A(t')\,\mathrm{d}t'\,A(t)\right] \\
&\quad + \frac{1}{3!}\Bigg[A(t)\left(\int_0^t A(t')\,\mathrm{d}t'\right)^2 + \left(\int_0^t A(t')\,\mathrm{d}t'\right)A(t)\left(\int_0^t A(t')\,\mathrm{d}t'\right) \\
&\quad + \left(\int_0^t A(t')\,\mathrm{d}t'\right)^2 A(t)\Bigg] + \ldots\Bigg) \\
&= \overleftarrow{T}\Bigg(A(t) + \frac{1}{2}A(t)\left[\int_0^t A(t')\,\mathrm{d}t' + \int_0^t A(t')\,\mathrm{d}t'\right] \\
&\quad + \frac{1}{3!}A(t)\Bigg[\left(\int_0^t A(t')\,\mathrm{d}t'\right)^2 + \left(\int_0^t A(t')\,\mathrm{d}t'\right)^2 \\
&\quad + \left(\int_0^t A(t')\,\mathrm{d}t'\right)^2\Bigg] + \ldots\Bigg) \\
&= A(t)\,\overleftarrow{T}\left(1 + \int_0^t A(t')\,\mathrm{d}t' + \frac{1}{2}\left(\int_0^t A(t')\,\mathrm{d}t'\right)^2 + \ldots\right) \\
&= A(t)\,\overleftarrow{T}\exp\left(\int_0^t A(t')\,\mathrm{d}t'\right) .
\end{aligned} \tag{12}$$

Clearly, for problems with time dependent Hamiltonian, the solution to the Schrödinger equation will be governed by a time-ordered exponential of this type (representing the time evolution operator).

Moreover, it is easy to see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\overrightarrow{T}\exp\left(\int_0^t A(t')\right)\,\mathrm{d}t' = \overrightarrow{T}\exp\left(\int_0^t A(t')\,\mathrm{d}t'\right)A(t) , \tag{13}$$

and this will become useful, too.

It is possible to write the time-ordered exponentials as Neumann series:

$$\begin{aligned}
\overleftarrow{T}\exp\left(\int_0^t A(t')\,\mathrm{d}t'\right) &= 1 + \int_0^t \mathrm{d}t'\,A(t') + \int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\,A(t_1)A(t_2) \\
&\quad + \int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\int_0^{t_2}\mathrm{d}t_3\,A(t_1)A(t_2)A(t_3) + \ldots ,
\end{aligned} \tag{14}$$

$$\begin{aligned}
\overrightarrow{T}\exp\left(\int_0^t A(t')\,\mathrm{d}t'\right) &= 1 + \int_0^t \mathrm{d}t'\,A(t') + \int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\,A(t_2)A(t_1) \\
&\quad + \int_0^t \mathrm{d}t_1\int_0^{t_1}\mathrm{d}t_2\int_0^{t_2}\mathrm{d}t_3\,A(t_3)A(t_2)A(t_1) + \ldots .
\end{aligned} \tag{15}$$

This can be seen either by taking the time derivative, which acts on the outermost integral only, because the inner ones have limits not containing t. Hence, the first operator satisfies the initial-value problem (3), the second the differential equation (13) plus an appropriate initial condition. Since the differential equations are first-order linear and one (operator-valued) initial condition is given, the solution is unique. (A mathematician might require additional conditions here to be able to prove uniqueness.)

Alternatively, we may exploit that for a function $f(t_1, t_2, \ldots t_n)$ that is totally symmetric in its arguments, we have

$$\frac{1}{n!}\int_0^t \mathrm{d}t_1 \int_0^t \mathrm{d}t_2 \ldots \int_0^t \mathrm{d}t_n\, f(t_1, t_2, \ldots t_n) = \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \ldots \int_0^{t_{n-1}} \mathrm{d}t_n\, f(t_1, t_2, \ldots t_n)\,. \tag{16}$$

As long as the time-ordering operator is in front of the exponential, the time-ordered products of operators appearing in its series expansion *are* totally symmetric functions. But in the right hand expression of (16), the times in the integral are ordered already, due to the integration limits. Then we have no difficulty rewriting the operator product with the desired sequence of times (on paper!) and may drop the time-ordering operator in front of the whole expression.

Finally, to conclude this section on time-ordered exponentials, I will cite, without proof, Feynman's disentanglement theorem:

$$\overleftarrow{T}\exp\left[\int_0^t \left(G(t') + H(t')\right)\mathrm{d}t'\right] = \overleftarrow{T}\exp\left[\int_0^t G(t')\mathrm{d}t'\right]\overleftarrow{T}\exp\left[\int_0^t \tilde{H}(t')\mathrm{d}t'\right]\,,$$

where

$$\tilde{H}(t) = \overrightarrow{T}\exp\left[-\int_0^t G(t')\mathrm{d}t'\right] H(t)\,\overleftarrow{T}\exp\left[\int_0^t G(t')\mathrm{d}t'\right] \tag{17}$$

and each time-ordering operator acts only on the exponential written after it. A way to prove this is to show that the left-hand side and the right-hand side satisfy the same first-order linear differential equation with the same initial condition (all exponentials are unity at $t = 0$). Note that $\overrightarrow{T}\exp\left[-\int_0^t G(t')\mathrm{d}t'\right]$ is the inverse of $\overleftarrow{T}\exp\left[\int_0^t G(t')\mathrm{d}t'\right]$. (It is almost trivial to show that it is a left inverse of the latter operator. If the operator is invertible, it must then also be the right inverse and is unique. At least for antihermitean operators $G(t)$, this is true, because then the exponential is unitary.)

As an aside, in his paper introducing this theorem, Feynman spoke of disentanglement of an "experimental factor" (instead of "exponential factor"), either an oversight or a testimony of his peculiar humour.

The pictures

Let us consider the Schrödinger equation with a time dependent Hamiltonian $H(t)$ and let us assume the Hamiltonian has a decomposition

$$H(t) = H_0(t) + H_1(t)\,. \tag{18}$$

The time dependent Schrödinger equation

$$|\dot{\psi}\rangle = -\frac{\mathrm{i}}{\hbar}H(t)\,|\psi\rangle \tag{19}$$

is then solved by

$$|\psi(t)\rangle = \overleftarrow{T}\exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^t H(t')\mathrm{d}t'\right]|\psi(0)\rangle = \overleftarrow{T}\exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^t \left(H_0(t') + H_1(t')\right)\mathrm{d}t'\right]|\psi(0)\rangle \ . \tag{20}$$

Note that this solution is also correct in the limit, where $H(t)$ happens to be time independent, i.e. $H(t) = H$, because then $\overleftarrow{T}\exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^t H(t')\mathrm{d}t'\right] = \exp\left[-\frac{\mathrm{i}}{\hbar}Ht\right]$. Thus, Eq. (20) covers this special case as well and hence is completely general.

Heisenberg picture

The state vector $|\psi(t)\rangle$ is not directly observable. Experiments only yield matrix elements of observables $\langle\phi(t)|O|\psi(t)\rangle$. So also the time dependence of observable quantities is only accessible via their matrix elements. It is then possible to assign the time dependence to the operator associated with the observable rather than to the state vectors. This is the Heisenberg picture, in which the state vector describes only initial conditions. It may change only on measurement (including the subsequent observation of a measured value, otherwise it does not even change then). Setting $U(t) = \overleftarrow{T}\exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^t H(t')\mathrm{d}t'\right]$, we can write

$$\begin{aligned}\langle\phi(t)|O_S|\psi(t)\rangle &= \langle U(t)\phi(0)|O_S|U(t)\psi(0)\rangle = \left\langle\phi(0)\middle|U^\dagger(t)O_SU(t)\middle|\psi(0)\right\rangle \\ &= \langle\phi(0)|O_H(t)|\psi(0)\rangle \ ,\end{aligned} \tag{21}$$

where O_S is the standard, usually time-independent, form of an operator in the Schrödinger picture, and

$$O_H(t) = U^\dagger(t)O_SU(t) = \overrightarrow{T}\exp\left[\frac{\mathrm{i}}{\hbar}\int_0^t H(t')\mathrm{d}t'\right]O_S\overleftarrow{T}\exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^t H(t')\mathrm{d}t'\right] \tag{22}$$

is its counterpart in the Heisenberg picture. O_S can be any operator, in particular, the Hamiltonian. We then have

$$H_H(t) = U^\dagger(t)H_SU(t) = \overrightarrow{T}\exp\left[\frac{\mathrm{i}}{\hbar}\int_0^t H(t')\mathrm{d}t'\right]H(t)\overleftarrow{T}\exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^t H(t')\mathrm{d}t'\right] \ . \tag{23}$$

If H is time independent, the time-ordered exponentials become standard exponentials $U^\dagger(t) = \exp\left[\frac{\mathrm{i}}{\hbar}Ht\right]$ and $U(t) = \exp\left[-\frac{\mathrm{i}}{\hbar}Ht\right]$ that commute with H and, hence, we have $H_H = H$. The Hamiltonian in the Heisenberg picture is equal to that of the Schrödinger picture and also time independent. This is no longer true, if $H_S = H(t)$ is genuinely time dependent and $[H(t), H(t')] \neq 0$. What form does the Heisenberg equation of motion take in the fully time dependent case? We have

$$\begin{aligned}\dot{O}_H(t) &= \frac{\mathrm{i}}{\hbar}\overrightarrow{T}\exp\left[\frac{\mathrm{i}}{\hbar}\int_0^t H(t')\mathrm{d}t'\right][H(t), O_S]\ \overleftarrow{T}\exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^t H(t')\mathrm{d}t'\right] \\ &\quad + \overrightarrow{T}\exp\left[\frac{\mathrm{i}}{\hbar}\int_0^t H(t')\mathrm{d}t'\right]\dot{O}_S\ \overleftarrow{T}\exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^t H(t')\mathrm{d}t'\right] \ ,\end{aligned} \tag{24}$$

where the second term is zero for most Schrödinger operators, but not for the Schrödinger Hamiltonian, if that is truly time dependent. Now this equation is not in a convenient form yet. We would like to have O_H in the commutator on the right-hand side and the commutator should not be surrounded by exponentials. Let us reformulate a bit:

$$\begin{aligned}
\dot{O}_H(t) &= \frac{\mathrm{i}}{\hbar}\Big(U^\dagger(t)H(t)O_S U(t) - U^\dagger(t)O_S H(t)U(t)\Big) + U^\dagger(t)\dot{O}_S U(t) \\
&= \frac{\mathrm{i}}{\hbar}\Big(\underbrace{U^\dagger(t)H(t)U(t)}_{H_H(t)}\underbrace{U^\dagger(t)O_S U(t)}_{O_H(t)} - U^\dagger(t)O_S U(t)U^\dagger(t)H(t)U(t)\Big) + U^\dagger(t)\dot{O}_S U(t) \\
&= \frac{\mathrm{i}}{\hbar}\left[H_H(t), O_H(t)\right] + U^\dagger(t)\dot{O}_S U(t)\,. \qquad (25)
\end{aligned}$$

For standard Schrödinger operators O_S (i.e. operators that are time independent), this becomes the Heisenberg equation of motion in its simplest form:

$$\dot{O}_H(t) = \frac{\mathrm{i}}{\hbar}\left[H_H(t), O_H(t)\right]\,, \qquad (26)$$

solved by

$$O_H(t) = \overleftarrow{T}\exp\left[\frac{\mathrm{i}}{\hbar}\int_0^t H_H(t')\mathrm{d}t'\right] O_H(0)\,\overrightarrow{T}\exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^t H_H(t')\mathrm{d}t'\right]\,. \qquad (27)$$

Since $O_H(0) = O_S$, this also gives us

$$\overleftarrow{T}\exp\left[\frac{\mathrm{i}}{\hbar}\int_0^t H_H(t')\mathrm{d}t'\right] = \overrightarrow{T}\exp\left[\frac{\mathrm{i}}{\hbar}\int_0^t H(t')\mathrm{d}t'\right]\,, \qquad (28a)$$

$$\overrightarrow{T}\exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^t H_H(t')\mathrm{d}t'\right] = \overleftarrow{T}\exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^t H(t')\mathrm{d}t'\right]\,. \qquad (28b)$$

For the Hamiltonian itself, the second term on the last line of Eq. (25) does not necessarily vanish, so we obtain

$$\dot{H}_H(t) = \frac{\mathrm{i}}{\hbar}\left[H_H(t), H_H(t)\right] + U^\dagger(t)\dot{H}(t)U(t) = U^\dagger(t)\dot{H}(t)U(t)\,, \qquad (29)$$

i.e., the first term vanishes. We may write a general form of the Heisenberg equation of motion as follows

$$\dot{O}_H(t) = \frac{\mathrm{i}}{\hbar}\left[H_H(t), O_H(t)\right] + \frac{\partial O_H}{\partial t}\,, \qquad \frac{\partial O_H}{\partial t} \equiv U^\dagger(t)\dot{O}_S(t)U(t)\,. \qquad (30)$$

The Heisenberg picture is characterized by the following features:

- State vectors (wave functions) are independent of time during the quantum evolution of a system.
- The Schrödinger equation does not hold.
- Instead, the Heisenberg equation of motion (30) is satisfied.

Interaction picture

In the interaction picture, we assign part of the time dependence of a matrix element to the wave functions and part to the operators. For example, we may take the time-ordered exponential involving only H_0 from (20) as belonging to the time evolution of operators. Set

$$U_0(t) = \overleftarrow{T} \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^t H_0(t')\mathrm{d}t'\right] , \tag{31}$$

then the disentanglement theorem gives us

$$\begin{aligned}\overleftarrow{T} \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^t \big(H_0(t') + H_1(t')\big)\,\mathrm{d}t'\right] &= \overleftarrow{T} \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^t H_0(t')\mathrm{d}t'\right] \overleftarrow{T} \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^t \tilde{H}_1(t')\mathrm{d}t'\right] \\ &= U_0(t)\overleftarrow{T} \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^t \tilde{H}_1(t')\mathrm{d}t'\right] ,\end{aligned}$$

where

$$\tilde{H}_1(t) = U_0^\dagger(t)H_1(t)U(t) . \tag{32}$$

Let us abbreviate the second exponential

$$\tilde{U}_1(t) \equiv \overleftarrow{T} \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^t \tilde{H}_1(t')\mathrm{d}t'\right] , \tag{33}$$

then we can rewrite a standard matrix element as follows

$$\begin{aligned}\langle\phi(t)|O_S|\psi(t)\rangle &= \left\langle U_0(t)\tilde{U}_1(t)\phi(0)\middle|O_S\middle|U_0(t)\tilde{U}_1(t)\psi(0)\right\rangle \\ &= \left\langle \tilde{U}_1(t)\phi(0)\middle|U_0^\dagger(t)O_S U_0(t)\middle|\tilde{U}_1(t)\psi(0)\right\rangle = \left\langle\tilde{\phi}(t)\middle|O_I(t)\middle|\tilde{\psi}(t)\right\rangle .\end{aligned} \tag{34}$$

Here, $|\tilde{\psi}\rangle$ satisfies a Schrödinger equation with the interaction Hamiltonian $\tilde{H}_1(t)$

$$|\dot{\tilde{\psi}}\rangle = -\frac{\mathrm{i}}{\hbar}\tilde{H}_1(t)|\tilde{\psi}\rangle , \tag{35}$$

whereas operators satisfy a Heisenberg equation involving H_{0H}:

$$\begin{aligned}&H_{0H}(t) \equiv U_0^\dagger(t)H_0(t)U_0(t) , \\ &\dot{O}_I(t) = \frac{\mathrm{i}}{\hbar}[H_{0H}(t), O_I(t)] + \frac{\partial O_I}{\partial t} , \qquad \frac{\partial O_I}{\partial t} = U_0^\dagger(t)\dot{O}_S U_0(t) .\end{aligned} \tag{36}$$

In principle, this may be used to do perturbation theory, if H_1 is small in some sense compared to H_0 and the problem with H_0 as time evolution operator is more easily solvable than the full problem. The standard case is the one, in which H_0 is time independent. Then $H_{0H} = H_0$ and $\left|\tilde{\psi}(t)\right\rangle = \exp\left(\frac{\mathrm{i}}{\hbar}H_0 t\right)|\psi(t)\rangle$ is a slowly varying function. Without the perturbation H_1, it would be time independent. Eigenstates of the unperturbed problem may be determined and used as basis of the Hilbert space. The perturbed Schrödinger equation has $\tilde{H}_1$ as Hamiltonian and remains time dependent. The goal of perturbation theory then is the prediction of transition rates between the eigenstates of the unperturbed problem (i.e., the problem with H_0 only) and a typical result is Fermi's golden rule.

This strategy is of little use, if H_0 becomes time dependent, because the time dependent eigenstates of H_0 (which of course exist) do not lead to simple expressions in an expansion of the full solution (because of the time-ordering of the exponential) and the transition rate from one *time-varying* state to another also does not look particularly useful. Nevertheless, the interaction picture exists and might be employed as the starting point of a perturbative scheme that is just more complicated in the fully time dependent case than in the one, where the zeroth-order Hamiltonian is time independent.

Moreover, because the calculations presented here are exact, not approximate, the interaction picture is equivalent to the Schrödinger picture, as the equality of matrix elements (34) certifies. The relationship between the wave function of the interaction picture and that of the Schrödinger picture is unitary and so is the relationship between operators in the two pictures. So the information content of both pictures is the same. Just some interpretations become different.

Consequences for interpretation

In an answer to my question starting this thread, it was suggested to avoid the Heisenberg picture in interpreting quantum mechanics, because an interpretation in terms of the wave function and the Schrödinger picture is much more accessible. I would still like to insist that interpretations of quantum mechanics should work in all three pictures, if they are to refer to reality. Of course, we can interpret the mathematical formalism in a particular picture. That helps in visualizing things, making qualitative assessments and developing intuition about the best way to understand new experimental situations.

Nevertheless, we must be aware that interpreting the Schrödinger equation in terms of a wave moving through configuration space and interacting with objects either via the Hamiltonian or the boundary conditions is not an interpretation of reality. For we can describe reality without the Schrödinger equation, e.g., by choosing the Heisenberg picture, in which there *is* no time dependent wave function. So we really interpret a layer of description that is between us and reality.

Alternatively, we could use the Feynman approach to quantum mechanics which has been linked here to the Schrödinger picture. In my opinion, the "ontological setup" of the Feynman approach is closer to the Heisenberg picture. What is calculated via path integrals in Feynman's quantum mechanics is the probability amplitude of a point-like entity (a particle) starting at x_1 at time t_1 to end at x_2 at time t_2. This Feynman propagator is similar, but not identical to, the wave function. In particular, it depends on initial coordinates in addition to final ones. (It is essentially a Green's function, satisfying the Schrödinger equation in terms of the final coordinates and its adjoint equation in terms of the initial ones.) But this propagator is not the dynamical object of the theory. Rather, the dynamical object consists of one or several point-like particles, exploring (potentially?) all possible paths between the initial and final positions. This is very similar to what the operator-valued dynamical variable (e.g., the position operator of one or more particles) does in the Heisenberg picture. Instead of using a single Hilbert space operator, Feynman sticks to a c-number description but must allow his particle(s) to explore many paths and whether this is "real" or "potential" only, is in the eye of the beholder.

But again, this is interpretation of a formalism, not of reality itself, to which we have very indirect access only. The peaceful coexistence of different interpretations that at first sight seem to contradict each other is possible, as long as we know that our interpretations refer to a particular picture only.

However, if we wish to go further, i.e., to say something more fundamental about the very nature of quantum objects and about how they interact, etc., then, I think, we should make sure that what we say makes sense in *all* mathematical representations of quantum mechanics. That does not exclude that our interpretation uses waves in one picture. But it must also say what the wave has to be replaced by in another picture that does not use waves. Therefore, it must explain the wave by something more fundamental that keeps its identity when the wave is transformed away by going to another picture.