Conventionality of simultaneity and velocity measurement

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Lack of awareness of the problem

Back in 2012, when I wrote my paper on how to resolve Selleri's paradox [1], I started its first version with an extensive discussion of the conventionality of simultaneity, in which I also explained why "measurements" of one-way velocities using only a single clock do not allow one to avoid choosing a synchronization. My previous experience with a paper [2], submitted to the same journal (*American Journal of Physics*), made me deem this necessary: I had gotten the impression that a substantial fraction of the referees of the journal were unaware of the subtleties of simultaneity in relativity. Two of the referees of [2] consistently asserted the correctness of my paper, whereas the third accepted its "general relativistic" part[1] but did not buy my "special relativistic" approach deriving the same results without the use of a metric. My attempts to explain to the adverse referee where he was wrong, by pointing out that the interpretation of length measurements of moving objects depends on the synchronization of clocks in the measuring system, were to no avail. He insisted that "everybody knows" that the relative velocities of two inertial systems are reciprocal (which is true only with Einstein synchronization) and that moving lengths could be measured "with a single clock" (which is only true after the synchronization has been fixed). So this referee did not have the slightest idea about the issues related to synchronization.

I found this a deplorable state of affairs. If even some *referees* of *Am. J. Phys.* have so little knowledge about an aspect of relativity that is emphasized by Einstein in his very first paper on the subject, what about the general *readership*? After sifting through several years of *Am. J. Phys.*, I realized that while there are many papers about relativity, there are hardly any about synchronization. There is some older stuff (for example by Alan Macdonald who understands the subject well) but nothing comprehensive. A paper by Cranor et al. [3], while giving a correct discussion, is about the twin paradox, not about synchronization. Its readers may be left with the impression that alternative synchronizations cannot arise except in rotating frames or possibly in situations where there is acceleration. In any case, the aforementioned referee did not learn enough from the Cranor paper (which he knew) to correctly assess my explanations although I gave all the necessary mathematics.

Therefore, I put some focus on synchronization in the first version of [1] (which even had a different title), with the Selleri paradox being but one example among several. This time the reaction of the referees was that my paper was too long and explained (trivial?) stuff in too much detail. So I threw out most of the discussion of conventionality of simultaneity and reduced the length of the paper by about a factor of two, now putting the emphasis on the paradox and its resolution. I added many references to leave no doubts about the sound experimental basis of special relativity (reinforced rather than opposed by the Sagnac effect). Not counting the references, the paper had a length of a little more than four pages, which then apparently was succinct enough to allow immediate publication.

But that left the readership of *Am. J. Phys.* largely uneducated about the conventionality of simultaneity. As to the members of *Research Gate*, the situation rather seems worse, as various discussion contributions from that side demonstrate. It may therefore be useful to exhibit the relevant points in writing. I do so using TeX, because some formulas are necessary that would be tedious to produce in HTML.

[1] By which the referees (and myself) meant the use of a metric to derive results. Since no gravity was taken into account in the description, it remained of course fully inside the realm of special relativity.

The notion of velocity

Before entering the topic of simultaneity and its ambiguities, I would like to briefly touch on the concepts of velocity and related spatiotemporal notions, which will motivate the necessity of considering simultaneity issues.

It should be obvious that in our current philosophical frameworks regarding space and time, velocity is not a primitive or fundamental entity; it rather is a derived notion. Velocity is, in its simplest form, distance covered per time interval, so it is defined in terms of more fundamental quantities, namely distance and time. Are these well-defined? In order to give quantitative meaning to a notion, which is what physics is largely about, its constituent notions must be quantifiable as well, so spatial and temporal distances must be quantitatively definable, *before* velocity is.[2]

A starting definition of velocity $\boldsymbol{v}$ might be: $\boldsymbol{v} = \Delta\boldsymbol{x}/\Delta t = (\boldsymbol{x}_2 - \boldsymbol{x}_1)/(t_2 - t_1)$, where $\boldsymbol{x}_1$ is the position of our moving object at time t_1 and $\boldsymbol{x}_2$ is its position at a later time t_2, with the (vectorial) distance $\Delta\boldsymbol{x} = \boldsymbol{x}_2 - \boldsymbol{x}_1$ and the time interval $\Delta t = t_2 - t_1$.[3] This is a definition for a *one-way* velocity: $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are usually different positions (unless the result is zero) and the motion from one to the other is monotonous, there is no see-sawing to and fro. A *two-way* velocity (better: speed) would be defined as $v = 2\left|\boldsymbol{x}_2 - \boldsymbol{x}_1\right|/(t_{1f} - t_{1i})$, where the motion is from $\boldsymbol{x}_1$ to $\boldsymbol{x}_2$ and back, with the starting time being t_{1i} and the final time t_{1f}. It is not a vector. We will focus on one-way velocities.

In order for our velocity definition to be complete, we need quantitative definitions of the two positions and the two times appearing in the velocity expression. How are positions and times defined? In order to have something usable (rather than an abstract definition such as "element of a vector space"), we should provide an operational procedure to measure each quantity. It need not be practical but must work in principle.

Dependency on frame of reference

As it turns out, positions are not absolute,[4] so their measurement requires a *frame of reference*. This can be thought of as any body that may be considered to be at rest and the shape of which

[2] Of course, any major theoretical edifice will have some undefined notions at its basis, which cannot be further clarified as language cannot explain itself. In Euclidean geometry, points and straight lines are essentially undefined. While Euclid gives "definitions" – a point is *that which has no part*, a straight line is a *line that lies evenly with the points on itself* – these do not play any role in the geometry that is erected on the basis of his five postulates. Only the properties of points and straight lines that are evoked in the postulates are relevant for the theory. The "definitions" are little more than an appeal to our intuition, so we may believe that there are objects for which Euclidean geometry provides an appropriate theory. As an example from physics we might quote mass and force in Newtonian mechanics, which Newton does not define in sufficient detail to make the definition relevant in the formulation of physical laws. Mass is introduced as a measure of the quantity of "stuff", with two identical objects having twice the mass of one. Otherwise, it is assumed that an intuitive idea of the concept of mass exists that will sharpen as the theory is developed. Force is introduced as the "cause of motion", which hardly is a definition allowing to quantify forces. Again, it is assumed that a prior intuition of the concept exists. Moreover, that concept gets modified later when pseudoforces are introduced that are not causes of motion but caused by motion. Newton's second axiom should *not* be taken as a definition of force, otherwise it would not be a physical law and could not be tested experimentally. You cannot falsify a *definition* by an experiment. But Newton's second law in its simplest form $\boldsymbol{F} = m\boldsymbol{a}$ *was* falsified experimentally when electrons were accelerated to near the speed of light and a relativistic generalization was needed.

[3] This definition is appropriate as long as the result does not vary strongly with time. It can be sharpened to give the *momentary* velocity by taking the limit $\Delta t \to 0$: $\boldsymbol{v} = \mathrm{d}\boldsymbol{x}/\mathrm{d}\boldsymbol{t}$.

[4] While Newton certainly postulated space to be absolute and gave his famous bucket experiment as a proof for the absoluteness of rotating motion, classical mechanics does not provide any physical law allowing one to determine a state of absolute rest.

allows one to define three fixed axes in space.[5] Experience tells us that only unaccelerated (i.e., in particular, non-rotating) bodies qualify as a rest frame.[6] We then introduce standard rulers defining a length unit and measure the positions $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ by laying out rulers parallel to the three axes from the origin of our frame of reference to the points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$. The path along which the rulers join the origin and the point to be measured consists of (possibly several) pieces parallel to the three axes, and to obtain the three coordinates of the vector $\boldsymbol{x}_i$, we simply count the number of unit rulers needed in each direction. For higher precision, it may be necessary to put some rulers with lengths that are a fixed fraction (1/10 th, 1/100 th, etc.) of our length unit.

The result for our velocity numerator will depend on the chosen frame of reference. It is no surprise that the values for $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ themselves depend on the frame of reference, because they must clearly depend on how far the origin located in our reference body is from the points. It is more interesting that also the difference Δx will in general be frame dependent.[7]

Dependency on synchrony

Next, we need a procedure for the determination of the two times appearing in the velocity expression. Similar to standard rulers used to measure distances we may introduce standard clocks to measure times. These will be a bit more complex to set up. Whereas for a standard ruler we may take a single rod made of almost any solid material, use it as length unit and then obtain more rulers simply by making identical copies[8] and integer multiples or fractions of it, a good standard clock will generally be a sophisticated apparatus, using a periodic process (such as an electromagnetic wave emerging from a certain hyperfine transition of a cesium-133 atom), counting a certain number of cycles as a time unit and determining, for an interval to be measured, how many such units fit in. The basic periodic process should be as independent of external conditions as possible and those conditions on which it does depend must be kept equal (atom at rest, no external magnetic and electric fields, etc.). It is believed that atomic transitions may serve as good standard clocks, measuring their own proper time.[9]

Obviously, we need such a clock (in principle) at each of the two different positions $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, where we wish to measure time and while their nature of being standard clocks will make their rates equal automatically,[10] we also have to make sure that their offset (the "origin" for time measurement[11]) is the same. That is, the clocks must be synchronized in order to make sure they show the same time. The problem then arises what it precisely *means* for clocks to be synchronized, if time is not absolute, i.e., if there is the possibility for time to run differently

[5] A parallelepiped or an ellipsoid with axes of different lengths would do.

[6] Accelerated frames of reference are of course possible, but we do not wish to base a definition of a *fixed* position in space such as $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ on fixed coordinates in such a frame.

[7] This is a consequence of the relativity principle asserting that there is no local experiment that would allow us to determine a body to be in a state of absolute rest. So two different observers may choose two different frames of reference that are moving with respect to each other. That motion cannot be accelerated, because acceleration of a frame *is* experimentally detectable. Once we have a complete definition of velocity, we will be able to characterize the motion of two acceptable rest frames with respect to each other as uniform or as motion at constant velocity.

[8] Which when used for measuring should be kept at identical conditions such as temperature, pressure, etc.

[9] Nonstandard clocks may be produced by providing an additional mechanism, decreasing or increasing their rate by a factor. This is applied with the GPS satellites, the proper time of which is running fast in comparison with earth based clocks, so a detuning mechanism is implemented to get them into synchrony with the latter. Therefore, they have a rate in orbit that is smaller than the rate of their proper time.

[10] They are at rest with respect to each other.

[11] This is analogous to the origin for position measurement, which is simply a fixed point of the frame of reference. Here, we need a "fixed point in time" to be the same.

in different places.[12]

In short, (two or more) distant clocks are synchronized, if they show the same time for events, each next to one of the clocks, happening *simultaneously.* That is, we need to give a definition of simultaneity at a distance. *Simultaneity at the same position* is trivial: two events at the same position are simultaneous, if they coincide. Coincidence of events is something we may assume to be easily ascertainable (within experimental uncertainties) – there is no problem (in principle) of establishing that two phenomena (that are localized and short-lived) happen at at the same place *and* time.

Simultaneity at a distance is a different thing. Its definition should not require a pre-defined notion of time in all of space. After all, we are in the process of *establishing* such a notion based on the concept of simultaneity at a distance. If we needed a notion of a common time everywhere to define simultaneity, we would end up in a logical circle, defining simultaneity via global time and global time via simultaneity. We may however extend a *local* definition of time to a *global* one by defining simultaneity and identifying different local times connected by a simultaneity relationship.

Simultaneity and causality

So how do we define simultaneity, i.e., what do we expect from a definition of simultaneity? We may start from an everyday notion that we develop towards a precise concept within the paradigm of Newtonian physics. Then we can study how far the approach carries within the less familiar relativistic world description.

Standard relationships of temporal ordering are provided by the notions *before* and *after.* We will certainly agree that in everyday situations we say an event A to happen *before* an event B, if A could have causally influenced B and B could not have causally influenced A (and that this is independent of *where* the two events happen).[13] Next, we define that A happens *after* B if and only if B happens before A. Finally, we may say that A happens simultaneously with B, iff A does not happen before *and* does not happen after B.

In fact, in Newtonian mechanics, this provides us with a temporal equivalence relation. Consider an event A^- infinitesimally earlier than A at the same position. It is certainly before A, because it can causally influence A but cannot be causally affected by A. But then it cannot be causally influenced by the distant B either, which does not happen before A. So A^- is before B. Similarly, consider an event A^+ infinitesimally later than A at the same position. It is certainly after A, because it can be causally influenced by A, but cannot affect A causally. But then it can be causally influenced by B (B does not happen after A, so it can affect A and A can affect A^+), but cannot itself influence the distant B (because it cannot even influence the close A). So A^+ is after B. Since the time interval between A^- and A^+ can be made arbitrarily small, there is only one event at the position of A ("on a world line containing A")

[12] In Newtonian mechanics, time is absolute by *postulate*, so it has to be the same in different places. In practical experiments, it may of course still be necessary to synchronize clocks at different places. But this is easy in principle: just as we can make identical copies of a standard ruler, we can make identical copies of a standard clock; since transporting them does not influence their rate (as they measure absolute time) or setting (which can be done before transporting them), putting them in different places provides us with a common time in these places. Of course, the premise of a transport-independent rate is no longer true as we leave the domain of validity of Newtonian mechanics.

[13] Note that this approach uses counterfactual arguments. It is not important, whether A *actually* could influence B – which might be prevented by countermeasures against a physical interaction between the events – but only whether A could in principle affect B. In Newtonian mechanics, this is always the case, if A does not happen later than B. If A happens at the same Newtonian time as B, a mutual influence between the two events is possible, in principle, because gravity travels at infinite speed in Newtonian mechanics.

that is simultaneous with B, namely A itself. Mutatis mutandis, the same kind of argument can be used to demonstrate that B is the only event at B's position that is simultaneous with A. Hence, simultaneity at a distance is unique and obviously symmetric. That it is also a transitive relation can be easily seen by considering a third event C that is at a different position both from A and B and simultaneous with B. That means B does not happen before and does not happen after C. But since A does not happen before B, it also does not happen before C, and since A does not happen after B, it also does not happen after C.[14] So A and C are simultaneous, too, and the simultaneity relation, being reflexive, symmetric, and transitive, is an equivalence relation. We can synchronize clocks (measuring absolute time) at a distance, simply by setting the time offset on each of them to the same value for a local event simultaneous with a trigger event at the position of some master clock (which itself shows the time value to which we set the others, at the moment of the trigger event). Alternatively, we can produce copies of the master clock at the same position, set them to the same time, and transport them to whereever we need to measure time afterwards (protecting the transported clocks against mechanical perturbations that might affect their rate).

The preceding discussion demonstrates how to define simultaneity on the basis of a *causal theory of time* in the Newtonian context. In Newtonian physics, all events are causally connectible, but if a pair of events is not simultaneous, the causal connectibility is only one-way. So simultaneity could be succinctly defined as two-way causal connectibility.

Things become a little different, if there is an upper limit to how fast causal influences can travel.[15] Assume that light signals[16] provide us with the fastest way of causal connection. The *before* relation then becomes equivalent to *"inside or on the past light cone"*, the *after* relation to *"inside or on the future light cone"*. For pairs of events, we do not only have the possibilities of being one-way causally connectible or two-way causally connectible but also that of not being causally connectible at all.[17]

The relation *before* as given above remains transitive, because if B is in the past light cone (or on its boundary) of C and A in the past light cone (or on its boundary) of B, then A is also in the past light cone (or on its boundary) of C, because the interior of the past light cone of B is a subset of the interior of the past light cone of C. Similarly, the relation *after* can be shown to be transitive. But the relation *not before*, meaning the same as *outside the past light cone* is clearly not transitive, as it encompasses spacelike and future events, and it is possible for B to be spacelike with respect to A and for C to be spacelike with respect to B with C being in the past light cone of A.[18] Nor is *not after* transitive and so the relation *not before and not after* is not transitive either: A is not before B and not after B simply means that A and B are spacelike (i.e., outside each other's past and future light cones), and this does not constitute an equivalence relation.

Eric Lord, knowing relativity and aware of the many misunderstandings and misrepresentations of the theory on Research Gate, suggested to avoid the notion of simultaneity at a distance

[14] The relations "not before" and "not after" are transitive in Newtonian physics. They are not transitive in special relativity.

[15] Note that in order to determine whether a signal travels faster or more slowly than another, we do not need a definition of velocity (we are still in the process of working that out in detail and noncircularly). If we send two signals from an observer O_A at the same time – that defines an event A – to an observer O_B, then the faster signal is the one that arrives earlier, i.e., if the first signal arrives at event B_1, the second at B_2, and B_1 is before B_2, then the first signal is faster.

[16] Or other signals traveling at the limit speed, such as gravitational wave signals.

[17] And two-way causal connectibility becomes rare as it applies only to pairs of coincident events. Spatially separated events are either one-way causally connectible or not causally connectible.

[18] This is most easily seen by an example. Assume we have introduced Minkowskian coordinates (ct,x,y,z) on a flat spacetime. Then set $A = (1,0,0,0)$, $B = (1,2,0,0)$ and $C = (0,0,0,0)$. Obviously A and B are spacelike, B and C are also spacelike, but C is in the past light cone of A, so A and C are definitely not spacelike.

altogether, as it did not have a place in the relativistic context. Rather, (future or past-oriented) timelike and spacelike are the only objective (and therefore meaningful) time ordering concepts within relativity. At the time he made his suggestion, I did not have a really good answer.[19] It simply seemed that people *wanted* to have an idea of simultaneity even for very distant events, where it does not make much sense. Now I have a better response: a notion of simultaneity is needed to even be able to define the concept of a one-way velocity.[20] And we certainly do not wish to drop the notion of velocity from our vocabulary.

Therefore, we need some definition of simultaneity in relativity, too. Simultaneity should be an equivalence relation on the set of pairs of events and it should satisfy the requirements coming from a causal theory of time, i.e., simultaneous events should be either coincident or not causally connectible. As we have seen, the latter condition is not sufficient to define an equivalence relation in a world the causal structure of which is governed by light cone geometry. Hence, we are at liberty (or under duty!) to introduce an additional criterion to produce a valid definition of a simultaneity relation.

Simultaneity definitions

Einstein was aware of this freedom. In his paper introducing the special theory of relativity [4], he emphasized that simultaneity of events at spatially separated points A and B is established *by definition*, requiring the "time" light needs to travel from A to B to be equal to the the "time" light needs to travel from B to A.[21] Clocks at A and B are synchronized, if

$$t_B - t_A = t'_A - t_B \,, \tag{1}$$

where t_A and t'_A are the departure and arrival times (on a clock at A) of a light signal sent from A to B and reflected back there at time t_B (on a clock of the same kind at B). The event at A that is simultaneous to the arrival of the signal at B happens at $t''_A = (t_A + t'_A)/2 = t_B$. As stated by Einstein himself even many years later, this statement about simultaneity is "neither a supposition nor a hypothesis about the physical nature of light, but a stipulation" [5]. In particular, Einstein did not make it a requirement on the (one-way) *speed* of light, thus avoiding circularity in his logic.

To state it clearly, Einstein did not *assume* the one-way speed of light to be the same in the directions from A to B and from B to A, as is sometimes stated. Rather, the equality of the two one-way speeds is a *consequence* of his definition of simultaneity.[22] Moreover, it is at this point only that we have an instance of a complete definition of a one-way velocity, because now all the ingredients in the numerator and denominator of the velocity expression are defined. The universality of the one-way speed of light then follows from Einstein's definition of simultaneity, as soon as it is established that the two-way speed of light is universal, i.e., with the formulation of the second postulate.[23]

[19] Of course, a reassessment of the notion of simultaneity was the foundation for the theory, without which it would have been difficult to develop, suggesting that the concept should not be given up at least in special relativity.

[20] Or, for that matter, of any time derivative, because that requires a synchronized time to be defined in a neighbourhood of the point where the derivative is to be evaluated.

[21] Here, A and B are not events but spatial positions (or world lines). I could rename them in order to avoid confusion, but I would like to keep Einstein's notation. And I do not want to change my notation above either, as many changes would be needed.

[22] And most likely, he chose his definition with this consequence in mind. Nevertheless, his own wording informs us that it is at least inaccurate to claim he made an assumption about the speed of light in his definition of simultaneity.

[23] Which happened *after* his definition of simultaneity at a distance [4], so Einstein did not have to indicate wether

That Einstein's definition is not compulsory, was discussed in some detail by Reichenbach [6]. He gave an alternative definition, containing a parameter ε that may take a value between 0 and 1. It is written as

$$t_B - t_A = \varepsilon(t'_A - t_A)\,, \tag{2}$$

which may be recast in the form $t_B = (1-\varepsilon)t_A + \varepsilon t'_A$ and that obviously reduces to (1) for $\varepsilon = 1/2$. As we shall see, this definition will lead, for $\varepsilon \neq 1/2$ to a different speed of light from A to B than from B to A, which means that we will not get the same simultaneity relation by using Eq. (2) to synchronize the clock at A with that at B as by using the same equation with the roles of A and B interchanged to synchronize the clock at B with that at A.[24] That is, the sychronization procedure is not symmetric for $\varepsilon \neq 1/2$. We *would* get the same simultaneity relation, however, if in addition to interchanging the roles of A and B, we would replace the parameter ε by $1-\varepsilon$ (which for $\varepsilon = 1/2$ means keeping the same value of ε as before, so the Einstein synchronization procedure *is* symmetric).

Before discussing more details, let us calculate the speeds of light in the directions $A \to B$ (which we indicate by a subscript +) and $B \to A$ (indicated by $-$). If the spatial separation is Δs, we have

$$c_+ = \frac{\Delta s}{t_B - t_A} = \frac{\Delta s}{\varepsilon(t'_A - t_A)}\,, \qquad c_- = \frac{\Delta s}{t'_A - t_B} = \frac{\Delta s}{(1-\varepsilon)(t'_A - t_A)}\,, \tag{3}$$

which, together with the result for the two-way velocity,

$$c = \frac{2\Delta s}{t'_A - t_A}\,, \tag{4}$$

leads to

$$c_+ = \frac{c}{2\varepsilon}\,, \qquad c_- = \frac{c}{2(1-\varepsilon)}\,. \tag{5}$$

It is easy to see that $1/c_+ + 1/c_- = 2/c$, which expresses the second postulate in the absence of the requirement of Einstein simultaneity.

To obtain a simultaneity relation in all of space, we may now use one clock at a fixed position A as the master clock and distribute clocks at many different positions B throughout space that are synchronized with the clock at A. This is using Reichenbach's prescription at face value and I will call the procedure *point-centered Reichenbach ε-synchronization.* The resulting simultaneity relation has the peculiarity that on a straight line containing A the speed of light is c_+ for light moving away from A and c_- for light returning to it. A light ray passing through A will change velocity from c_- to c_+, and this holds for all directions. It is clear that this makes the position of A distinguished and that therefore the same procedure centered at different

the postulate refers to one-way or two-way speeds. Had he given his two postulates before defining simultaneity, the second postulate could have referred to two-way speeds only, because one-way speeds remain undefined until simultaneity is defined.

[24] As I perceive it, in most discussions of the synchronization procedure, it is tacitly assumed that it is the clock at B that is reset after the trip of the light ray from A to B and back. If the time shown by the clock at B on arrival of the light ray was $t_{B\,\mathrm{old}}$, then synchronization with that at A means resetting it after the full round trip of the light to its *actual reading plus* $t_B - t_{B\,old}$. Clearly, this cannot be done at time t'_A, because after the return of the light signal, only A has all the information (the times t'_A and t_A) needed to calculate t_B. So at least one more signal has to be sent to B, informing them about t'_A (they can already know t_A, of course). Therefore, the way the procedure is described (i.e., without the last signal), it can be used to synchronize the clock at A with that at B, not vice versa. In the case $\varepsilon \neq 1/2$, we will however wish to synchronize many clocks with A, using Eq. (2), so the third signal will be tacitly implied.

points will lead to different synchronies. While point-centered Reichenbach synchronization enables the definition of a global time and hence leads to an equivalence relation that defines simultaneity of distant events, the operational procedure is, as we have seen, not symmetric, nor is it transitive, unless, of course, $\varepsilon = 1/2$, i.e., it reduces to Einstein synchronization. The prescription cannot directly be applied to any pair of clocks, rather every clock must be synchronized with the clock at A.

However, the basic approach described by Eq. (2) for a pair of clocks can be generalized easily in a way not distinguishing a single point and giving the resulting simultaneity relation the property of translational invariance. I will call this a *directional (Reichenbach) ε-synchronization*. The approach is straightforward in a $1+1$ dimensional spacetime (the spatial part of which we identify with the x axis). Use the synchronization rule with parameter ε for clocks to the right of A (i.e., having $x > x_A$) and with parameter $1-\varepsilon$ for clocks to the left of A (i.e., having $x < x_A$). Obviously, this will set the speed of light equal to c_+ for right-moving light rays and equal to c_- for left-moving ones. Arbitrary pairs of clocks along the line can be synchronized by either synchronizing the right one with the left one using the ε rule or the left one with the right one using the $1-\varepsilon$ rule. Obviously, the synchrony so obtained will be the same whether we synchronize all clocks with that at A directly or start a chain of pairwise clock synchronizations $B \leftarrow A$, $C \leftarrow B$, etc.[25]

Suppose now that $\check{t}$ corresponds to an Einstein synchronized time. If we take the direction $A \to B$ to be the x direction of our frame of reference, with A at $x = 0$ and $x_B > 0$, moreover set the "Reichenbach time" t at $x = 0$ according to $t = \check{t}$, then a light signal sent to B from A will reach B at $\check{t}_B = \check{t}_A + x_B/c$ (because $\check{t}$ is Einstein synchronized) and at $t_B = t_A + 2\varepsilon x_B/c = \check{t}_A + 2\varepsilon x_B/c$. Subtracting the two expressions, we find $t_B = \check{t}_B + (2\varepsilon - 1)x_B/c$. This holds for any position B, so the relationship between the Reichenbach time t and the Einstein time $\check{t}$ at position $x > 0$ is generally

$$t = \check{t} + \frac{2\varepsilon - 1}{c} x \,. \tag{6}$$

It is easy to see that the *same* relation holds for negative x, if we use the Reichenbach prescription with $1-\varepsilon$ (i.e., $t_B = t_A - 2(1-\varepsilon)x_B/c$). Moreover, Eq. (6) can be taken to define a directional Reichenbach synchrony in all of three-space,[26] and we may then derive the relationship between any velocity $\boldsymbol{v}$ as measured with Einstein synchronization and the corresponding velocity $\tilde{\boldsymbol{v}}$ resulting for this particular Reichenbach synchronization:

$$\tilde{\boldsymbol{v}} \equiv \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\check{t} + \mathrm{d}x\,(2\varepsilon - 1)/c} = \frac{\mathrm{d}\boldsymbol{x}/\mathrm{d}\check{t}}{1 + ((2\varepsilon - 1)/c)(\mathrm{d}x/\mathrm{d}\check{t})} = \frac{\boldsymbol{v}}{1 + ((2\varepsilon - 1)/c)\, v_x} \,. \tag{7}$$

You can easily check that if $\boldsymbol{v}$ is oriented along the x direction and the Einstein speed is c, we recover $\tilde{v}_x = c_+ = c/(2\varepsilon)$ for $x > 0$ (where $\mathrm{d}x/\mathrm{d}\check{t} = c$) and $\tilde{v}_x = -c_- = -c/(2(1-\varepsilon))$ for $x < 0$ (where $\mathrm{d}x/\mathrm{d}\check{t} = -c$).

A different way to interpret the synchrony established by Eq. (6) is to realize that it makes the simultaneity of the frame considered identical to Einstein simultaneity in a frame S' moving relative to it at a particular velocity. Since $\check{t}$ is an Einstein synchronized time, the time t' of a certain frame moving at velocity v parallel to the x direction is given by a Lorentz transformation as $t' = \gamma(\check{t} - vx/c^2)$, with $\gamma = (1 - v^2/c^2)^{-1/2}$, i.e. $t'/\gamma = \check{t} + ((2\varepsilon - 1)/c)x$,

[25] If we do not start the chain by synchronizing some clock with A, we will still get the same synchrony, but the time coordinate established may differ from one where the chain is started at A by an additive constant.

[26] This is equtivalent to synchronizing a clock located on the ray with direction cosine $\cos\alpha$ referred to the x axis with that at the origin, using as parameter $\varepsilon_\alpha = (1 + (2\varepsilon - 1)\cos\alpha)/2$. Clocks in the yz plane will therefore be Einstein synchronized with that at the origin (because $\alpha = \pi/2$).

provided $v = -c(2\varepsilon - 1)$. Hence, the time t is proportional to t' with a constant (positive) proportionality factor, so the sets of hyperplanes given by $t' =$ const. and $t =$ const. are identical. We could then make *all* inertial systems have the same simultaneity relations as the Einstein synchronized system S' by defining simultaneity in a system moving at velocity $\boldsymbol{u}$ with respect to S' using a directional ε synchronization with the orientation given by $\boldsymbol{u}$ and its parameter by $\varepsilon = (1 + |\boldsymbol{u}|/c)/2)$. (This would give rise to Tangherlini's [7] absolute Lorentz transformation and S' might be considered an absolute frame, defining simultaneity for all others. Of course, it would be only pseudo-absolute, because *any* inertial frame could be taken as the absolute one, not just S'.)

We have thus defined an alternative synchronization in flat spacetime that we may use as a case in point when we wish to examine how functional relationships change in comparison with the standard synchrony (which is the one chosen by Einstein). Of course, much more general modifications of the simultaneity relation are possible. Consider a time

$$t^* = \check{t} + f(\boldsymbol{r})\,. \tag{8}$$

Any resynchronization of this type will leave the round-trip time of light invariant on any closed path. Causality requirements impose an additional condition. For a light signal moving from $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$ the time taken is

$$\mathit{\Delta} t^* = \mathit{\Delta}\check{t} + f(\boldsymbol{r}_2) - f(\boldsymbol{r}_1) = \frac{|\boldsymbol{r}_2 - \boldsymbol{r}_1|}{c} + f(\boldsymbol{r}_2) - f(\boldsymbol{r}_1) \tag{9}$$

and this must be positive.[27] The same holds for the time light takes from $\boldsymbol{r}_2$ to $\boldsymbol{r}_1$. Both relations taken together yield the condition

$$|f(\boldsymbol{r}_2) - f(\boldsymbol{r}_1)| < \frac{|\boldsymbol{r}_2 - \boldsymbol{r}_1|}{c}\,, \tag{10}$$

which for $t^* = \check{t} + ax$ means that $|a| < 1/c$, and for $a = (2\varepsilon - 1)/c$ this translates to $0 < \varepsilon < 1$.

An even more general way (than that given by Eq. (8)) to generate a synchrony on a spacetime is to use an arbitrary spacelike *foliation* of spacetime. A foliation of spacetime is a subdivision into disjoint codimension-1 hypersurfaces the union of which gives the whole spacetime; it is spacelike, if each hypersurface is spacelike. The hypersurfaces define equivalence classes and their being spacelike ensures that these classes yield an appropriate simultaneity relation. Note that in a *general* spacetime, a global foliation of this kind may not be possible.[28] For a flat spacetime – which is all we are concerned with in special relativity – such a foliation always exists.

Before closing this section, I would like to mention that in order for one-way velocities to be definable, we only need a *locally* valid definition of simultaneity, because a derivative with respect to time can be defined as soon as we have a well-defined time variable in a small neighbourhood of where the derivative is needed. The simultaneity definition plays an essential role in the extension of the definition of a time coordinate along a world line to the surroundings of that world line. It is well-known that extensions of local coordinates to a larger domain may be limited by the appearance of coordinate singularities; in the case of the time coordinate, such a singularity would prevent global validity of the simultaneity relation beyond the singular point. If a local simultaneity relation can be maintained along the full path considered (let us call it $\mathcal{P}$), a velocity may still be well-defined along it, but due to the failure of global

[27] Or non-negative, if we are willing to admit infinite signal velocities. In that case the range of ε includes the values $\varepsilon = 0$ and $\varepsilon = 1$.

[28] The Goedel universe [8], containing closed timelike curves, does not admit such a foliation.

simultaneity, the average velocity between the endpoints of the path will not be given by a simplified expression of the form $\Delta s/\Delta t$, with t being a globally valid time coordinate, but will rather take the form

$$\langle v\rangle = \frac{\int_{\mathcal{P}} v\,\mathrm{d}t}{\int_{\mathcal{P}} \mathrm{d}t}\,, \tag{11}$$

where the time integration is along the path and involves the local time variable(s) compatible with the local simultaneity relation.

This situation arises in the circular geometry leading to the Sagnac effect: locally, Einstein simultaneity can be imposed everywhere along the circumference of the circle, but the corresponding time coordinate develops a discontinuity when extended around the full circle. The speed of light is c everywhere along the rim and in both directions and its average according to (11) is c, too. But when the velocity is calculated as $\Delta s/\Delta t$, where Δt refers to the local time at the position from which the light was emitted and to which it returns, the result will be different from c and depend on the direction, with or against the sense of rotation, taken by the light ray. This happens because the local time cannot be simultaneous with the time along the light path in its entirety. Instead, there is at least one point, beyond which simultaneity gets lost and the local time differs (by an additive constant) from the time along the light path. Details are given in my small treatise *Sagnac effect and uniform speed of light* of August 21, 2023.

In the current exposition, I would like to discuss some possible attempts to refute the alleged dependence of velocities on synchronization experimentally. Of course, if the logical situation is the way I explained it, such a feat must fail. If velocities are not defined before simultaneity is, then they cannot be determined experimentally without at least an implicit specification of a synchrony. It may be useful to demonstrate this explicitly with concrete examples.

Velocity measurement using the Doppler effect

When thinking about ways to measure one-way velocities using a single clock, the first thing that may come to mind is the Doppler effect. Suppose the moving object sends a monochromatic light signal towards ourselves, the observers, the frequency of which is known (because the emitting device on the object was prepared beforehand). Then by measuring the frequency of the signal that we receive, we should be able to infer the velocity of the object from the Doppler shift between emission and reception, using no more than a single clock, right?[29] Well, let us find out...[30]

Consider ourselves being at rest in system S, in which we have synchronized our clocks according to the Einstein procedure, establishing the time coordinate $\check{t}$ of Eq. (6). Assume the object the unknown velocity v of which we wish to measure to be at rest in the system S'. Choose the x direction to coincide with the direction of motion of S'. Let ω' and $k' = \omega'/c$ be the (angular) frequency and wave number of the light wave emitted by our object in frame S'. Measure the frequency ω in S. What will be the velocity?

The Lorentz transformations connecting the two systems read (if we suppress two uninteresting spatial coordinates)

$$x' = \gamma\left(x - v\check{t}\right)\,, \qquad t' = \gamma\left(\check{t} - \frac{v}{c^2}x\right)\,, \tag{12}$$

[29] Or even no clock at all, if we measure the wavelength instead of the frequency.

[30] A similar example to what I will discuss now was in the original version of my paper [1], but taken out before publication, in order to avoid lengthy and “trivial” discussions.

with $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$ and

$$x = \gamma\left(x' + vt'\right) \ , \qquad \check{t} = \gamma\left(t' + \frac{v}{c^2}x'\right) \ . \tag{13}$$

To derive a relationship between the measured frequency and the velocity, we exploit that the phase of the emitted light wave is a relativistic invariant. Writing for a component $A(x', t')$ of the wave (assumed to move from $x' > 0$ towards negative x')

$$A(x', t') \propto \mathrm{e}^{\mathrm{i}(\omega' t' + k' x')} \ , \tag{14}$$

we may therefore set

$$\omega' t' + k' x' = \omega \check{t} + \check{k} x \ , \tag{15}$$

where we denote the wavenumber in S with an inverted caret for reasons that will become transparent later. Replacing x' and t' on the left-hand side with their expressions given by the Lorentz transformation (12), we obtain an equation that must be identically satisfied in $\check{t}$ and x, yielding expressions for ω and $\check{k}$ in terms of ω' and k'. These read

$$\omega = \gamma\left(\omega' - vk'\right) \ , \qquad \check{k} = \gamma\left(k' - \frac{v}{c^2}\omega'\right) \ , \tag{16}$$

meaning that ω', k' transform like x, $\check{t}$. Inserting the dispersion relation $\omega' = ck'$ that holds in S', we find

$$\omega = \gamma\left(1 - \frac{v}{c}\right)\omega' = \left(\frac{1 - v/c}{1 + v/c}\right)^{1/2} \omega' \ , \qquad \check{k} = \gamma\left(1 - \frac{v}{c}\right) k' = \left(\frac{1 - v/c}{1 + v/c}\right)^{1/2} k' \ , \tag{17}$$

The first of these is of course the well-known formula for the relativistic longitudinal Doppler effect, which we rederived in order to have a template for a similar derivation within a Reichenbach synchronized system. From the second and first formulas together, we note that the dispersion relation in S has the same form as in S', $\omega = c\check{k}$, which is no surprise. (Both systems are Einstein synchronized.)

Solving the first equation from (17) for v, we obtain a prescription of how to calculate the velocity from the measured frequency

$$\boxed{v = c\,\frac{1 - (\omega/\omega')^2}{1 + (\omega/\omega')^2}} \ . \tag{18}$$

ω may be measured using a single clock (and the emitted frequency ω' could be tuned before the experiment using the same clock), so the claim that we did not need two clocks to measure a one-way velocity looks justified. However, Eq. (18) was derived assuming Einstein synchronization in S (where we wanted to determine the velocity). So the result is not likely to be independent of synchronization,[31] a question that we will explore further now.

To do so, we derive the Doppler effect formula for a time variable t that is Reichenbach synchronized according to Eq. (6). Note that it is perfectly feasible to set up such a synchrony in a laboratory experiment. A way to achieve it would be to first define a Cartesian coordinate system in our laboratory hall. Then we could install a grid of atomic clocks along the coordinate

[31] Since we relied on the assumption of a particular synchronization, it might be argued that what we really did was to use as the second clock (needed for the determination of a one-way velocity) a virtual device, the reading of which could be inferred via the synchronization assumption from that of the clock actually in use.

axes and parallels to them, say with a mesh size of 10 m $\times$ 10 m $\times$ 10 m (actually one clock at the origin and one at a known distance of the order of 10 m along each axis would suffice, i.e., four base clocks). Clocks in the yz plane are synchronized with the one at the origin via the Einstein procedure, clocks in the x direction with respect to one of these are synchronized with the Reichenbach ε prescription (for some $\varepsilon \neq 1/2$). All time measurements in the experiment would then be made with local radio clocks that work in the style of GPS clocks, except that they receive their signal not from satellites but from the grid clocks in the hall. With four signals, each local clock can calculate its position and time based on the known anisotropic velocity of light along each spatial direction.[32] In such a lab, measurement of time independent quantities would give the same results as in any ordinary lab (which would sport an Einstein synchronized clock system), even time intervals at *fixed* positions would remain the same, but time intervals measured between different positions would normally be different. No physical phenomenon would of course run differently. What would change is the *description* of certain dynamical phenomena, because we would use a different time coordinate.[33]

Of course, the phase of a light wave would still be invariant, so we can equate the phase in S' and that in the now Reichenbach synchronized system S:

$$\omega' t' + k' x' = \omega t + kx \,, \tag{19}$$

where in writing ω we anticipate that the frequency measured at a fixed position in S (i.e. with one clock) is independent of which of the two synchronizations under consideration we use. However, we write k for the wave number instead of $\check{k}$, because we cannot expect the wavelength to remain unchanged. (The wavelength is the distance between two successive maxima of the oscillation at a fixed time. But the meaning of "fixed time" at the – different – positions of the maxima is synchronization dependent.)

Let us introduce the abbreviation

$$a_\varepsilon = \frac{2\varepsilon - 1}{c} \,. \tag{20}$$

Then we have $\check{t} = t - a_\varepsilon x$ and can easily derive, using the Lorentz transformation (12), the coordinate transformations between (x', t') and (x, t):

$$x' = \gamma(x - v(t - a_\varepsilon x)) = \gamma(1 + a_\varepsilon v)(x - \tilde{v} t) \,, \tag{21}$$

where

$$\tilde{v} = \frac{v}{1 + a_\varepsilon v} \,. \tag{22}$$

$\tilde{v}$ has been introduced, as it clearly has the meaning of the velocity of system S' with respect to S, when the time in S is directionally Reichenbach synchronized with parameter ε. This is trivial to see: set $x' = 0$, this implies $x = \tilde{v} t$. It is then useful to express all quantities that depend on v by $\tilde{v}$:

$$v = \frac{\tilde{v}}{1 - a_\varepsilon \tilde{v}} \quad \Rightarrow \quad \gamma = \left(1 - \frac{\tilde{v}^2/c^2}{(1 - a_\varepsilon \tilde{v})^2}\right)^{-1/2} = \frac{1 - a_\varepsilon \tilde{v}}{\left[(1 - a_\varepsilon \tilde{v})^2 - \tilde{v}^2/c^2\right]^{1/2}} \,. \tag{23}$$

[32] Alternatively, the local clock could calculate its time and position within an Einstein synchronized coordinate system and then apply Eq. (6) to obtain its directionally Reichenbach ε-synchronized time.

[33] Hence, simultaneity at a distance is not a physical relation but a coordinate-like relation. It involves an element of choice like the choice of coordinates.

Moreover,

$$t' = \gamma\left(t - a_\varepsilon x - \frac{v}{c^2}x\right) = \gamma\left(t - \left(a_\varepsilon + \frac{\tilde{v}/c^2}{1-a_\varepsilon\tilde{v}}\right)x\right) , \tag{24}$$

and, eliminating v from (21), using $1 + a_\varepsilon v = 1 + a_\varepsilon\tilde{v}/(1-a_\varepsilon\tilde{v}) = 1/(1-a_\varepsilon\tilde{v})$, we finally get:

$$x' = \frac{\gamma}{1-a_\varepsilon\tilde{v}}(x - \tilde{v}t) , \qquad t' = \gamma\,(t - b_\varepsilon x) , \quad b_\varepsilon \equiv a_\varepsilon + \frac{\tilde{v}/c^2}{1-a_\varepsilon\tilde{v}} \tag{25}$$

It is seen immediately that for $\varepsilon = 1/2$, i.e., $a_\varepsilon = 0$, this reduces to the standard Lorentz transformation with $\tilde{v} = v$.

The inverse transformation will not actually be needed in this section, but is given here for reference (and use in a later section):

$$x = \gamma\left(x' + \frac{\tilde{v}}{1-a_\varepsilon\tilde{v}}t'\right) , \qquad t = \gamma\left(\frac{t'}{1-a_\varepsilon\tilde{v}} + b_\varepsilon x'\right) . \tag{26}$$

Note that the first of these equations tells us that S is moving with respect to S' at velocity $-\tilde{v}/(1-a_\varepsilon\tilde{v})$, which is not the negative of $\tilde{v}$ (but is, of course, equal to $-v$). As stated in the introduction, the velocities of S' in S and of S in S' are not reciprocal, if one of the systems is not Einstein synchronized.

Using (25) in (19), we are able to derive the Doppler effect formula. We have

$$\omega'\gamma(t - b_\varepsilon x) + k'\frac{\gamma}{1-a_\varepsilon\tilde{v}}(x - \tilde{v}t) = \omega t + kx \tag{27}$$

and collecting coeffidients of t and x, we arrive at

$$\omega = \gamma\left(\omega' - \frac{\tilde{v}}{1-a_\varepsilon\tilde{v}}k'\right) , \tag{28}$$

$$k = \gamma\left(\frac{k'}{1-a_\varepsilon\tilde{v}} - b_\varepsilon\omega'\right) , \tag{29}$$

showing that ω' and $k'/(1-a_\varepsilon\tilde{v})$ transform like x and t. Using $\omega' = ck'$ again,[34] we find for the frequency

$$\omega = \gamma\omega'\left(1 - \frac{\tilde{v}/c}{1-a_\varepsilon\tilde{v}}\right) \tag{30}$$

and, using the definition of γ (Eq. (23)), we get

$$\frac{\omega}{\omega'} = \frac{1 - a_\varepsilon\tilde{v} - \tilde{v}/c}{\left[(1-a_\varepsilon\tilde{v})^2 - \tilde{v}^2/c^2\right]^{1/2}} = \left(\frac{1 - a_\varepsilon\tilde{v} - \tilde{v}/c}{1 - a_\varepsilon\tilde{v} + \tilde{v}/c}\right)^{1/2} . \tag{31}$$

This can be easily solved for $\tilde{v}$ (square, multiply with the denominator of the right-hand side, collect terms linear in $\tilde{v}$)

$$\tilde{v} = \frac{1 - (\omega/\omega')^2}{a_\varepsilon\,(1 - (\omega/\omega')^2) + (1 + (\omega/\omega')^2)\,/c}$$

[34] The dispersion relation for ω is *not* $\omega = ck$. We have $k = \gamma\,(k'/(1-a_\varepsilon\tilde{v}) - (a_\varepsilon c + \tilde{v}/(c(1-a_\varepsilon\tilde{v})))\,k')$. Then, $k = (\gamma k'/(1-a_\varepsilon\tilde{v}))\left(1 - a_\varepsilon c + a_\varepsilon^2\tilde{v}c - \tilde{v}/c\right) = (\gamma k'/(1-a_\varepsilon\tilde{v}))(1 - a_\varepsilon\tilde{v} - \tilde{v}/c)(1 - a_\varepsilon c)$ and many terms cancel when we take the ratio between frequency and wave number: $\omega/k = c/(1-a_\varepsilon c) = c/(2(1-\varepsilon)) = c_-$. This is of course completely consistent with our previous calculation of the speed of light for "left-running" waves.

$$\Rightarrow \quad \boxed{\tilde{v} = c\,\frac{1-(\omega/\omega')^2}{1+(\omega/\omega')^2}\,\frac{1}{1+a_\varepsilon c\frac{1-(\omega/\omega')^2}{1+(\omega/\omega')^2}}}\,. \tag{32}$$

Obviously, this will reduce to (18), if $a_\varepsilon = 0$. For general ε, we obtain the relationship

$$\tilde{v} = \frac{v}{1+a_\varepsilon v}\,, \tag{33}$$

which shows that the velocities $\tilde{v}$ and v correspond to the same state of motion, the former for system S with directional Reichenbach synchronization, the latter for S with Einstein synchronization.

Hence, if the same experiment is done twice in the same system S but with the two different synchronizations, the measured frequency will be the same in both instances, but it will be considered as measurement of different velocities (that however correspond to the same physical state of motion). The reason is that the actual quantity measured is not a velocity but a frequency and that theoretical reasoning is required to turn the frequency result into a velocity measurement. In practice, this kind of theory-weighted indirect evaluation is frequent with all modern experiments, as the observation of very microscopic or very distant objects requires indirect methods that are far from straightforward input to our senses. (And even what appears to be direct input often is already interpreted data the moment we become aware of it.) To obtain the desired result from the observed quantity, we usually need theory. In the case of a velocity measurement via the Doppler effect, this theory provides the connection between the actually measured quantity, a frequency, and the quantity to be determined, a velocity. That connection is different for different synchronizations. Therefore, even though frequencies can be measured at one position with the help of a single clock, we cannot use this to determine a one-way velocity *independent of synchronization*.

The *same* Doppler shift has to be interpreted as originating from different velocities of S' on the basis of different synchronizations. Physically, the two situations are nevertheless indistinguishable. Velocities have a gauge degree of freedom that they inherit from the corresponding ambiguity of synchronization, so they take a definite value only after fixing of the gauge.

Velocity of a moving rod

Another suggestion to measure a one-way velocity using just a single clock is based on the following idea: suppose you know the length L of a train; then you can measure its velocity by positioning yourself next to the track, starting your stopwatch when its front end passes you and stopping it when its back end passes you. If the time interval on your stopwatch reads Δt, then the train's velocity was $L/\Delta t$, assuming the train did neither accelerate nor decelerate during its passage by your position.

This approach clearly works within Newtonian mechanics, and it is based on the implicit assumption that the length L of the train is velocity independent.[35] If it was not, you would not know the length unless you knew the velocity, the quantity you wish to determine. However, if you know the velocity *dependence* of the length, it may still be possible to extract the velocity of an object from a measurement of its length and of the passage times of the two ends at a given position.

To turn this into a high precision laboratory experiment, let us replace the train by a (reasonably rigid) rod of rest length L. We measure L before setting the rod in uniform motion. Call

[35] Another assumption (that we will not call into question) is that all points of the train move at the same velocity.

the rest system of the rod S' and the positions of its left and right ends x'_l and x'_r, respectively. The laboratory system is S and we measure the times $\check{t}_1$ and $\check{t}_2$ when the right and left ends of the (right-moving) rod coincide with a predetermined position x_1 (that initially is to the right of x_r). The conditions determining the two times are then $x_1 = x_r(\check{t}_1)$ and $x_1 = x_l(\check{t}_2)$. As the naming of the laboratory times suggests, we work with Einstein synchronized clocks first and will consider the more complicated case of a directional Reichenbach ε-synchronization later. The goal is now to develop a formula for the velocity v of our rod that depends on the measured quantities L and $\Delta t = \check{t}_2 - \check{t}_1$ only. In the Einstein synchronized case we could easily guess a formula using our knowledge about the relativistic length contraction. Nevertheless, we will approach the matter with somewhat more rigor, because this will clarify how to proceed within other synchronizations.

Let the right end of the rod coincide with x_1 at time $\check{t}_1$. Then the transformations (13) give us:

$$\begin{aligned} & x_1 = x_r(\check{t}_1) = \gamma(x'_r + v t'_1)\,, \qquad \check{t}_1 = \gamma(t'_1 + \frac{v}{c^2}x'_r) \\ \Rightarrow \quad & x_1 = \gamma\left(x'_r + v\left(\frac{1}{\gamma}\check{t}_1 - \frac{v}{c^2}x'_r\right)\right) = \gamma\left(x'_r\left(1 - \frac{v^2}{c^2}\right)\right) + v\check{t}_1 = \frac{x'_r}{\gamma} + v\check{t}_1\,. \end{aligned} \tag{34}$$

Moreover, if the left end of the rod coincides with x_1 at time $\check{t}_2$, we have by an analogous calculation:

$$x_1 = x_l(\check{t}_2) = \frac{x'_l}{\gamma} + v\check{t}_2\,, \tag{35}$$

and subtracting (35) from (34), we find

$$0 = x_r(\check{t}_1) - x_l(\check{t}_2) = \frac{x'_r - x'_l}{\gamma} + v(\check{t}_1 - \check{t}_2) = \frac{L}{\gamma} - v\Delta t\,, \tag{36}$$

which yields

$$v = \frac{L}{\gamma \Delta t}\,, \tag{37}$$

which is an implicit equation for the velocity, as γ depends on v. It has a simple interpretation: in system S (with Einstein synchronization), the length that has moved past x_1 in the measuring time interval is L/γ instead of L, due to the Lorentz contraction. So the velocity must be $L/(\gamma\Delta t)$, rather than $L/\Delta t$. Fortunately, the algebra is simple enough to calculate v in explicit form:

$$\begin{aligned} & v^2 = \frac{L^2}{\Delta t^2}\left(1 - \frac{v^2}{c^2}\right) \qquad \Rightarrow \qquad v^2\left(1 + \frac{L^2}{c^2\Delta t^2}\right) = \frac{L^2}{\Delta t^2} \\ & \boxed{v = \frac{L}{\Delta t}\frac{1}{\left(1 + \dfrac{L^2}{c^2\Delta t^2}\right)^{1/2}}}\,. \end{aligned} \tag{38}$$

A measurement of the velocity v would then be achievable by measuring the rest length L of the rod and the time interval Δt at one position (i.e., a single clock would suffice) passed by the moving rod, and evaluating the result using formula (38). However, that formula is based on assuming Einstein synchronization of clocks in S.

Let us now pretend that the experimenters in S use directional ε-synchronization and, for simplicity, that the rod moves parallel to the x axis determinig the orientational anisotropy of the synchronization.[36] Then the procedure to obtain a velocity formula is essentially the same as before, except that we now need the transformation (26) instead of (13).

Let the right end of the rod coincide with x_1 at time t_1:

$$\begin{aligned} & x_1 = x_r(t_1) = \gamma(x_r' + \frac{\tilde{v}}{1-a_\varepsilon \tilde{v}} t_1') \,, \qquad t_1 = \gamma \left(\frac{t_1'}{1-a_\varepsilon \tilde{v}} + b_\varepsilon x_r' \right) \,, \\ \Rightarrow \quad & t_1' = \left(\frac{t_1}{\gamma} - b_\varepsilon x_r' \right) (1 - a_\varepsilon \tilde{v}) \,, \\ & x_1 = \gamma \left(x_r' + \tilde{v} \left(\frac{t_1}{\gamma} - b_\varepsilon x_r' \right) \right) = \gamma x_r' \left(1 - b_\varepsilon \tilde{v}\right) + \tilde{v} t_1 \,. \end{aligned} \tag{39}$$

The parenthesis containing b_ε can be rewritten in a more useful form:

$$\gamma\left(1 - b_\varepsilon \tilde{v}\right) = \gamma \left(1 - a_\varepsilon \tilde{v} - \frac{\tilde{v}^2/c^2}{1 - a_\varepsilon \tilde{v}} \right) = \left(1 - a_\varepsilon \tilde{v}\right) \gamma \left(1 - \frac{\tilde{v}^2/c^2}{\left(1 - a_\varepsilon \tilde{v}\right)^2} \right) \underset{(23)}{=} \frac{1}{\gamma} \left(1 - a_\varepsilon \tilde{v}\right) \,. \tag{40}$$

Inserting this in the last equation from (39), we obtain

$$x_1 = x_r(t_1) = \frac{x_r'}{\gamma} \left(1 - a_\varepsilon \tilde{v}\right) + \tilde{v} t_1 \,, \tag{41}$$

which bears a lot of resemblance to the last expression in (34) (and in fact reproduces it for $a_\varepsilon = 0$).

A completely analogous calculation for the left end x_l' arriving at x_1 at time t_2 provides

$$x_1 = x_l(t_2) = \frac{x_l'}{\gamma} \left(1 - a_\varepsilon \tilde{v}\right) + \tilde{v} t_2 \,, \tag{42}$$

and subtracting (42) from (41) we get

$$0 = x_r(t_1) - x_l(t_2) = \frac{x_r' - x_l'}{\gamma} \left(1 - a_\varepsilon \tilde{v}\right) + \tilde{v}(t_1 - t_2) = \frac{L}{\gamma} \left(1 - a_\varepsilon \tilde{v}\right) - \tilde{v} \Delta t \,, \tag{43}$$

where Δt has the same value as in the calculation for the Einstein synchronisation, because it refers to a time difference at one position (x_1) and the Einstein time $\check{t}$ and the Reichenbach time t differ by a constant offset at a fixed position, so the difference $\check{t}_2 - \check{t}_1 = t_2 - t_1$.

We arrive at an implicit equation for $\tilde{v}$,

$$\tilde{v} = \frac{L}{\gamma \Delta t} (1 - a_\varepsilon \tilde{v}) \,, \tag{44}$$

where now γ should be expressed as a function of $\tilde{v}$ (via Eq. (23)). Again, we can solve explicitly for $\tilde{v}$ via a quadratic equation:

$$\begin{aligned} & \tilde{v} = \frac{L}{\Delta t} \left(1 - \frac{\tilde{v}^2}{c^2 (1 - a_\varepsilon \tilde{v})^2} \right)^{1/2} (1 - a_\varepsilon \tilde{v}) \\ \Rightarrow \quad & \tilde{v}^2 = \frac{L^2}{\Delta t^2} \left((1 - a_\varepsilon \tilde{v})^2 - \frac{\tilde{v}^2}{c^2} \right) \end{aligned}$$

[36] In the Einstein synchronized case, we could simply choose the x axis to lie along the direction of motion of the rod, because all directions are equivalent due to isotropy of space and the synchronization. In the case of $\varepsilon \neq 1/2$, the synchronization and the direction of motion of the rod both define a direction in space and coincidence of these two directions is but a special case. The general case can of course be treated, too, but is more complicated, so we prefer to restrict generality here.

$$\Rightarrow \qquad \left(1+\frac{L^2}{c^2\mathit{\Delta} t^2}-\frac{L^2 a_\varepsilon^2}{\mathit{\Delta} t^2}\right)\tilde{v}^2+2\frac{L^2}{\mathit{\Delta} t^2}a_\varepsilon\tilde{v}-\frac{L^2}{\mathit{\Delta} t^2}=0\,, \tag{45}$$

with the positive solution of the quadratic equation given by

$$\tilde{v}=\frac{-\dfrac{L^2}{\mathit{\Delta} t^2}a_\varepsilon+\dfrac{L}{\mathit{\Delta} t}\sqrt{1+\dfrac{L^2}{c^2\mathit{\Delta} t^2}}}{1+\dfrac{L^2}{c^2\mathit{\Delta} t^2}-\dfrac{L^2 a_\varepsilon^2}{\mathit{\Delta} t^2}}=\frac{L}{\mathit{\Delta} t}\frac{\sqrt{1+\dfrac{L^2}{c^2\mathit{\Delta} t^2}}-a_\varepsilon\dfrac{L}{\mathit{\Delta} t}}{1+\dfrac{L^2}{c^2\mathit{\Delta} t^2}-\dfrac{a_\varepsilon^2 L^2}{\mathit{\Delta} t^2}}=\frac{L}{\mathit{\Delta} t}\frac{1}{\sqrt{1+\dfrac{L^2}{c^2\mathit{\Delta} t^2}}+a_\varepsilon\dfrac{L}{\mathit{\Delta} t}}\,, \tag{46}$$

and we end up with the neat explicit formula

$$\boxed{\tilde{v}=\frac{L}{\mathit{\Delta} t}\frac{1}{\left(1+\dfrac{L^2}{c^2\mathit{\Delta} t^2}\right)^{1/2}+a_\varepsilon\dfrac{L}{\mathit{\Delta} t}}}\,. \tag{47}$$

Clearly, this reduces to (38) for $a_\varepsilon = 0$. Moreover, it is easily verified that $\tilde{v}$ from (47) and v from (38) satisfy the relationship $\tilde{v} = v/(1 + a_\varepsilon v)$.

Again, we see that in S with the indicated Reichenbach synchronization, measurement results of L for the rest length of the rod and $\mathit{\Delta} t$ for the time interval in the "single-clock velocity measurement" produce the "measured velocity" $\tilde{v}$ rather than v. Since the *same* values for L and $\mathit{\Delta} t$ (measurable independently of the synchronization) lead to *different* values for the velocity (either v or $\tilde{v}$), it is logically inevitable that *without* a synchronization specified, the experiment does not yield *any* meaningful value for the velocity. (The naive $L/\mathit{\Delta} t$ is incorrect, except for sufficiently small velocities, where the Newtonian limit is approached. Even then, the relative error of the Newtonian result in comparison with the Reichenbachian one is of order $L/(c\mathit{\Delta} t)$, if $a_\varepsilon \neq 0$, not even quadratic in the small quantity.)

The Sagnac effect

As a final example, I would like to consider the Sagnac effect, which can be used to measure an angular velocity by an interference experiment. Stefano Quattrini claimed that, consequently, it is possible to obtain the rotational velocity of a Sagnac gyroscope independent of any synchronization via observation of the phase shift of the interference pattern (between rest state and rotating state of the gyrometer), and that this experiment does not need any clocks. He is dead wrong about the synchronization independence.

I will not discuss this at the same level of detail as the methods from the two preceding sections. A description of the Sagnac effect from scratch in a directionally ε-synchronized system might become pretty complicated. However, from experience with our two preceding examples we may obtain a velocity formula valid in the context of a directional Reichenbach ε-synchronization in a rather direct way, without the detour of the Lorentz transformations (where now arbitrary directions of the velocity within the plane would have to be considered, so we would need the Lorentz transformations with more than one spatial dimension involved).

In my treatise of 21 August 2023 on the Sagnac effect, I derived the phase shift formula within the lab system in a somewhat sloppy way (actually following other authors in this respect...), evading discussions of the longitudinal Doppler effect that affect the waves along the ring. There is no such effect in the corotating frame, so the derivation within that frame is more rigorous. The end formula is nevertheless correct and agrees with literature results. Still, I would like to briefly discuss the Doppler effect here, so as to not completely sweep it under the rug.

For visualization, it is useful to imagine the light source on the disk to be a point source emitting spherical waves.[37] While surfaces of constant phase of the emitted light signal will be spherical in any inertial frame of reference (due to the universal speed of light),[38] these spheres will be concentrical only in a local inertial frame comoving with the source. For the lab observer, seeing a rotating disk, when two successive wave crests are emitted in the "forward" (i.e., corotating) direction, while the first will have traveled a distance $c\Delta t$ (where Δt is the inverse of the frequency) from its emission point, the second is not emitted from the same point but from a point $v\Delta t$ to which the source has followed the first crest. Therefore, the wavelength (the distance between two crests) will not be $c\Delta t$ but $(c-v)\Delta t$, i.e., it will be shortened by a factor of $1-v/c$. In addition, the frequency in the lab frame is reduced by a factor $\sqrt{1-v^2/c^2}$ due to time dilation, meaning Δt and, concomitantly, the wavelength is increased by the inverse of this factor. The combined effect is a relativistic Doppler effect in the forward direction, decreasing the wavelength by a factor of $((1-v/c)/(1+v/c))^{1/2}$. Considering the counterrotating direction, we find that while a crest travels $c\Delta t$, the source moves $v\Delta t$ in the opposite direction, so the distance between two successive crests increases by a factor $1+v/c$. The time dilation effect is the same as before, resulting in a Doppler effect in the backward direction that increases the wavelength by a factor $((1+v/c)/(1-v/c))^{1/2}$. Therefore, the interference pattern resulting when the two waves meet again after having circled the disk, will consist, within any region at rest in the inertial frame of the disk center, of a superposition of two waves of *different* frequencies. As a consequence, it will not be stationary,[39] but drift at constant velocity. In fact, the velocity turns out to be exactly the rotation velocity of the disk rim, i.e., in a reference frame attached to the disk, the interference pattern will be a standing wave, it will not move.

Its wavelength turns out to be the emission wavelength (in the source frame), modified in the inertial frame by the factor $\sqrt{1-v^2/c^2}$, which is the length contraction of the standing wave.[40] This is understandable, as the standing pattern in a disk stationary frame would correspond to a receiver that moves away from the source in the forward direction at the same speed as the source itself moves towards it, so the Doppler effect at the receiving end of the signal precisely cancels the Doppler effect of the source.[41] When looking towards the oncoming signal in the counterrotating direction, the receiver follows the source in a way to keep the distance constant, which again cancels the Doppler effect. This is why Kevin Brown in his "Reflections on Relativity" (`https://www.mathpages.com/rr/s2-07/2-07.htm`) emphasizes that there is no Doppler effect in a Sagnac device.

Then the result for the phase shift in a Sagnac gyrometer rotating at angular frequency Ω (as measured in the lab) is

$$\Delta\vartheta = \gamma^2 \frac{8\pi}{\lambda c} A\Omega = \frac{1}{1-\Omega^2 R^2/c^2} \frac{8\pi^2 R^2}{\lambda c} \Omega \,, \tag{48}$$

[37] The wave packets traveling along the circumference will in any case behave the same way as the parts of a spherical wave "cut out" by the disk perimeter.

[38] At this point, our discussion is in terms of the standard synchronization used in special relativity, i.e., Einstein synchronization.

[39] This is what a direct calculation shows.

[40] Of course, this is not the wavelength of the interference pattern on a screen. It is much too small to be visible to the eye, with optical frequencies. If the pattern is visualised on a screen, the latter is typically oriented almost parallel to a surface of constant phase. If it makes a very small angle α with such a surface, the visible pattern has a trace wavelength $\lambda/\sin\alpha \gg \lambda$, which is the distance between constant phase surfaces intersecting the plane of detection.

[41] The distance of source and receiver remains constant along the arc.

where A is the area enclosed by the light path and the second formula holds for a circular path with radius R (i.e., $A = \pi R^2$). If I recall correctly, Quattrini gave his corresponding formula only for the nonrelativistic limit, i.e., without the prefactor γ^2. I do not wish to neglect this factor, however, as we have seen in the last section (and could verify in the section preceding that one as well) that in the Newtonian limit, i.e., if all terms of linear or higher order in the ratio between the velocity and the speed of light are neglected, then there will be no difference between the results for Einstein synchronization and directional Reichenbach ε-synchronization.[42] Since it is this difference we are interested in, we should accurately take into account relativistic effects.

We can easily solve Eq. (48) for the (positive) rotation velocity $v = \varOmega R$ of the interferometer:[43]

$$v = c\left\{\left[\left(\frac{4\pi^2 R}{\lambda\varDelta\vartheta}\right)^2 + 1\right]^{1/2} - \frac{4\pi^2 R}{\lambda\varDelta\vartheta}\right\} . \tag{49}$$

According to Quattrini, this would imply a velocity measurement independent of synchronization, as it could be done without two clocks (in fact, not even a single clock seems to be necessary to measure the quantities on the right-hand side, i.e., R, $\varDelta\vartheta$ and the wavelength λ).

But of course, Einstein synchronization was assumed in deriving (49), from the very beginning: the time difference between the two light rays going around the disk in opposite directions, from which the phase shift can be inferred, was calculated using the speed of light to be c in either direction, in the inertial system where the disk center is at rest. Moreover, the wavelength on the right-hand side is also a quantity taken to satisfy $\nu\lambda = c$, where ν is the frequency of the light; Einstein synchronization is implicit here as well.

In order to assess the value of Quattrini's claim, we may consider the velocity expression that holds under the assumption of a different synchronization. If that gives a different velocity for the *same* measured quantities, then the claim is clearly invalid.

To make things more interesting, let us imagine a world, a planet called Rivusdives, where clocks have been synchronized from ancient times according to a particular Reichenbach ε-prescription. This might have its origin in ancient religious perceptions. Suppose, for example, that the night sky of Rivusdives sports the spectacular view of three close-by cepheids aligned along a straight line, all oscillating with exactly the same period. Let their distances be 2 ly (light years), 4 ly, and 6 ly, and the offset of their oscillation maxima be 1 and 2 weeks, respectively.[44] In the mythology of the people on Rivusdives, these three brilliant stars that oscillate almost synchronously might become the symbol of a god or of a goddess's tool. If the Rivusdivians were as inclined as the Maya on Earth to fashion their calendars (and their timekeeping in general) according to astronomical phenomena, they might use the oscillation period (which could be a few to somewhat above a hundred days) as the basic unit of their time reckoning, in particular, if it happens to be close to an integer fraction of the length of their year. It is not unconceivable that they would develop the idea of the mightiest goddess of all manifesting herself under the sign of trinity (there are similar religious beliefs on Earth...). Clearly, the fact that the three stars do not seem to oscillate exactly synchronously must be due to some imperfection on the side of the measly believers. Of course, the Rivusdivians would find a solution as their astronomical capabilities increase: that the three celestial clocks do not seem to be synchronous comes from the fact that the speed of light is finite and so they

[42] The velocity of our moving rod will then be given by $L/\varDelta t$ in both synchronizations.

[43] Clearly, $\varDelta\vartheta$ and $\varOmega$ must have the same sign, so we require the positive solution to the quadratic equation for v (for positive $\varDelta\vartheta$).

[44] This is an ambiguous statement. To make it precise, take it to be valid in the rest frame of the three stars (which have no proper motion) with the standard Einstein synchrony.

are seen at different points of their oscillation period, even though they have the same phase at the same time. Taking into account the running time of the speed of light, the Rivusdivians would find that they can get the three star-clocks into synchrony, but only if they assume the time light takes to such a star to be slightly different from the time it takes on the way back. That is, they could synchronize them assuming a directional Reichenbach ε-synchronization with $\varepsilon \neq 1/2$. Now, modern times have come, but there are still vestiges of the influence of the olden times, so the religiously motivated convention for simultaneity has been kept (just as the seven-day week has been kept in our society in spite of the decimal system having taken over in most other calculational activities).

In the modern world, Rivusdivians would have high-precision atomic clocks based on a GPS like system with satellite based clocks replaced by a grid of master clocks, aligned with the direction inscribed into space by the alignment of the three cepheids, which would give the anisotropy direction of the synchrony. Of course, the clocks of the grid would have to be mounted on rotating platforms that compensate for the rotation of the planet itself, so that the grid axes would remain parallel and orthoghonal to the alignment of the cepheids.

Formulating laws of nature, Rivusdivians would note some peculiarities. For example, a disk set in rotation by a short impulse along its rim would, as long as the friction between its axle and the bearing is negligible, rotate at constant angular velocity, if it was oriented in the yz plane, but the angular velocity would vary in the same experiment, if it was oriented in the xy plane. Our theoretical knowledge allows us to deduce why. The disk in the yz plane would be described with a time coordinate that is isotropically synchronized in that plane, so it is equivalent to Einstein synchronization there. However, the disk rotating in the xy plane would have an angular velocity $\tilde{\Omega}$ that it calculable from its – constant – angular velocity Ω in an Einstein synchronized frame according to (assuming the center of the disk to be at x_0)

$$\tilde{\Omega} = \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\mathrm{d}\varphi}{\mathrm{d}\check{t} + a_\varepsilon \mathrm{d}x} = \frac{\mathrm{d}\varphi/\mathrm{d}\check{t}}{1 + a_\varepsilon \mathrm{d}(R\cos\varphi + x_0)/\mathrm{d}\check{t}} = \frac{\Omega}{1 - a_\varepsilon R\Omega \sin\varphi}\,, \tag{50}$$

that is, the angular velocity is oscillatory in time! The scientists of that world with its cepheid goddess would also note that this oscillation is purely kinematical. It would not be visible – the disk would seem to rotate completely uniformly, the light speeds from different parts of the disk conspiring so harmoniously that the impression to the eye (arising due to a combination of light *arriving* at the same time at the eye, not due to light *sent* at the same time) would be that of a uniformly rotating disk. Also, measurements of stresses inside the disk would not reveal any deviation from cylindrical symmetry, whereas $\tilde{\Omega}$ from (50) breaks that symmetry explicitly. Nevertheless, the non-uniformity would be measurable by surrounding the disk with a set of directionally ε-synchronized clocks at equidistant arclengths, each of which notes the time a marker painted on the rim at a fixed point of the disk passes the clock. Time intervals at which successive clocks are passed would not be exactly equal but have a φ dependent modulation.

Another peculiarity of the world described would be that mechanical clocks that Rivusdivians would have been able to produce before their GPS based electronic ones and that did not exchange radio signals with the master clocks to continually resynchronize themselves with the grid would have a tendency to lose synchronization with the master clocks on journeys along the x direction. On the other hand, the same clocks would easily keep their synchronization with the grid when moved along a plane parallel to the yz plane, as long as the motion was not too fast.

Finally, as the science of Rivusdives continued to develop, someday a genius named Unuslapis might enter the stage and suggest that the goddess-given anisotropy of the world would disappear if a different synchronization was introduced that did not even need master clocks but could be done completely internally. It would make a rotating disk behave the same no matter what direction its axis of rotation was pointing. Slow clock transport would not lead to any desynchronization depending on the direction of motion. Some scientists would embrace the new world view, rendering some theoretical aspects much simpler than the preceding theories of natural science. Others would be opposed, claiming that simultaneity must be a physical given and one was not at liberty to choose a synchronization. The battle between the different factions would continue even a hundred years after the first paper of Unuslapis in which he declared synchronization to be definable in other ways than the one given by the trinitarian goddess.

One reason, why I am telling this story (apart from it being fun) is that it gave me the opportunity to introduce Eq. (50). The "natural" state of a rotating disk in directional Reichenbach ε-synchronization is not that with constant angular velocity. Rather, the rotation rate of he disk circumference is a function of the position angle φ. Now, normally one cannot determine an infinite set of values (the function values for different angles φ) from a finite set of measurements, so it would seem difficult to measure the angular velocity of the disk and, with it, the velocity of its rim in a Reichenbach synchronized world. How could a single value of the phase shift in a Sagnac experiment determine the infinitely many different velocities of the disk circumference? Of course, Rivusdivian scientists would have an answer to this - the angular velocity of Eq. (50) has a well-known (to them) functional dependence on the angle, so all a measurement must determine is the single parameter $\mathit{\Omega}$ that makes the relation quantitative.[45] The vectorial velocity of a point of the disk rim is given according to Eq. (7) by

$$\tilde{\boldsymbol{v}} = \frac{\boldsymbol{v}}{1 + a_\varepsilon\, v_x} = \frac{\boldsymbol{v}}{1 - a_\varepsilon v \sin\varphi}\,, \tag{51}$$

where $x = x_0 + R\cos\varphi$ implies $v_x = \mathrm{d}R\cos\varphi/\mathrm{d}\check{t} = -R\mathit{\Omega}\sin\varphi = -v\sin\varphi$. We are interested in the absolute value

$$\tilde{v} = |\tilde{\boldsymbol{v}}| = v/(1 - a_\varepsilon v\sin\varphi)\,. \tag{52}$$

(Note that $|a_\varepsilon v| < 1$ always.)

All we need now to convert Eq. (49) into an expression for the rotation velocity of the disk measured on Rivusdives (for a disk rotating in the xy plane) is a way to express the wavelength λ by a measurable quantity in that world. A problem that seems to pose itself is that this wavelength is simply a constant in the standard expression, uniquely determined by the frequency of the light source at rest. On Rivusdives, the wavelength of a light source is not a constant at constant frequency, it is direction dependent, since the speed of light is direction dependent itself. However, we can in fact measure the constant λ on Rivusdives as well, simply by measuring the wavelength in a similar experiment in the yz plane. Then the following will certainly be a correct formula in the particular Reichenbach ε-synchronization of Rivusdives:

$$\tilde{v} = c\left\{\left[\left(\frac{4\pi^2 R}{\lambda\mathit{\Delta}\vartheta}\right)^2 + 1\right]^{1/2} - \frac{4\pi^2 R}{\lambda\mathit{\Delta}\vartheta}\right\}\Bigg/\left(1 - a_\varepsilon c\sin\varphi\left\{\left[\left(\frac{4\pi^2 R}{\lambda\mathit{\Delta}\vartheta}\right)^2 + 1\right]^{1/2} - \frac{4\pi^2 R}{\lambda\mathit{\Delta}\vartheta}\right\}\right)\,. \tag{53}$$

The quantities R, $\mathit{\Delta}\vartheta$ and λ are all measurable, the first two being independent of synchronization and the last determinable as discussed. Of course, the constant c is measurable, too,

[45] The functional dependence would be different for different angles of the axis of rotation of the disk with the anisotropy axis of the synchronization, but the scientists of Rivusdives would have figured that out as well and take it into account after measuring the relevant angle, which would be easy as they would know the anisotropy direction given by the three cepheids.

as the two-way speed of light. Hence, measurement of the phase shift $\Delta\vartheta$ (plus the three other parameters, which can be measured before the Sagnac experiment) allows us to determine the angle dependent velocity of the rim of the disk rotating in the xy plane on Rivusdives. Of course, this velocity will not in general agree with the velocity measured in the case of Einstein synchronization, according to Eq. (49). Hence, the claim of that measurement being independent of synchronization is wrong, even though the four quantities to be measured for the determination of the right-hand side will be the same in both synchronizations. The point is (again!) that the theoretical results connecting the quantities *actually* measured with the quantity to be determined are different in different synchronizations. (In the Newtonian limit $\lambda\Delta\vartheta/(4\pi^2 R) \ll 1$, however, we obtain $v = \tilde{v} = c\lambda\Delta\vartheta/(8\pi^2 R)$.)

It may be useful to point out that Quattrini has taken two contradictory positions in our various discussions of the Sagnac effect. The first is the aforementioned one, viz. that measurement of the phase shift in the Sagnac effect allows us to determine the rotation velocity independent of any synchronization, the second the claim that the Sagnac effect proves the existence of a preferred frame of reference in the rotating system *viz.* a non-Einstein synchronized frame, in which the speed of light is constant but different in the two directions. (That synchronization, which I have termed central synchronization, is global on the circumference and can even be made global on the whole disk, provided a non-standard time coordinate is admitted.)

Why do these two claims contradict each other? Well, suppose the first was true. Then the determination of the rotation velocity via Eq. (49) would be the unique possible velocity result compatible with the experimental phase shift $\Delta\vartheta$. If it is obtainable without prior (implicit or explicit) setting of the synchronization, then it must set the synchronization itself, which would then be physical and not conventional. But the synchronization so proven to be the correct one would be Einstein synchronization. All derivations of the Sagnac phase shift in the non-rotating system that I know of assume that system to be an (Einstein synchronized) inertial system.[46] Since the result applies to all inertial systems in which the center of our rotating disk could be at rest, we would thus have proven that the only physically valid synchronization for such a system would be Einstein synchronization. Because Einstein synchronization in inertial systems moving at different velocities implies the relativity of simultaneity, this relativity would be physical as well. Meaning that there would be no absolute synchronization. Moreover, for local inertial systems no other than Einstein synchronization would be admissible as well (which could be proven by considering a disk small enough that it fits into the local system). Therefore, when going to a description of the rotating system in terms of many local inertial frames along the rim, the only admissible synchronization for these would also be Einstein synchronization. And I have shown in *Sagnac effect and uniform speed of light* that indeed such a description is possible for the Sagnac effect. But that is not the description advocated by Quattrini's second claim, where he rather favors a non-Einsteinian "preferred" synchronization. However, as we have seen, this would not be allowed as a consequence of his first claim...

Discussion and Conclusions

Occasionally, the statement is made that it is impossible to measure the one-way speed of light. This is not accurate: obviously, one *can* measure the one-way speed of light as soon as the synchronization is fixed, i.e., as soon as a working definition of simultaneity at a distance is given. On the other hand, without a simultaneity definition (at least an implicit one), *no* one-way velocity can be measured at all – the problem is not just one of the velocity of light. This is something Veritasium has not understood in his video *Why No One Has Measured the Speed Of Light* (`https://www.youtube.com/watch?v=pTn6Ewhb27k`), when he says in the

[46] As the velocity of light along the circumference is taken to be c in both directions.

introduction: "We can't measure the speed of light the same way we measure the speed of anything else". In fact, we can. Either we cannot measure both speeds (if we don't have a synchronization) or we can measure both of them (if the synchronization is fixed). The title of the video implies a falsehood,[47] unless we argue that it has become correct after the redefinition of the meter in terms of the speed of light, which sets the limiting speed for causal interactions to a fixed value and turns any measurement of that speed into a gauge measurement for our length unit instead.

A typical argument against the measurability of the one-way speed of light is that if we use Einstein synchronization to synchronize two clocks and measure the speed of light with these two clocks on the basis of the definition of velocity (i.e., measuring their distance and the time difference between the emission of a light signal from one clock and its arrival at the other), then it is a tautology that the measured speed must be c, the known two-way speed, so we have not measured but set the one-way speed. That may be true for the first measurement immediately after synchronization of the clocks. But it does not hold for later following measurements. There is no need to synchronize the same clocks a second time and subsequent measurements of the speed of light using these clocks (kept at their positions) will tell us that the one-way speed of light does not change with time. And this result is a genuine measurement result!

Of course, all of this begs the question what we should consider a valid measurement. The three experiments discussed all do not measure the velocity directly, but some quantities from which the velocity can be inferred via known theoretical relationships. In fact, I know of no experiment that measures a velocity directly. The most direct way I can imagine is to measure the distance between starting point and arrival point and the departure and arrival times and to use the definition of velocity. What has been measured is positions and times, but the theoretical relationship producing a velocity from these raw data is a definition and as such infallible, so this kind of measurement might be accepted as not being indirect.

An answer to this question might be gathered from an anecdote told by Heisenberg. The young Heisenberg had a very positivist view as to how to construct a physical theory and he wanted to apply this to the new quantum mechanics that he was going to develop. During a visit of Einstein, Heisenberg told him to believe himself that all concepts entering a theory should be based on observable quantities (thus barring metaphysics). Einstein was doubtful and pointed out: It is theory that decides what is observable. Heisenberg was duly impressed and subsequently softened his positivist attitude.

There can be little doubt that Einstein was right. Only theory can tell you what you can expect to observe in a particular experiment. Now measurements are simply particular observations, viz. observations with a quantitative outcome. So it is theory that tells you what is measurable and how. If you have a valid theory giving you the velocity from the measurement of some other quantities, then that measurement constitutes a velocity measurement, albeit possibly a relatively indirect one. In all three experiments discussed, this was the case. We had a theoretical formula giving the velocity from the two measured frequencies in the Doppler measurement,[48] we had one in the case of the moving rod, measuring the rest length and the time interval the rod took to pass a stationary clock, and finally we had a theoretical prediction for the rotation velocity of the circumference of a Sagnac interferometer, with the essential measured quantity being the phase shift. However, the theoretical formula was synchronization dependent. While I gave the result for Einstein synchronization at first and separately in all cases, only the formula for the directional Reichenbach ε-synchronization was actually

[47]The two-way speed of light has been measured many times and with Einstein synchronization this implies a measurement of the one-way speed of light in the same sense as our three examples constitute velocity measurements.

[48]The method is used in traffic control to detect the violation of speed limits via radar Doppler shifts.

necessary, because it includes Einstein synchronization as the special case $\varepsilon = 1/2$. The important point is that the result of the velocity measurement was synchronization dependent even though the directly measured quantities were not.[49]

Some authors,[50] when discussing synchronization issues in special relativity emphasize the "circularity" of Einstein's definition by claiming that synchronization relies on the equality of the one-way speeds of light whereas the universality of the one-way speed of light follows from Einstein synchronization. This is a misunderstanding, as I have pointed out before. Einstein synchronization does not assume anything about the speed of light – its definition only involves times, not speeds. It better had not, because (one-way) speeds are not defined before synchronization is.

Synchronization is *logically prior* to the definition of any velocity. It does not need velocities for its definition, but the velocity definition requires a synchronized time. Note that the times appearing in the definition of synchronization are local times at A and B, not instances of an already synchronized global time (where global means extended over a region containing A and B), so there is no circularity there either. (Of course, *after* a resetting of the clock at A or B in order to synchronize the two times, they do constitute instances of a common time, extending over a whole region.)

This *logical* priority of a definition of synchronization over that of a velocity is not understood, ignored or even denied by many, although it actually clarifies the issue. Why is this so? Because velocity is a concept that everybody believes to understand, whereas far fewer persons have thought in depth about synchronization. In fact, *historically*, the definition of velocity precedes definitions of synchronization.[51] Newton's absolute time made thinking about synchrony obsolete; its absoluteness implied its objectivity, if it was observable at all.[52] Therefore, Newtonian time provides us with a global definition of simultaneity,[53] which is implied in the definition of velocities, essentially without thinking. Then everybody thinks they know what kind of beast a velocity is and since this knowledge did not involve simultaneity or the necessity of synchronization, they do not buy the logical priority of this concept, on a gut level.

General relativity has taught us otherwise. It is, among other things, a theory of spacetime. In it, time is not absolute. Spacetime is.[54] A spacetime point is describable by a four-vector, which is an objective entity, in principle. Time is one component of the representation of such a four-vector and, as such, not objective.[55] If time is not objective, observers in different

[49] This holds for the first two examples. In the Sagnac effect case, the wavelength of light was obtained from a measurement in an Einstein synchronized plane. I simply did not want to develop an even more complicated formula, in which this wavelength was expressed via an angle dependent wavelength in arbitrary directional ε-synchronizaton.

[50] For example Ohanian [9].

[51] This is true, if we don't count in St. Augustine's polemics against the validity of horoscopes, where he invoked simultaneity of two distant births by having two wanderers walking from one place towards the other at the same speed and meeting in the middle. The two newborns should have the same horoscopes but if one was born into a rich house, the other into a poor one, they were not likely to have the same destiny... But that "definition" is not as well-known as scientific velocity definitions from Galilei's time.

[52] Absoluteness means being-so independent of anything else. Objectivity means observer invariance. So if an entity is absolute and observable, it is also objective. An example for an absolute and non-objective entity is the absolute time of Lorentz ether theory, which gives that theory an absolute simultaneity. Unfortunately, this absolute time is unobservable and, hence, not objective.

[53] An in-depth discussion of how this can be derived on the basis of the notion of causal connectibility was given above.

[54] One might argue against the absoluteness of spacetime by pointing out that *matter tells spacetime how to curve*, so spacetime is dynamically dependent on its matter content, which makes it non-absolute. But this dynamics is an internal dynamics resulting from an interpretation of spacetime geometry in terms of a splitting in space and time. On the other hand, there is no external time, in which the four-dimensional spacetime would vary. It is only static geometry and its absoluteness is to be understood in this sense.

[55] Remember the simple example of a force in Newtonian mechanics. It is a three-vector and objective, having

places may experience time differently, i.e., experience different times. A velocity definition, in which the two times appearing are not instances of a common time but actually different and unrelated, would not make much sense. It could take arbitrary values that are not related to each other by some transforation law. Therefore, it is necessary to define a common time at the two positions appearing in the velocity definition. This implies that it must be possible to say what it means that the time is the same at the two positions, hence the necessity of a simultaneity definition.

Since time is but a coordinate in most contexts of general relativity, a common time will be given by a coordinate that is global on a patch of spacetime. Obviously, there will be many possible choices for such a coordinate, as there are only a few restrictions for a coordinate to qualify as a time coordinate. Hypersurfaces of constant time must be spacelike. If they are, they already define a simultaneity relation. The conventionality of simultaneity then follows from the freedom of choice of coordinates describing spacetime.

If considered from this general relativistic viewpoint, there can be no doubt about the question of conventionality of simultaneity. Presumably, most if not all experts on general relativity will agree that simultaneity at a distance is not a matter of physical fact but a matter of convention.

Arguments against the conventionality of simultaneity have been given by persons who typically know the special theory of relativity well and insist on Einstein synchronization having more than conventional significance in inertial systems. The question of simultaneity in accelerated or gravitating systems is often disregarded, although it seems a bit unsatisfactory to answer the question of conventionality of simultaneity only for inertial systems.

There are also adherents of the idea that simultaneity is not conventional and that Einstein synchronization is not the "correct" way to establish simultaneity, advocating for example the existence of a hidden preferred synchronization that reestablishes absolute simultaneity (which would make Lorentzian ether theory a theory representing truth more closely than special relativity). Usually, followers of these ideas do not have a comprehensive grasp of special relativity. In particular, they are unaware of the (covariant) formulation of special relativity in arbitrary coordinates. Their supporting arguments are often based on incorrect application of the theory (purporting to show its "absurd consequences").

References

[1] K. Kassner, Ways to resolve Selleri's paradox, *Am. J. Phys.* **80**, 1061-1066 (2012).

[2] K. Kassner, Spatial geometry of the rotating disk and its non-rotating counterpart, *Am. J. Phys.* **80**, 772-781 (2012).

[3] M. B. Cranor, E. M. Heider, and R. H. Price, A circular twin paradox, *Am. J. Phys.* **68**, 1016–1020 (1999).

[4] A. Einstein, Zur Elektrodynamik bewegter Körper, *Ann. Phys.* **322**, 891-921 (1905). English translation in *The Principle of Relativity* (Methuen, 1923, reprinted by Dover Publications, New York, 1952), pp. 35 – 65, "On the electrodynamics of moving bodies".

[5] A. Einstein, *Relativity: The Special and the General Theory, 15th ed.* (Crown, New York, 1952).

a well-defined length and direction. Its x component, however, depends on the orientation of the coordinate system and, as such, is not objective, as it depends on the observer's chosen coordinate frame.

[6] Hans Reichenbach, *Axiomatik der relativistischen Raum-Zeit-Lehre* (Friedrich Vieweg & Sohn, Braunschweig, 1924). Maria Reichenbach's translation: *Axiomatization of the Theory of Relativity* (University of California Press, Berkeley, 1969).

[7] F. R. Tangherlini, *The Velocity of Light in Uniformly Moving Frame*, Ph.D. thesis, Stanford University, 1958.

[8] K. Gödel, An Example of a New Type of Cosmological Solutions of Einstein's Field Equations of Gravitation, *Rev. Mod. Phys.* **21**, 447–450 (1949).

[9] H. Ohanian, The role of dynamics in the synchronization problem, *Am. J. Phys.* **72**, 141-148 (2004).