Light deflection by the sun

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After my discussion with Charles Francis about the two "contributions" to light deflection in the gravitational field of the sun following from the equivalence principle – one by refraction and one by acceleration – I decided to check whether I cannot show that these two contributions correspond in fact to the same physics. This turned out to be relatively simple, but to my surprise, it yielded a method to compute, via Fermat's principle, the full contribution to the deflection, including both the equivalence principle and the spatial curvature contribution, without ever encountering a Christoffel symbol!

Charles's refraction calculation is based on Fermat's principle of geometric optics. The variation of the local speed of light due to time dilation leads to an effective refractive index different from 1 of the vacuum. While Charles's calculation is straightforward, I will start from the general principle, because I want a pretty general proof.

Fermat's principle requires

$$\delta F = 0 \qquad \text{for} \qquad F = \int \frac{\mathrm{d}s}{c(s)}\,, \tag{1}$$

where $\mathrm{d}s$ is the length element of the geometrical path of the light ray, and $c(s)$ is the local speed of light. The endpoints of the path are supposed to be given. In the following, I will slightly abuse notation, in taking c for the universal speed of light and $c(x)$ for the coordinate speed of light corresponding to some point x.

If, instead of using geometrical optics to describe light, we take a Newtonian point of view that light consists of particles governed by some energy-conserving equations of motion, we have, instead of Fermat's principle, the Hamiltonian principle of least action:[1]

$$\delta S = 0 \qquad \text{for} \qquad S = \int L \mathrm{d}t\,, \tag{2}$$

where L is the Lagrangian of the problem.

To make these principles work, we need to specify the speed of light in the first and the Lagrangian in the second case.

The speed of light in a gravitational potential Φ that we assume to be weak, varies due to time dilation at different potential heights. That dilation can be calculated using the equivalence principle. The result is:

$$c(r) = c\left(1 + \frac{\Phi}{c^2}\right)\,. \tag{3}$$

It is easy to check that this agrees with Charles's result, if we take for Φ the potential of a spherical mass distribution:

$$\Phi = -\frac{GM}{r}\,. \tag{4}$$

[1] Actually, we only require the action to be stationary.

The acceleration a at radius r with Charles's sign convention will then be $a = \Phi'(r) = \frac{GM}{r^2}$, and we obtain for the the speeds of light at two close-by radii $R + \delta R$ and R

$$\begin{aligned} c(R+\delta R) &= c\left(1+\frac{\Phi(R+\delta R)}{c^2}\right) = c\left(1+\frac{\Phi(R)}{c^2}+\frac{\Phi'(R)}{c^2}\,\delta R\right) \\ &= c(R)\left(1+\frac{\Phi'(R)}{c^2(1+\Phi(R)/c^2)}\,\delta R\right) \approx c(R)\left(1+\frac{a\delta R}{c^2}\right)\,, \end{aligned} \tag{5}$$

where we have used the weak-field assumption to neglect the $\Phi(R)$ in the denominator (it gives only a second-order contribution in terms of the potential) and the identity with Charles's expression on *http://rqgravity.net/Gravitation#BendingFromEEP* becomes clear, if we remember that there $c = 1$. Of course, I will not repeat the calculation given there, which would be boring. All we have to discuss at this point is that Fermat's principle takes the form:

$$\delta \int \frac{\mathrm{d}s}{c\left(1+\frac{\Phi}{c^2}\right)} = 0\,. \tag{6}$$

Let us now consider Hamilton's principle. To obtain the Lagrangian $L = T - V$, we need the kinetic and the potential energies T and V of the photon. The kinetic energy is $h\nu$, the potential energy $(h\nu/c^2)\Phi$ (for weak fields). Hence,

$$L = h\nu\left(1-\frac{\Phi}{c^2}\right)\,, \tag{7}$$

so the action becomes:

$$S = \int h\nu\left(1-\frac{\Phi}{c^2}\right)\mathrm{d}t\,. \tag{8}$$

Now we know that there will be time dilation, so both $\mathrm{d}t$ and ν vary along the path, if the photon changes its distance r to the sun's center. But the product $\nu\mathrm{d}t$ is relativistically invariant and may be replaced by $\nu_0\mathrm{d}t_0$, where the subscript refers to a distant observer (at whose position time dilation effects have become negligible already). Moreover, we can write $\mathrm{d}t_0 = \mathrm{d}s/c$. Then the action may be written

$$S = h\nu_0\int\left(1-\frac{\Phi}{c^2}\right)\frac{\mathrm{d}s}{c} = h\nu_0\int\frac{\mathrm{d}s}{c\left(1+\frac{\Phi}{c^2}\right)} \tag{9}$$

and Hamilton's principle becomes

$$h\nu_0\;\delta\int\frac{\mathrm{d}s}{c\left(1+\frac{\Phi}{c^2}\right)} = 0\,, \tag{10}$$

which obviously is identical to (6). So at least in the weak-field limit both principles will always produce the same trajectories. I presume that this remains true for strong fields, but then the Lagrangian will be more difficult to formulate. (I took a form, in which the only relativistic element was the formula $h\nu/c^2$ for the "inertial mass" of the photon.)

Because it is fun, let us calculate the deviation from a straight line of the photon's path using Fermat's principle. Before embarking on this, let me point out, that the formula

$c(s) = c\left(1+\frac{\Phi}{c^2}\right)$ coming from the equivalence principle is not right in the Schwarzschild metric. The isotropic form of the latter reads

$$\mathrm{d}s^2 = k_1^2 c^2\,\mathrm{d}t^2 - k_2^2\left(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2\right)\ ,$$

with

$$\begin{aligned} k_1 &= \frac{1-\frac{GM}{2rc^2}}{1+\frac{GM}{2rc^2}}\ , \\ k_2 &= \left(1+\frac{GM}{2rc^2}\right)^2\ , \end{aligned} \tag{11}$$

from which we deduce a speed of light

$$c(r) = \frac{k_1}{k_2}c \approx c\left(1-\frac{2GM}{rc^2}\right) = c\left(1+\frac{2\Phi}{c^2}\right)\ . \tag{12}$$

I have taken the isotropic line element in order not to have to discuss any possible direction dependence of $c(r)$. It is obvious that in the approximation from the equivalence principle, where all of the change of the light speed is due to time dilation, such an anisotropy cannot arise, but in order to avoid any speculations about locally anisotropic speeds of light when I will later go *beyond* the equivalence principle, I prefer to use the isotropic form of the line element.

Two remarks are in order. The speed of light predicted by general relativity is obtained by that from the equivalence principle simply by doubling the potential (rather, squaring the factor $1+\Phi/c^2$). Hence, the speed of light is *not* only determined by time dilation in the full theory. This is what caused a lot of confusion to Robert Shuler and made him present his tower puzzle. Had he done a small calculation using the line element, no confusion would have had to come up. (When the usual form of the Schwarzschild line element is taken, the calculation gives the same result in the radial direction as obtained here isotropically.)

Furthermore, once we have calculated the result using the equivalence principle, we can also obtain the full result, simply by doubling Φ. Because our calculation is weak field, the result will be linear in Φ, so doubling Φ means also doubling the deflection angle. Therefore, Fermat's principle is able to give the full deflection angle, if only the correct speed of light is inserted! This should settle once and for all that there is no room of interpretation allowing to *add* the results from Fermat's and from Hamilton's principle.

With these preliminaries, let us proceed to the calculation. It is slightly more complicated than that of Charles, but the approach seems more fundamental to me.

We can write the functional F to be minimized in several ways:

$$F = \int \frac{\sqrt{1+y'(x)^2}}{c\left(1+\frac{\Phi(r)}{c^2}\right)}\,\mathrm{d}x = \int \frac{\sqrt{r'(\varphi)^2+r(\varphi)^2}}{c\left(1+\frac{\Phi(r)}{c^2}\right)}\,\mathrm{d}\varphi = \int \frac{\sqrt{1+r^2\varphi'(r)^2}}{c\left(1+\frac{\Phi(r)}{c^2}\right)}\,\mathrm{d}r\ , \tag{13}$$

corresponding to Cartesian and polar coordinates, respectively ($x = r\cos\varphi$, $y = r\sin\varphi$). Which of these three forms is the most advantageous one? I think, the last, because in it the integrand depends on $\varphi'(r)$ but not on φ, i.e., φ is a cyclic coordinate, and the

Euler-Lagrange equations of the variational problem simplify considerably. Let us call the integrand $f(\varphi'(r), \varphi(r), r)$, then we have, because of $\partial f/\partial\varphi = 0$:

$$\frac{\mathrm{d}}{\mathrm{d}r}\frac{\partial f}{\partial\varphi'} = 0 \qquad\qquad \frac{\partial f}{\partial\varphi'} = \text{const.} \tag{14}$$

Naming the constant b/c, we find

$$\frac{\varphi'(r)r^2}{\sqrt{1+r^2\varphi'(r)^2}} = b\left(1+\frac{\Phi(r)}{c^2}\right) . \tag{15}$$

This looks like a pretty complicated differential equation for the polar angle $\varphi(r)$ of the photon's trajectory. Fortunately, we know that $\Phi/c^2 \ll 1$, and we know that the solution for $\Phi(r) = 0$ must give straight lines. So, we first determine the solution for that case and then treat $\Phi(r)$ perturbatively.

Setting $\Phi(r) = 0$, we can solve algebraically for $\varphi'(r)$

$$\varphi'(r)^2 = \frac{b^2/r^4}{1-b^2/r^2} . \tag{16}$$

We want our solution to start at $r = \infty$ with $\varphi = \varphi_0 = 0$ and pass the origin above the axis, which means that φ increases for decreasing r, so we take the square root with the *negative* sign:

$$\mathrm{d}\varphi = -\frac{b/r^2}{\sqrt{1-b^2/r^2}}\,\mathrm{d}r . \tag{17}$$

This can be solved easily with the substitution $u = b/r \Rightarrow \mathrm{d}u = -b/r^2\,\mathrm{d}r$, and the solution reads $\varphi = \varphi_0 + \arcsin u$, giving

$$r\sin(\varphi-\varphi_0) = b . \tag{18}$$

As expected, this is a straight line. It makes an angle φ_0 with the x axis and passes the origin at the minimal distance b. Setting $\varphi_0 = 0$, we have a parallel to the x axis at a distance b (taken > 0) above it. A less familiar form of (18) would be

$$\varphi = \varphi_0 + \arcsin\frac{b}{r} . \tag{19}$$

This also has the disadvantage that $\varphi(r)$ is not a unique function, there are two walues of φ for each r. Therefore, while it is easiest to set up and solve the differential equation for φ, we should always look for the inverse function $r(\varphi)$.

Now we return to the full equation (15). It can be cast in the form

$$\varphi'(r) = -\frac{\frac{b}{r^2}\left(1+\frac{\Phi(r)}{c^2}\right)}{\sqrt{1-\frac{b^2}{r^2}\left(1+\frac{\Phi(r)}{c^2}\right)^2}} . \tag{20}$$

Introducing the variable $u = b/r$ again and expanding all expressions in powers of Φ/c^2, we end up with

$$\mathrm{d}\varphi = \mathrm{d}u \left(\frac{1}{(1-u^2)^{1/2}} - \frac{GM}{bc^2} \frac{u}{(1-u^2)^{3/2}} \right) , \tag{21}$$

which can be immediately integrated

$$\varphi - \varphi_0 = \arcsin \frac{b}{r} - \frac{GM}{bc^2} \frac{1}{\sqrt{1 - \frac{b^2}{r^2}}} . \tag{22}$$

We set $\varphi_0 = 0$ again, which this time however does not mean setting $\varphi(r = \infty)$ equal to zero; instead the ray starts at an angle $-\frac{GM}{bc^2}$. To invert (22), we set $b/r = \sin\varphi$ in the small term (containing the factor GM/bc^2), where it does not hurt, and solve for b/r:

$$\frac{b}{r} = \sin \left(\varphi + \frac{GM}{bc^2} \frac{1}{\cos\varphi} \right) . \tag{23}$$

From this equation, we can read off the bending angle. For $r \to \infty$, the argument of the sine must go to zero in one direction and to π in the other. Since we fixed the ray to start at large x which corresponds to a starting angle $\varphi_1 \approx 0$, we obtain $\cos\varphi_1 \approx 1$ and $\varphi_1 = -\frac{GM}{bc^2}$ (as noted before). For the escape of the ray to infinity, we must have $\varphi = \varphi_2 \approx \pi$, hence $\cos\varphi_2 \approx -1$, hence $\varphi_2 = \pi + \frac{GM}{bc^2}$. The deflection angle then is

$$\tilde{\alpha} = \varphi_2 - \pi - \varphi_1 = \frac{2GM}{bc^2} = \frac{2\Phi(b)}{c^2}. \tag{24}$$

Using $G = 6.674 \times 10^{-11}\ \frac{\mathrm{m}^3}{\mathrm{kg\,s}^2}$, $M = 1.989 \times 10^{30}$ kg, $b = 7 \times 10^8$ m (the radius of the sun, rounded, as I found only the equatorial diameter, and the difference between that and the radius to the poles is not really small), $c = 3 \times 10^8\ \frac{\mathrm{m}}{\mathrm{s}}$, I find $\tilde{\alpha} = 4.414 \times 10^{-6}$ in radians, which has to be divided by π and multiplied by $180{\times}3600$ to obtain its value in arcseconds. This gives

$$\tilde{\alpha} = 0.869'' ,$$

which is a decent result for the contribution by the equivalence principle.

If we now take the speed of light as obtained from the Schwarzschild metric, i.e., we use (12) instead of (3) as the speed of light in Fermat's principle, the calculation is the same except that Φ is replaced by 2Φ everywhere. Therefore, we obtain for the full deflection angle

$$\alpha = 1.738'' ,$$

which is close enough to Einstein's result to be satisfactory (my sun radius is probably not very precise).

Note that the effect of spatial curvature is included in this second result, due to the fact that we did not take the speed of light as predicted by the equivalence principle but the correct value from the Schwarzschild metric.

Does this work more generally, i.e., can we predict the result that alternative theories of gravitation give? Apparently, yes. As an example, consider the (weak-field limit of the) Brans-Dicke theory, which gives the following metric for a spherically symmetric mass distribution:

$$\mathrm{d}s^2 = \left(1 - \frac{2GM}{\rho c^2}\right) c^2\,\mathrm{d}t^2 - \frac{1}{1 - \frac{2\gamma GM}{\rho c^2}}\,\mathrm{d}\rho^2 - \rho^2\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2\right)\,, \tag{25}$$

where

$$\gamma = \frac{\omega + 1}{\omega + 2} \tag{26}$$

is smaller than one for the admissible positive values of the parameter ω from the Brans-Dicke theory. (The Brans-Dicke theory turns into general relativity for $\omega \to \infty$.)

As I said before, we want isotropic light speeds, so we transform this to an isotropic spatial part of the metric first. Using

$$\rho = r\left(1 + \frac{\gamma GM}{2rc^2}\right)^2\,, \qquad r^2 = x^2 + y^2 + z^2\,, \tag{27}$$

we can transform the line element into

$$\mathrm{d}s^2 = k_1^2 c^2\,\mathrm{d}t^2 - k_2^2\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right)\,,$$

with

$$\begin{aligned} k_1 &= \left[\left(\frac{1 - \frac{\gamma GM}{2rc^2}}{1 + \frac{\gamma GM}{2rc^2}}\right)^2 + \frac{2GM(\gamma - 1)}{\left(1 + \frac{\gamma GM}{2rc^2}\right)^2 rc^2}\right]^{1/2}\,, \\ k_2 &= \left(1 + \frac{\gamma GM}{2rc^2}\right)^2\,. \end{aligned} \tag{28}$$

The calculation of the speed of light from this gives, for $GM/rc^2 \ll 1$:

$$c(r) = c\left(1 - \frac{(1+\gamma)GM}{rc^2}\right)\,, \tag{29}$$

from which we can immediately obtain the deflection angle

$$\alpha_{\mathrm{BD}} = (1+\gamma)\frac{2GM}{bc^2} = (1+\gamma) \times 0.869''\,. \tag{30}$$

Since γ can be determined from this to be close to 1 experimentally, a lower limit is obtained for the value of ω. In fact, by the year 2003, the lower bound for ω had been driven up to 40000 according to Wikipedia[2] – which supports general relativity rather than the BD theory.

[2] But not from light-bending measurements.