The Sagnac effect as a confirmation of relativity

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In my preceding essay *Sagnac effect and uniform speed of light* (SEUSL) (https://wasd.urz.uni-magdeburg.de/kassner/research_gate_pres/sci_edu_res_gate/the_sagnac_effect.html) I showed that special relativity describes the effect both in the inertial frame in which the axis of rotation of the measuring apparatus is at rest and in the corotating frame, for a circularly shaped light path, with light propagating *in vacuum* (or in a medium with refractive index one). Results in the corotating frame were demonstrated for two different synchronizations, clarifying that the physics of the Sagnac effect does not lead to preference for one of them.

I did not discuss whether relativity is *necessary* to explain the effect, I only showed that it *can* explain it, so it is *sufficient*. Moreover, I gave little attention to the fact that present-day applications of the Sagnac effect often guide the light by a glass fiber, i.e., that it does not travel through vacuum. Knowing that the effect does not depend on the refractive index quantitatively, I suggested a discussion of the Sagnac effect with light moving through a medium would involve only minor modifications. In a sense, this is true, but it turns out that the Sagnac effect with light moving in a material medium is not explicable without a relativistic theory or at least such an explanation would be difficult. As a consequence, the Sagnac effect as it is employed in present-day ring laser gyroscopes, not only does not contradict special relativity, but also favors it over non-relativistic theories!

In the following, I will briefly recall the derivation of the time difference for the "vacuum" Sagnac effect in the inertial frame, without discussing phase shifts. Then I will consider how a non-relativistic approach gives the same result and that the non-relativistic version of the theory also works approximately in the rotating frame. A discussion of the Sagnac effect with light moving in a material with refractive index larger than one will then show that relativity has no problems with this generalization, whereas the non-relativistic theory does not work anymore. Obvious modifications that would make it work again are excluded by experiment.

The "vacuum" Sagnac effect

For simplicity, we consider the Sagnac effect on a rotating disk with radius R. Call the angular frequency of (counterclockwise) rotation ω, then the velocity at the circumference is $v = \omega R$. Light rays are sent around the disk, guided along its circumference. We discuss the single-loop case, i.e. light travels only once around the disk before being made to interfere with light going in the opposite direction.[1]

The laboratory frame is an inertial system, in which the disk center is at rest. For the light ray sent along the corotating direction, we have a race between the ray and the emitter on the disk. The time of flight is the time t_+ it takes for the light ray to overtake the emitter once, which is describable as follows

$$ct_+ = 2\pi R + \omega R t_+ \qquad \Rightarrow \qquad t_+ = \frac{2\pi R}{c(1-\omega R/c)} \,. \tag{1}$$

The ray in the counterrotating direction takes a smaller time:

$$ct_- = 2\pi R - \omega R t_- \qquad \Rightarrow \qquad t_- = \frac{2\pi R}{c(1+\omega R/c)} \,. \tag{2}$$

[1] In real Sagnac interferometers, the guiding glass fiber may take several loops before the two light rays are reunited; this will simply multiply the time difference and phase shift by an integer factor (if all loops are equal), increasing the sensitivity of the apparatus.

The time difference is

$$\Delta t = t_+ - t_- = \frac{2\pi R}{c}\frac{2\omega R/c}{1-\omega^2R^2/c^2} = \gamma^2\frac{4\pi R^2\omega}{c^2} = \gamma^2\frac{4A\omega}{c^2}\,, \tag{3}$$

where $\gamma = \left(1-\frac{\omega^2R^2}{c^2}\right)^{-1/2}$ is the standard Lorentz factor and $A = \pi R^2$ the area enclosed by the loop.

The main assumption entering this derivation is that the (vacuum) speed of light is constant and the same along both paths. This is true in special relativity (with Einstein synchronization), so special relativity is sufficient to derive the result.

However, it is also true in any other theory where light satisfies the Maxwell equations in the rest frame of the disk center, which means it would be true in a stationary-ether theory that was popular before the advent of relativity, provided the inertial system considered was the ether rest frame (with Einstein synchronization).[2] This was why Sagnac believed his experiment had proven the existence of the ether and why he became an opponent of special relativity (apparently not realizing that the latter could explain the result equally well).

For an observer sitting on the disk next to the emitter, the time difference between the arrival of the two light rays is, according to special relativity, smaller by a factor of $1/\gamma = \sqrt{1-\omega^2R^2/c^2}$

$$\Delta\tau = \tau_+ - \tau_- = \frac{t_+ - t_-}{\gamma} = \gamma\frac{4\pi R^2\omega}{c^2}\,, \tag{4}$$

as one would expect from relativistic time dilation and as I have shown in SEUSL in about four different ways (if I count correctly). The simplest way to understand this result in the corotating frame is to first note that a clock moved around the loop by slow transport[3] will lag by a certain time interval $\Delta\tau_g = 2\pi\gamma R^2\omega/c^2$ with respect to an identical clock on return to it, if the motion is *along* the direction of rotation, and will be ahead of the clock that stayed in place, by $\Delta\tau_g$, if the motion is *against* the direction of rotation. This means that for any kind of signal sent around the disk with the *same* speed in both directions,[4] the signal in the corotating direction will take a time interval $\Delta\tau_g$ more on the stationary clock than the time calculated from its velocity, whereas the signal in the counterrotating direction will take $\Delta\tau_g$ less, so the total time difference between the arrival of both signals will be, for light (which has the same local speed c in both directions, due to local Einstein synchronization) $\Delta\tau = 2\Delta\tau_g$, in agreement with Eq. (4). Note that the same time difference would also apply to two sound signals, if the air in which the sound moved was rotating along with the disk (e.g. by being confined to a tunnel built on the disk circumference), because that would guarantee that the speed of sound with respect to the disk is the same in both directions.

So that is the relativistic explanation. What about an explanation in terms of the non-relativistic stationary-ether theory? Well, since the speed of light is c in the rest frame of the disk center, the speed of light along the corotating direction will be $c - v = c - \omega R$ by virtue of the Galilei transformations and $c + v = c + \omega R$ along the counterrotating direction. (These are speeds in frames moving relative to the ether, so the velocity of the frame is added or subtracted from that of the light signal.) Then the time taken by the corotating signal is

$$\tilde{\tau}_+ = \frac{2\pi R}{c-\omega R}\,, \tag{5}$$

[2] Einstein synchronization is not distinctive for special relativity. It is also implicit in Newton's absolute time, in all frames of reference where the Maxwell equations hold. Which would be just the ether rest frame.

[3] Which means that stationary clocks along the way can be locally Einstein synchronized by setting their reading equal to that of the slowly transported clock.

[4] Where velocity evaluation is based on the locally Einstein synchronized time as discussed in detail in SEUSL.

the one by the counterrotating signal reads

$$\tilde{\tau}_- = \frac{2\pi R}{c + \omega R} , \tag{6}$$

with the difference

$$\Delta\tilde{\tau} = \frac{2\pi R}{c - \omega R} - \frac{2\pi R}{c + \omega R} = \frac{4\pi R^2 \omega}{c^2 - \omega^2 R^2} = \gamma^2 \frac{4\pi R^2 \omega}{c^2} . \tag{7}$$

Here we have taken the non-relativistic circumference of the disk (its relativistic circumference in the corotating disk frame is larger by a factor of γ) to calculate the round trip times. The ether theory result Eq. (7) is not exact, because it is a factor γ larger than the correct result Eq. (4), but it is acceptable as a non-relativistic approximation. In practical applications, $\omega R \ll c$ and even the factor γ^2 in Eq. (3) is often neglected.[5]

The Sagnac effect with light traveling in a tangible material

When light is guided, say, in a glass fiber, then the speed of light with respect to that fiber will be the vacuum speed of light divided by the refractive index. A typical refractive index for glass is 1.5, so the speed of light would be $c_m \approx 2 \times 10^8$ m/s, where the subscript m stands for material (to avoid the subscript g [for glass] which we already have in $\Delta\tau_g$).

Now the speeds of the light going around the disk in opposite directions will not be equal in the inertial frame. Within special relativity, we may calculate the corotating and counterrotating velocities using the addition theorem for velocities. The velocity of light with respect to the corotating fiber loop will be $\pm c_m$, the velocity of the loop is $v = \omega R$, so we get

$$c_{m+} = \frac{c_m + \omega R}{1 + \omega R\, c_m/c^2} , \tag{8}$$

$$-c_{m-} = \frac{-c_m + \omega R}{1 - \omega R\, c_m/c^2} . \tag{9}$$

The calculation of the round trip times then follows the same recipe as in the vacuum case, i.e., we solve the equation of motion for the light ray catching up with the moving emitter,

$$\begin{aligned} c_{m+}\, t_{m+} = 2\pi R + \omega R\, t_{m+} \quad &\Rightarrow \quad \frac{c_m + \omega R - \omega R - \omega^2 R^2 c_m/c^2}{1 + \omega R\, c_m/c^2}\, t_{m+} = 2\pi R \\ t_{m+} &= \frac{2\pi R}{c_m\,(1 - \omega^2 R^2/c^2)} \left(1 + \frac{\omega R\, c_m}{c^2}\right) = \gamma^2 \frac{2\pi R}{c_m} + \gamma^2 \frac{2\pi R^2 \omega}{c^2} , \end{aligned} \tag{10}$$

and the one for the light ray running head-on towards the emitter

$$\begin{aligned} c_{m-}\, t_{m-} = 2\pi R - \omega R\, t_{m-} \quad &\Rightarrow \quad \frac{c_m - \omega R + \omega R - \omega^2 R^2 c_m/c^2}{1 - \omega R\, c_m/c^2}\, t_{m-} = 2\pi R \\ t_{m-} &= \frac{2\pi R}{c_m\,(1 - \omega^2 R^2/c^2)} \left(1 - \frac{\omega R\, c_m}{c^2}\right) = \gamma^2 \frac{2\pi R}{c_m} - \gamma^2 \frac{2\pi R^2 \omega}{c^2} . \end{aligned} \tag{11}$$

The difference in arrival times then is given by

$$\Delta t_m = t_{m+} - t_{m-} = \gamma^2 \frac{4\pi R^2 \omega}{c^2} , \tag{12}$$

[5] That factor should not be considered of relativistic origin. The naming is just a shortcut, but the same factor arises whether we evaluate Δt from a relativistic or a non-relativistic calculation. In the case of $\Delta\tilde{\tau}$, γ^2 even arises in a completely non-relativistic calculation, whereas the single γ in $\Delta\tau$ involves relativity indeed.

which is the *same* result as for vacuum and therefore independent of the refractive index of the material. Note that this is corroborated by experiments, so the result is correct. The magnitude of the Sagnac effect does not depend on whether the light used goes through vacuum (i.e., is guided by mirrors, a bit difficult for a circular path) or through a glass fiber, or some other material. Of course, the round trip time for each ray does depend on the speed of light in the material, it is about 3/2 times that for vacuum, if the refractive index is 1.5. But the difference in arrival times for the two counterpropagating rays is the same in both cases. The wavelength of the interference pattern will also depend on the material, because it is proportional to the speed of light (in the material). However, the phase shift is a non-dimensional number, the ratio between an optical path difference and the wavelength. Both the numerator and denominator will contain the speed of light, so the phase shift should also be independent of whether the experiment is done in vacuum or with a light-guiding material.

Clearly, our derivation was relativistic as it used the relativistic velocity addition formula. Can we get the same result from a non-relativistic approach? Looking back to the stationary-ether theory, we first note that the speed of light in the material is either determined by the material alone or somehow by a combined influence of the ether and the material. In any case, the velocity of light with respect to the material should be $\pm c_m$.[6] So let us assume that the velocity of light in the arrangement where the material moves is given with the help of the Galilei transformations. (What else?) The velocities of light in the inertial frame are then given by

$$\tilde{c}_{m+} = c_m + \omega R \,, \tag{13}$$

$$-\tilde{c}_{m-} = -c_m + \omega R \,. \tag{14}$$

The equations of motion determining the catch-up times of the light rays are then

$$\tilde{c}_{m+}\,\tilde{t}_{m+} = 2\pi R + \omega R\,\tilde{t}_{m+} \quad \Rightarrow \quad c_m\,\tilde{t}_{m+} = 2\pi R \quad \Rightarrow \quad \tilde{t}_{m+} = \frac{2\pi R}{c_m}\,, \tag{15}$$

$$\tilde{c}_{m-}\,\tilde{t}_{m-} = 2\pi R - \omega R\,\tilde{t}_{m-} \quad \Rightarrow \quad c_m\,\tilde{t}_{m-} = 2\pi R \quad \Rightarrow \quad \tilde{t}_{m-} = \frac{2\pi R}{c_m}\,. \tag{16}$$

The round-trip times are equal, their difference is zero, hence the Sagnac effect is not predicted!

Finally, let us consider the rotating frame. The relativistic calculation becomes trivial here, if we assume Einstein synchronization. It is in this synchronization that the speed of light in the material will be c_m for both directions. (In any other synchronization, if will be different for the forward and backward directions.) But we have already seen that if a signal is sent in opposite directions along the circumference with the same speed (which can be different from the vacuum speed of light) then the difference in arrival times back at the position of emission after completion of a full loop by each will be given by twice the detuning time $\Delta\tau_g$ of a clock transported around the loop by slow transport. Hence, we may infer

$$\Delta\tau_m = \tau_{m+} - \tau_{m-} = 2\Delta\tau_g = \gamma\frac{4\pi R^2\omega}{c^2}\,, \tag{17}$$

which agrees with the result from Eq. (4) and is by the time dilation factor $1/\gamma$ smaller than the result from Eq. (12), as it must.

The result from the non-relativistic theory is as easily calculated and consistent with the result from Eqs. (15) and (16): since the velocity of light is $\pm c_m$ with respect to the fiber material, i.e. in the moving frame, light takes the time $2\pi R/c_m$ for either loop around the disk, hence there is a zero time difference; no Sagnac effect, contrary to what the experiment says.

[6] That is what the Maxwell equations with material dependent dielectric coefficient and permeability predict.

In summary, it appears that in order to explain the Sagnac effect with light being guided by a material, we actually *need* special relativity. This may be traced back to the fact that the speed of light is now determined by the material and relative to the material rather than by an unknown ether.

It is possible to modify the ether theory to enable it to predict the Sagnac effect, but then new problems arise. In my treatment, I assumed that it is the rest frame of the material in which the speed of light becomes equal in both directions. This corresponds to "complete ether drag" by the material. The situation changes a little if we assume that the ether is dragged along only partially by the material, an idea originally suggested by Fresnel [1]. According to him, if the material moved at velocity v relative to the ether rest frame, then the speeds of light in the parallel and antiparallel directions of the motion would be $c_{m\pm} = c_m \pm av$ with $a = 1 - 1/n^2$ where n is the refractive index of the material. If we use these velocities in our inertial-frame calculation for the round-trip times of light, we find

$$t^*_{m+} = \frac{2\pi R}{c_m + (a-1)\omega R} = \frac{2\pi R}{c_m - \omega R/n^2}\,, \tag{18}$$

$$t^*_{m-} = \frac{2\pi R}{c_m - (a-1)\omega R} = \frac{2\pi R}{c_m + \omega R/n^2}\,, \tag{19}$$

$$\Delta t^*_m = t^*_{m+} - t^*_{m-} = 2\pi R \frac{2\omega R/n^2}{c_m^2 - \omega^2 R^2/n^4} = \frac{4\pi R^2 \omega}{n^2 c_m^2 - \omega^2 R^2/n^2} = \frac{4\pi R^2\omega}{c^2 - \omega^2 R^2/n^2}\,, \tag{20}$$

which is *not* quite the correct result. If we had set $a = 1 - 1/n$ instead, the result would have given the right magnitude for the Sagnac effect, but would not explain the null result of Arago's experiment,[7] for which Fresnel had developed the ether drag formula. So the formula would be in conflict with another experimental result. It seems therefore difficult to develop an ether theory of the standard type (i.e., luminoferous ether and Galilean transformations between inertial systems) that would be in agreement with the "material" version of Sagnac's experiment and not contradict other known results.

Note that a particular ether theory, *Lorentzian ether theory*, is not excluded by these considerations, because the ether in it has a peculiar behaviour, different from that in standard ether theories. In fact, Lorentzian ether theory is simply special relativity with some metaphysical baggage (or garbage?) added. Lorentzian ether theory is so constructed that its empirical predictions agree with those of special relativity. Only the interpretation is different. Measurable lengths and times are apparent quantities, different from true lengths and times that are invariant and may in principle be determined in the absolute ether frame. The latter agrees with a particular inertial frame, but it is not determinable with which one. The difference between apparent and true quantities is due to an interaction of objects and their properties with the ether. No mechanism is given for that interaction and its most fishy aspect is that it introduces symmetries into the world (Lorentz invariance) that make the ether frame unidentifiable. So Lorentzian ether theory is both a hidden-variable theory and a conspiracy theory (albeit without personal conspirators).

Reference

[1] A. Fresnel, Lettre de M. Fresnel à M. Arago sur l'influence du mouvement terrestre dans quelques phénomènes d'optique, *Annales de Chimie et de Physique* **9**, 57–66 (Sep. 1818), 286–7 (Nov. 1818)

[7] This is described in the Wikipedia article *Aether drag hypothesis*.