My favourite derivation of the Schwarzschild geometry

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The purpose of this text is to give a fairly complete description of the steps needed in deriving the Schwarzschild geometry from the field equations. Not all the details of the calculation will be shown, but I will indicate what approach is taken and where there are choices to be made.

What do we wish to obtain? The space-time metric outside a spherically symmetric time independent mass distribution. Ultimately, we may consider the distribution localized so strongly that it becomes a mass point and our solution is valid in all of space-time except at the world line of that mass point. Since the metric will be a vacuum solution, the field equations reduce to the requirement that the Ricci tensor vanishes.

We require time independence and spherical symmetry. Hence, the coefficients of the metric must not depend on the time coordinate. Moreover, to implement spherical symmetry, we take standard angular coordinates ϑ and φ to coordinatize each two-dimensional subspace at fixed time and radial coordinates (the "radial" is already an interpretation). Then the spatial part of the line element will contain the expression $\mathrm{d}\vartheta^2 + \sin\vartheta^2 \mathrm{d}\varphi^2$ but cannot have other dependencies on either ϑ or φ; moreover, the mirror symmetries that are part of spherical symmetry exclude any mixed terms containing $\mathrm{d}\vartheta$ or $\mathrm{d}\varphi$, hence the elements of the metric tensor connecting time and angular (or radial and angular) coordinates must be zero.

Therefore, the most general line element compatible with spherical symmetry and time independence may be written in the form:

$$\mathrm{d}s^2 = g_{ik}\mathrm{d}x^i\mathrm{d}x^k = -F(\tilde{r})\,\mathrm{d}\tilde{t}^2 + 2K(\tilde{r})\,\mathrm{d}\tilde{t}\mathrm{d}\tilde{r} + G(\tilde{r})\,\mathrm{d}\tilde{r}^2 + H(\tilde{r})\left(\mathrm{d}\vartheta^2 + \sin\vartheta^2\,\mathrm{d}\varphi^2\right)\,, \quad (1)$$

containing four functions of the radial coordinate $\tilde{r}$. However, these may not all be independent. We can eliminate two of them by appropriate redefinitions of the radial and time coordinates. With the metric (1), the surface of a sphere with radius $\tilde{r}$ (obtained by integrating the surface element $\sqrt{g_2}\,\mathrm{d}\vartheta\mathrm{d}\varphi = H(\tilde{r})\sin\vartheta\,\mathrm{d}\vartheta\mathrm{d}\varphi$ over the whole solid angle), becomes $4\pi H(\tilde{r})$. Choosing as new radial coordinate[1]

$$r = \sqrt{H(\tilde{r})}\,, \quad (2)$$

this area becomes $4\pi r^2$ and our line element turns into

$$\mathrm{d}s^2 = -\tilde{F}(r)\,\mathrm{d}\tilde{t}^2 + 2\tilde{K}(r)\,\mathrm{d}\tilde{t}\mathrm{d}r + \tilde{G}(r)\,\mathrm{d}r^2 + r^2\left(\mathrm{d}\vartheta^2 + \sin\vartheta^2\,\mathrm{d}\varphi^2\right)\,, \quad (3)$$

where $\tilde{K}(r) = K(\tilde{r})\,\mathrm{d}\tilde{r}/\mathrm{d}r$ and $\sqrt{\tilde{G}(r)} = \sqrt{G(\tilde{r})}\,\mathrm{d}\tilde{r}/\mathrm{d}r$ and $\tilde{F}(r) = F(\tilde{r})$. We may therefore state to have chosen the radial coordinate so that it is equal to the square root of the surface, divided by 4π, of the sphere about the origin having that radius. The same choice of radial coordinate obviously obtains if instead we require r to be $1/2\pi$ of the circumferenc of a corresponding circle centred at the origin.

A second simplification, reducing the number of independent functions on which our metric depends to two, is obtained by an appropriate choice of the time coordinate. This can be seen

[1] $H(\tilde{r})$ must be positive for the metric to have the right signature.

setting

$$\tilde{t} = t + w(r) \qquad \Rightarrow \qquad \mathrm{d}\tilde{t} = \mathrm{d}t + w'(r)\,\mathrm{d}r\,, \tag{4}$$

which transforms Eq. (3) into

$$\begin{aligned} \mathrm{d}s^2 = &-\tilde{F}(r)\,\mathrm{d}t^2 + 2\left[\tilde{K}(r) - \tilde{F}(r)w'(r)\right]\,\mathrm{d}t\mathrm{d}r \\ &+ \left[\tilde{G}(r) - \tilde{F}(r)w'(r)^2 + 2\tilde{K}(r)w'(r)\right]\mathrm{d}r^2 + r^2\left(\mathrm{d}\vartheta^2 + \sin\vartheta^2\,\mathrm{d}\varphi^2\right)\,. \end{aligned} \tag{5}$$

If we choose $w'(r) = \tilde{K}(r)/\tilde{F}(r)$, the off-diagonal term of the metric will become zero, leaving us with only two radial functions to be determined, viz. the prefactors of $\mathrm{d}t^2$ and $\mathrm{d}r^2$.

But let us first discuss the meaning of this *choice*. It makes the metric *time-orthogonal*, resulting in a line element that contains only squares of differentials. Since propagation of light may be described by setting the line element equal to zero, the time interval light will take to run from one end of a spatial line element to the other will not depend on the direction of the motion. Therefore, for any (non-moving) spatial curve with endpoints A and B light will take exactly the same time to pass from A to B *along the curve* as from B to A. Nothing more is guaranteed, neither does time-orthogonality imply an isotropic velocity of light nor a constant universal speed. As a denotation referring more directly to this particular property of the coordinate velocity of light (than the equivalent time-orthogonality), we might introduce the notion of *bidirectional symmetry*. The interesting question then is whether this feature that, according to Einstein's own words is "neither a supposition nor a hypothesis about the physical nature of light, but a stipulation" [1], should be considered desirable, in general.

Let us assume we have a horizon somewhere in space-time, i.e. a spatial surface from which light cannot escape in one direction. If this is to be described via fixing a spatial coordinate, then the horizon property means that the coordinate velocity of light along the coordinate increasing in the forbidden direction must be equal to zero. If it were larger than zero, light could move beyond the horizon, if it were smaller than zero, it would be smaller in a whole environment of the surface, meaning that light could not escape even from a slightly shifted surface, so the surface would actually be already "inside" or "below" the horizon. Now, if light has the bidirectional symmetry property (which is not really a property of light but a property following from our coordinate choice [1]) then this means that the coordinate velocity of light also in the other direction, i.e. towards the "interior" of the horizon must be zero. Therefore, light approaching the horizon from the "outside" must slow down. This is clearly counterintuitive. We tend to think that light cannot escape from a horizon in one direction *because* of a strong gravitational pull opposing motion in that direction. The *same* pull should rather make light go *faster* in the opposite direction than slow it down. Since we are talking *coordinate speeds* here, which depend on the choice of coordinates, we are certainly allowed to make a choice that leads to non-intuitive behaviour. But is it desirable?

An *advantage* of bidirectional symmetry of the velocity of light is that we can use light to synchronize distant clocks in a procedure that is called Einstein synchronization. Essentially, to any event an observer assigns the time on his clock that corresponds to the arithmetic mean of the departure and arrival times (on the same clock) of a light signal he sent to the event and that was immediately reflected back so he could receive it again and record its arrival time. Clearly, any event on a horizon can only be assigned the time infinity in such a set-up because a light signal will *never* return – pictorially, return only after infinite time – from the horizon, and the mean value of a finite and an infinite time remains infinite. But of course, this is a consequence of our choice that light should have a bidirectionally symmetric velocity. This will automatically render any time coordinate so established *singular* on a horizon, which is a *disadvantage*.

Therefore, I will not choose $w(r)$ in such a way as to render the metric time-orthogonal. Nevertheless, I would still like to deal with only two independent radial functions, not with three. Are there other, possibly more favourable choices?

Well, why don't I choose $w(r)$ so as to make the prefactor of $\mathrm{d}r^2$ in Eq. (5) equal to one? If feasible that would render the spatial line element of a slice at constant t of our space-time Euclidean, which looks like a nice feature to have. The condition to ensure this is

$$\tilde{G}(r) - \tilde{F}(r)w'(r)^2 + 2\tilde{K}(r)w'(r) = 1\,, \tag{6}$$

which after solving for $w'(r)$ turns into

$$w'(r) = \frac{\tilde{K}(r)}{\tilde{F}(r)} \pm \left(\frac{\tilde{K}(r)^2}{\tilde{F}(r)^2} + \frac{\tilde{G}(r)-1}{\tilde{F}(r)}\right)^{1/2} . \tag{7}$$

An acceptable function $w(r)$ is obtained by simple integration. If we choose the $+$-sign, it will be monotonously increasing as long as $\tilde{F}(r)$ and $\tilde{K}(r)$ are positive. We need not really worry about these details, because at this moment we are looking only for an acceptable *form* of the metric. The functions appearing in it will be determined by the field equations. If we find a solution of those, then we are sure that we have solved the physical problem as well, no matter how we got there.

With the simplification introduced by the coordinate transformation of time, we may then write the general form of our line element as follows:

$$\mathrm{d}s^2 = -f(r)\,c^2\mathrm{d}t^2 + 2k(r)\,c\,\mathrm{d}t\mathrm{d}r + \mathrm{d}r^2 + r^2\left(\mathrm{d}\vartheta^2 + \sin\vartheta^2\,\mathrm{d}\varphi^2\right)\,, \tag{8}$$

which contains only two independent function as desired.

The next step is to write down the field equations for the metric following from (8), i.e.,

$$(g_{ij}) = \begin{pmatrix} -f(r)\,c^2 & k(r)\,c & 0 & 0 \\ k(r)\,c & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\vartheta \end{pmatrix} . \tag{9}$$

This involves a number of lengthy calculations. We should obtain the Christoffel symbols from the metric and their derivatives with respect to the coordinates, then write down the Riemann curvature tensor, take its trace to get the Ricci tensor, and finally set the components of the Ricci tensor equal to zero, thus stating the vacuum field equations. Doing this by hand would be the most tedious part of the derivation and of course pretty error prone. Fortunately, nowadays there are computer algebra systems such as Maple that can do this work for you.

Plugging the metric into Maple's engine[2] and cranking the handle, I get the following non-

[2]This was the version Maple 17.

trivial components of the Ricci tensor:

$$R^t_t = -\frac{1}{4r\,(f+k^2)^2}\left[(2rf''(r)+4f'(r))\left(f(r)+k(r)^2\right)-f'(r)(f'(r)+2k(r)k'(r))\right] \,, \tag{10a}$$

$$R^t_r = \frac{1}{cr\,(f+k^2)^2}\left[f'(r)+2k(r)k'(r)\right]k(r)\,, \tag{10b}$$

$$R^r_r = -\frac{1}{4r\,(f+k^2)^2}\Big[2rf''(r)\left(f(r)+k(r)^2\right)-rf'(r)(f'(r)+2k(r)k'(r)) + 4k(r)^2f'(r)-8k(r)k'(r)f(r)\Big] \,, \tag{10c}$$

$$R^\vartheta_\vartheta = R^\varphi_\varphi = \frac{1}{2r^2\,(f+k^2)^2}\Big[2k(r)^2\left(f(r)+k(r)^2\right)-rf'(r)\left(f(r)+2k(r)^2\right) + 2rf(r)k(r)k'(r)\Big] \,, \tag{10d}$$

where I have suppressed the argument r of $f(r)$ and $k(r)$ in the prefactors.

We then have to solve the field equations

$$R^t_t = R^t_r = R^r_r = R^\vartheta_\vartheta = R^\varphi_\varphi = 0 \tag{11}$$

subject to the boundary conditions

$$\lim_{r\to\infty} f(r) = 1\,, \tag{12a}$$

$$\lim_{r\to\infty} k(r) = 0\,, \tag{12b}$$

requiring the metric to become Minkowskian at infinity.

Calculating the radial velocity of light from our metric (setting $\mathrm{d}s^2 = 0$ and $\mathrm{d}\vartheta = \mathrm{d}\varphi = 0$ in (8)) we obtain

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + 2k(r)\,c\,\frac{\mathrm{d}r}{\mathrm{d}t} - f(r)c^2 = 0 \quad\Rightarrow\quad \frac{\mathrm{d}r}{\mathrm{d}t} = c\left(\pm\sqrt{f(r)+k(r)^2}-k(r)\right)\,. \tag{13}$$

For positive $f(r)$, the $+$-sign gives $\frac{\mathrm{d}r}{\mathrm{d}t} > 0$, hence corresponds to an outgoing light ray, whereas the $-$-sign produces $\frac{\mathrm{d}r}{\mathrm{d}t} < 0$, describing an infalling ray. In order for the absolute value of the infalling velocity to be larger than that of the outgoing one, which corresponds to our intuition that gravity will help light to move inward and hinder it moving outward, we have to choose the sign of $k(r)$ positive. (As we shall see later, we do have a choice in that regard.)

Equations (10) and (11) are four distinct nonlinear ordinary differential equations for the two functions $f(r)$ and $k(r)$, which looks like too many. But as physicists we are hopeful that the problem is well-posed for physical reasons. In fact, we know that there must be a solution in the very-weak-field limit (the Newtonian one) and so only two of these four equations should eventually turn out independent.

A good solution strategy is to start with the simplest equation, i.e., by requiring the component from Eq. (10b) to vanish. This gives

$$f'(r) + 2k(r)k'(r) = 0\,, \tag{14}$$

which can be immediately integrated to yield

$$f(r) + k(r)^2 = \text{const.} \tag{15}$$

and the constant is obtained from (12) taking $r \to \infty$, hence we have

$$k(r)^2 = 1 - f(r)\,, \tag{16}$$

which means that the three remaining equations must all have the same solution. It is comforting to know at this point already that out of the two possible square roots of (16), we may restrict ourselves to the positive one.

Inserting (16) and (14) into (10a), (10c), and (10d), we obtain, using (11)

$$\begin{aligned}
&0 = rf''(r) + 2f'(r) \quad \Rightarrow \quad \frac{f''(r)}{f'(r)} + \frac{2}{r} = 0 \quad \Rightarrow \quad \ln f'(r) + \ln r^2 = \ln A = \text{const.}\\
&\quad \Rightarrow \quad f'(r) = \frac{A}{r^2} \quad \Rightarrow \quad f(r) = 1 - \frac{A}{r}\,,
\end{aligned} \tag{17a}$$

$$\begin{aligned}
&0 = 2rf''(r) + 4(1 - f(r))f'(r) + 4f'(r)f(r) \quad \Rightarrow \quad rf''(r) + 2f'(r) = 0\\
&\qquad \text{(same equation)}\,,
\end{aligned} \tag{17b}$$

$$\begin{aligned}
&0 = 2(1 - f(r)) - rf'(r)(2 - f(r)) - rf(r)f'(r) \quad \Rightarrow \quad 1 - f(r) - rf'(r) = 0\\
&\quad \Rightarrow \quad \frac{1}{r} - \frac{f'(r)}{1 - f(r)} = 0 \quad \Rightarrow \quad \ln r + \ln(1 - f(r)) = \text{const.}\\
&\quad \Rightarrow \quad r(1 - f(r)) = B = \text{const.} \quad \Rightarrow \quad f(r) = 1 - \frac{B}{r} \quad \Rightarrow \quad B = A\,.
\end{aligned} \tag{17c}$$

So we indeed find a solution that depends on one free parameter, A:

$$f(r) = 1 - \frac{A}{r}\,, \qquad k(r) = \sqrt{\frac{A}{r}}\,, \tag{18}$$

where we have chosen $k(r)$ to have positive sign and obviously we must have $A \geq 0$.

Using the equivalence principle and introducing a potential $\Phi(r)$ so that the gradient of this potential gives the local proper acceleration, one can show that $f(r) = \mathrm{e}^{2\Phi(r)/c^2}$ (if the potential is normalized the standard way, i.e., goes to zero as $r \to \infty$) [2]. Requiring this potential to reduce to the Newtonian limit as $r \to \infty$, we obtain:

$$f(r) \sim 1 + \frac{2\Phi(r)}{c^2} \sim 1 - \frac{2GM}{c^2 r} \qquad (r \to \infty)\,, \tag{19}$$

whence $A = 2GM/c^2$, where M is the mass of the central body, and therefore

$$\boxed{f(r) = 1 - \frac{2GM}{rc^2}}\,, \qquad \boxed{k(r) = \sqrt{\frac{2GM}{rc^2}}}\,, \tag{20}$$

Our metric or line element, obtained by solution of the field equations, then reads:

$$\boxed{\mathrm{d}s^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2\mathrm{d}t^2 + 2\sqrt{\frac{2GM}{rc^2}}\, c\,\mathrm{d}t\mathrm{d}r + \mathrm{d}r^2 + r^2\left(\mathrm{d}\vartheta^2 + \sin\vartheta^2\,\mathrm{d}\varphi^2\right)}\,. \tag{21}$$

This is the Schwarzschild metric, though not in standard Schwarzschild coordinates.

Let us discuss some of its salient features. There is a radius $r_S = 2GM/c^2$, where $f(r)$ becomes zero. The metric is however not singular there, because the off-diagonal terms prevent its determinant from taking on the value zero. Nevertheless, the sign change of $f(r)$ is significant. The coordinate speed of light of Eq. (13) becomes zero at r_S for the outgoing ray. The general expression simplifies to

$$\frac{\mathrm{d}r}{\mathrm{d}t} = c\left(\pm 1 - \sqrt{\frac{r_S}{r}}\right)\,, \tag{22}$$

which shows that for $r < r_S$, there is no outgoing light ray anymore, as both velocities are smaller than zero. Moreover, coordinate stationary observers are impossible at $r < r_S$, because $\mathrm{d}r = \mathrm{d}\vartheta = \mathrm{d}\varphi = 0$ would imply $\mathrm{d}s^2 > 0$, hence a spacelike interval, for the world line of such an observer. However, world lines of observers must be timelike. Therefore, any observer at $r < r_S$ must move, and the motion must be towards smaller r, because otherwise the observer would move faster than light in the outward direction. This demonstrates that there is an event horizon at $r = r_S$.

It is easy to show that an observer falling freely from some finite value r_0 towards $r = r_S$ will hit the horizon at finite proper time *and* that the global time t of the metric describing that event will also remain finite.[3] Now the conversion factor between the rate of proper time of each coordinate stationary observer and the rate of the global time is just $\sqrt{f(r)}$, which is defined and finite outside the event horizon. (Inside, there are no coordinate stationary observers anyway.) But this means that if the falling observer O_{in} hits the horizon at finite time t, also the proper times of all the coordinate stationary observers remaining outside will be finite the moment he hits the horizon.

Weird as this may seem to someone accustomed only to the standard form of the Schwarzschild metric (which is obtained from the metric (21) via the coordinate transformation $t \to t_s = t + w(r)$ with $w'(r) = \frac{\sqrt{r_S/r}}{c(1-r_S/r)}$),[4] there is no contradiction with the statement made in the context of this standard form that an *infinite* amount of proper time will pass for external observers until the time the infalling observer hits the horizon. For what do we want to compare here? In the past, the infalling observer O_{in} and some external coordinate stationary one O_{ext} may have set their clocks to the same time when the former met the latter on his way in. So at the intersection of their world lines they have the same proper time. But when the infalling observer hits the horizon, he is far away from the coordinate stationary one. How can the two proper times they have on their clocks "now", be compared? (In fact, that is not quite the right question, which would be how can they be *defined*?) Well, one way is to use the global time that is defined for both of them. If O_{in} hits the horizon at time t_{hit}, corresponding to his proper time $\tau_{\mathrm{in,hit}}$, then the corresponding proper time of O_{ext} will be $\tau(t_{\mathrm{hit}}) = \tau_{\mathrm{ext,hit}}$. Since t_{hit} is finite, this must be finite as well. If instead we use the more artificial Schwarzschild time coordinate t_s for comparison, then we will find that O_{in} still hits the horizon at $\tau_{\mathrm{in,hit}}$, because this is a point coincidence. But the time $t_s = t_s(\tau_{\mathrm{in,hit}}) = t_{s\,\mathrm{hit}}$ is infinite, because it is defined so that events on a horizon cannot have finite time coordinates, due to the insistence on bidirectional symmetry of the speed of light. Of course, the proper time of the external observer

[3] In fact, for an observer starting his journey at the inward speed he would have attained by falling from infinity to r_0, it turns out that his proper time coincides with the global time during his fall.

[4] Since the transformation fixes only the derivative of $w(r)$, it is possible to choose a radius r_0 (preferably distant from the centre), where $t_s = t$.

corresponding to this time $t_{s\,\text{hit}} = \infty$ has to be infinite, too... But the event on O_{ext}'s world line taken to correspond to the moment when O_{in} hits the horizon is a *different* event in the case where we define correspondence via t than in the case where we define correspondence via t_s. So there is not *one* event at O_{ext}'s end corresponding to O_{in}'s crossing. Instead we get to choose, which one we consider as "corresponding". This freedom of choice is not in any way affected by the fact that proper time is a relativistic invariant. The proper time between two *fixed* events on a world line of some observer is the same for all observers. But we do not have two fixed events on O_{ext}'s world line. Only the event where O_{ext} and O_{in} meet up is fixed. The crossing of the event horizon by O_{in} does not fix any event on O_{ext}'s world line.

Using light signals in an attempt to establish a relationship between the proper times of O_{ext} and O_{in} will not change this result. If we wish to know what time of O_{ext} corresponds to the time of O_{in} and have O_{in} send light signals all the time he is falling, then of course these signals will take longer and longer to arrive at O_{ext}'s position. But to compare times, O_{ext} has to subtract from the arrival time of a light signal the time it took the light to cover the distance from O_{in} to his position. Naturally, this time will also go to infinity. In the end, O_{ext} has to take the limit of a difference of two times (the time he receives the light signal and the time the signal took), both approaching infinity, to calculate the time $\tau_{\text{ext,hit}}$. Now the arrival proper time of the signal on O_{ext}'s clock is the same in both coordinate systems, since it is a point coincidence. But the running time of the light is different, because the coordinate velocity of light differs in both coordinate sets. In fact, the signal will take a much shorter time in Schwarzschild coordinates, because the outgoing light signal is about twice as fast in these near the horizon than in the coordinates describing (21). The closer to the horizon the signal is sent, the more dominant this behaviour of the speed of light in its vicinity becomes. Therefore, O_{ext}, when reckoning in Schwarzschild coordinates will subtract a much smaller number from the arrival time of the light (which does approach infinity) than when reckoning in the so-called Gullstrand-Painlevé coordinates (21). The net effect is that the difference of two infinite numbers will go to infinity in the first case and remain finite in the second.

There are other advantages of Gullstrand-Painlevé coordinates. For example, it is very difficult to discuss the question in Schwarzschild coordinates whether a black hole will evaporate from under an observer due to Hawking radiation, before he can fall in, whereas the answer is unambiguous in Gullstrand-Painlevé coordinates. The point is that the evaporation time, even though very long[5] is *finite*. Moreover, it is a time calculated for a distant observer receiving the radiation, so this time is independent of the coordinates used as long as they become Minkowskian far from the centre.

With Schwarzschild coordinates, there seem to be two options. If we assume the infinite time it takes our infalling observer to reach the horizon to be real, the horizon should start shrinking under him and disappear, before he can reach it. An infalling observer does not see Hawking radiation,[6] so the black hole would seem to disapppear below him and he would find himself propelled into the far future (by about 10^{67} years). Alternatively, he would have

[5] About 10^{67} years for a black hole the size of the sun, not accounting for the delay by the fact that there is a microwave backgrund that still increases the mass of black holes having temperatures below 2.75 K; black holes exceeding the sun's mass have temperatures below 60 nK

[6] Which the external observer may rationalize by noting that the infalling one is length contracted and Hawking radiation created by a kind of tunnel effect will appear to start its existence somewhat outside the horizon, with the infalling observer hovering over the horizon closer to it than the locus where the radiation is generated. Note that the wavelength of Hawking radiation is a few Schwarzschild radii.

to fall through the horizon in finite time. This seems implausible given that the horizon is retreating from him and he would need infinite time to reach a non-retreating horizon. Things are even worse: if we try to describe what happens using Schwarzschild coordinates, we will inevitably run into contradictions. Suppose an observer tries to just touch the horizon at one moment by firing his rocket at full throttle. Normally, on touching it, he would fall in. But with good timing, he might touch it so that due to the fact that the horizon is shrinking, he will not fall in, because a split-second after his having touched it, the horizon will be lower. (If this seems unplausible, just consider a lightwave hovering at the horizon.) So he can escape (or the lightwave could). In his local coordinates, this means that at some time τ_1 the horizon was at position r_1 and at some later time τ_2 at position $r_2 < r_1$. Now how to describe this in the terms of the distant observer? The event (τ_1, r_1) must be at time *infinity* for him, because events on the horizon cannot have another time coordinate if the speed of light is bidirectionally symmetric. At the local time $\tau_2 > \tau_1$, the radial coordinate r_1 is outside the horizon. Therefore, its description by the external observer must be that it is at a *finite* time. Light can get to (τ_2, r_1) and back. So the *later* local time which is connected by a timelike (or null) trajectory with the former, will be *earlier* for a distant observer than the first time. Of course, we can avoid this contradiction by saying by the time the horizon retreats, the black hole, being dynamic, cannot be described by the Schwarzschild metric anymore. But this means we cannot answer our question before we can solve the complicated time-dependent problem of just how the black hole evaporates.

On the other hand, with Gullstrand-Painlevé coordinates, everything is pretty simple. Suppose our infalling observer starts 5 light years from the black hole with an inward (coordinate) velocity of about one tenth of the speed of light. It is then easy to convince oneself that with these initial conditions his inward coordinate velocity will increase towards the horizon, so he will arrive there in a little less than fifty years of coordinate time. This is a fraction of 2×10^{-65} of the time it takes the black hole to evaporate (which is also coordinate time, because far from the centre coordinate time and proper time of coordinate stationary observers are the same). In a time span of fifty (or even a million) years, this evaporation will not be perceptible,[7] i.e., the Schwarzschild geometry is still an exceedingly good approximation, even though the problem formally is dynamic. But Gullstrand-Painlevé coordinates on the Schwarzschild geometry predict that the infalling observer crosses the horizon within a time of somewhat less than fifty years. Since the approximation of a stationary geometry is still good enough to be indistinguishable from reality at that time, the observer will fall in, long before Hawking radiation becomes appreciable. So we have an unambiguous answer – there is no evaporation of the black hole under the feet of our observer; he will be long dead before evaporation becomes visible, because he crosses the horizon some 10^{67} years before.[8]

[1] A. Einstein, *Relativity: The Special and the General Theory* 15th ed. (Crown, New York, 1952)

[2] K. Kassner, *Classroom reconstruction of the Schwarzschild metric*, to appear in Eur. J. Phys. **36**, November 2015, at present available as preprint arXiv:1502.00149 (2015)

[7] At a temperature of 6×10^{-8} K, energy is not radiated away very fast.
[8] $10^{67} - 50$ (or even $10^{67} - 10^6$ for that matter) is still 10^{67} to a very good approximation.