Zariski's problem on periodicities in the dimensions of linear systems

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1 Introduction

This thesis is concerned with the following question raised by Zariski in [Zar62]:

Question 1. *Let D be an effective divisor on a nonsingular projective surface S over an arbitrary field $\Bbbk$. Is it true that there exists a polynomial $p \in \mathbb{Q}[x]$ of degree ≤ 2 and a periodic function $\lambda : \mathbb{N} \to \mathbb{Q}$ such that for $n \gg 0$:*

$$h^0(S, \mathcal{O}_S(nD)) = P(n) + \lambda(n)?$$

Zariski himself proved the answer to be "yes" in case $\kappa(D) \leq 1$, where $\kappa(D)$ denotes the Kodaira–Iitaka dimension of D (i.e. $\kappa(D) + 1$ is the transcendence degree of the algebra $\bigoplus_{n \geq 0} H^0(S, O_S(nD))$ over $\Bbbk$). The remaining case $\kappa(D) = 2$ was treated by Čutkosky and Srinivas in [Cut93]. They proved Zariski's problem for fields $\Bbbk$ of characteristic 0 and for finite fields.

The aim of this thesis is to provide an easily accessible presentation of Cutkowski and Srinivas's results, assuming only basic knowledge of algebraic geometry as treated in [Har77]. Nevertheless, in order to keep the presentation compact, we don't prove every result needed. For instance, results on the Picard scheme will be stated in detail, but we refer to the literature for proofs.

Notations and Conventions: By *variety* we mean an integral separated scheme of finite type over some field $\Bbbk$ (not necessarily algebraically closed). *Subvarieties* will always be closed. Any inclusion morphism of a subvariety will be denoted by j (it will always be clear from context, which inclusion is meant). The term *surface* will be used exclusively for nonsingular, projective varieties of dimension 2 over an algebraically closed field $\Bbbk$. The line bundle corresponding to a Cartier divisor D on a variety X will be denoted $\mathcal{O}_X(D)$. If C is an effective divisor on a surface S, then the closed subscheme of S associated to C (with ideal sheaf $\mathcal{O}_S(-C)$) will also be denoted by C and we write $\mathcal{O}_C(D) := j^*\mathcal{O}_S(D)$ for any divisor D on S. The intersection pairing on a surface S will be denoted by $C \cdot D$ for divisors C and D on S, and we write $C^2 := C \cdot C$. If C and D are divisors on S, we say that $C \leq D$ if $D - C$ is effective. For a coherent $\mathcal{O}_X$-module $\mathcal{L}$ on a variety X we write $h^i(\mathcal{L}) := h^i(X, \mathcal{L}) := \dim_{\Bbbk} H^i(X, \mathcal{L})$. We write $h^i(D) := h^i(\mathcal{O}_S(D))$ for a divisor D on a surface S. If X is a variety and Y is a subvariety of X, then for any $\mathcal{O}_Y$-module $\mathcal{M}$ we will also denote by $\mathcal{M}$ the corresponding $\mathcal{O}_X$-module $j_*\mathcal{M}$ by abuse of notation. The canonical divisor on a nonsingular projective variety X is denoted K_X.

Contents

2 Zariski decomposition

The aim of this section is to introduce the Zariski decomposition, which will play an essential role in the investigation of Zariski's problem. It states, roughly speaking, that effective divisors on surfaces can be decomposed into a positive and a negative part (in a sense to be defined below).

2.1 Basic facts about divisors

We begin by proving some basic results about divisors on surfaces. Most of the results are true in much more general settings, but we concentrate on the cases needed later on.

Lemma 2. *Let C be an effective divisor on a surface S and let D be any divisor on S. Then, as $\mathcal{O}_S$-modules: $\mathcal{O}_C(D) = \mathcal{O}_C \otimes \mathcal{O}_S(D)$.*

Proof. By definition, $\mathcal{O}_C(D) = j^*\mathcal{O}_S(D)$ (as $\mathcal{O}_C$-modules), so this just follows from projection formula (see e.g. [Har77], (Ex. II.5.1), pp. 123–124). □

Proposition 3. *Let C and D be effective divisors on a surface S. Then there is a short exact sequence of $\mathcal{O}_S$-modules:*

$$0 \to \mathcal{O}_D(-C) \to \mathcal{O}_{C+D} \to \mathcal{O}_C \to 0.$$

Proof. We consider the following diagram of $\mathcal{O}_S$-modules.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_S(-C-D) \arrow[r, equal] \arrow[d] & \mathcal{O}_S(-C-D) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_S(-C) \arrow[r] \arrow[d] & \mathcal{O}_S \arrow[r] \arrow[d] & \mathcal{O}_C \arrow[r] \arrow[d, equal] & 0 \\
 & \mathcal{O}_D \otimes \mathcal{O}_S(-C) \arrow[d] & \mathcal{O}_{C+D} \arrow[d] & \mathcal{O}_C \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Here, the second row and the first two columns come from the standard exact sequences for the effective divisors C, D and $C + D$. By Lemma 2: $\mathcal{O}_D \otimes \mathcal{O}_S(-C) = \mathcal{O}_D(-C)$. So by commutativity of the diagram the statement follows from the nine lemma. □

Proposition 4. *Let C be an effective divisor on a surface S. Then for any line bundle $\mathcal{L}$ on C and for all $i \geq 0$: $h^i(S, \mathcal{L}) = h^i(C, \mathcal{L})$.*

Proof. Recall, that referring to the $\mathcal{O}_S$-module $\mathcal{L}$, we mean $j_*\mathcal{L}$. So the result follows from [Har77], (Ex. III.8.2), p. 252, because $j : C \to S$ is an affine morphism. $\square$

We will need a version of the one-dimensional Riemann–Roch theorem on divisors of a surface, i.e. on certain curves that may be singular and reducible.

Proposition 5. *Let C be an effective divisor on a surface S. Then for any divisor D on S:*

$$\chi(S, \mathcal{O}_C(D)) = \chi(C, \mathcal{O}_C(D)) = C \cdot D + 1 - p_a(C),$$

where χ denotes the Euler characteristic and p_a the arithmetic genus.

Proof. The first equality follows from Proposition 4. In case C is a nonsingular irreducible curve, the second equality follows from the usual Riemann–Roch theorem for curves and the fact that $\deg_C(\mathcal{O}_C(D)) = (C \cdot D)$ (see [Har77], §V.1). If C is an arbitrary effective divisor, it is linearly equivalent to a difference $C_1 - C_2$ of effective divisor, that are nonsingular irreducible curves on S (loc. cit.). By Proposition 3 we get a short exact sequence of $\mathcal{O}_S$-modules:

$$0 \to \mathcal{O}_C(-C_2) \to \mathcal{O}_{C+C_2} \to \mathcal{O}_{C_2} \to 0$$

We tensor with $\mathcal{O}_S(D + C_2)$ and use additivity of the Euler characteristic to get

$$\chi(\mathcal{O}_C(D)) = \chi(\mathcal{O}_{C+C_2}(D + C_2)) - \chi(\mathcal{O}_{C_2}(D + C_2)). \tag{2.1}$$

Applying additivity of the Euler characteristic to the short exact sequences

$$0 \to \mathcal{O}_S(D + C_2 - C_1) \to \mathcal{O}_S(D + C_2) \to \mathcal{O}_{C_1}(D + C_2) \to 0$$

and

$$0 \to \mathcal{O}_S(D - C) \to \mathcal{O}_S(D + C_2) \to \mathcal{O}_{C+C_2}(D + C_2) \to 0$$

we get $\chi(\mathcal{O}_{C+C_2}(D+C_2)) = \chi(\mathcal{O}_S(D+C_2)) - \chi(\mathcal{O}_S(D-C)) = \chi(\mathcal{O}_{C_1}(D+C_2))$, so together with (2.1) we have:

$$\begin{aligned}\chi(\mathcal{O}_C(D)) &= \chi(\mathcal{O}_{C_1}(D + C_2)) - \chi(\mathcal{O}_{C_2}(D + C_2)) \\ &= C_1 \cdot (D + C_2) + 1 - p_a(C_1) - (C_2 \cdot D) + C_2 - 1 + p_a(C_2),\end{aligned}$$

where we have used the already shown case of nonsingular irreducible curves. Now by [Har77], (Ex. V.1.3), pp. 366–367:

$$p_a(C) = p_a(C_1 - C_2) = p_a(C_1) - p_a(C_2) + (C_2 - C_1) \cdot C_2 + 1.$$

This finally shows $\chi(\mathcal{O}_C(D)) = C \cdot D + 1 - p_a(C)$. $\square$

Definition 6. *Let S be a surface and D a divisor on S. If $H^0(S, O_S(nD)) = 0$ for all $n > 0$, then we define $\kappa(D) := -\infty$. Otherwise we consider the graded $\Bbbk$-algebra $R(S, D) := \bigoplus_{n\geq 0} H^0(S, O_S(nD))$ and define $\kappa(D) :=$* tr.deg $R(S, D) - 1$.

We state the following characterizations of $\kappa(D)$ without proof:

Theorem 7. *Let S be a surface over $\Bbbk$ and D a divisor on S such that $\kappa(D) \neq -\infty$.*

(i) For every $n > 0$ such that $h^0(nD) > 0$ let $\varphi_n : S \to \mathbb{P}_{\Bbbk}^{h^0(nD)}$ be the rational map associated to the linear system $|nD|$ and let X_n be the closure of the image of φ_n. Then $\kappa(D) = \max \dim X_n$.

(ii) There exist integers α and β such that $\alpha n^{\kappa(D)} \leq h^0(nD) \leq \beta n^{\kappa(D)}$ for n sufficiently divisible. This property determines the integer $\kappa(D)$ uniquely.

Proof. [Iit82], §10, pp. 298–301. □

Corollary 8. *Let S be a surface and D a divisor on S. Then $\kappa(D) \leq 2$.*

Proof. This is immediate from Theorem 7, part (i). □

2.2 $\mathbb{Q}$-divisors

It turns out that for the existence of the Zariski decomposition it is necessary to enlargen slightly our class of divisors.

Definition 9. *Let S be a surface. Let* Div(S) *denote the group of divisors on S. A $\mathbb{Q}$-**divisor** is an element of the group* Div$(S) \otimes \mathbb{Q}$. *In contrast, elements of* Div(S) *are referred to as **integral divisors**, or just divisors.*

Remark 10. *The relation $\leq$ and the intersection pairing on* Div(S) *extend to* Div$(S) \otimes \mathbb{Q}$. *We observe that* $(\mathrm{Div}(S) \otimes \mathbb{Q})/\mathrm{Div}(S)$ *is a torsion group, i.e. for any $\mathbb{Q}$-divisor D on S some multiple nD ($n \geq 0$) is an integral divisor. So a $\mathbb{Q}$-divisor D on S is just a formal sum $\sum_{i=1}^{r} a_i C_i$, where $a_i \in \mathbb{Q}^*$ and C_i are integral curves on S. If all $a_i \geq 0$ (i.e. $D \geq 0$), then D is **effective** and the C_i are called **components** of D.*

Definition 11. *Let S be a surface. A $\mathbb{Q}$-divisor D is **ample**, if nD is an ample integral divisor for some integer $n > 0$.*

Definition 12. *Let S be a surface. To a $\mathbb{Q}$-divisor D on S we associate an $\mathcal{O}_S$-module $\mathcal{O}_S(D)$ by defining*

$$\Gamma(U, \mathcal{O}_S(D)) := \{f \in K(S) \mid ((f) + D)|_U \geq 0\} \cup \{0\}.$$

Remark 13. *For a $\mathbb{Q}$-divisor D denote by $\lfloor D \rfloor$ the maximal integral divisor (w.r.t. $\leq$) such that $D - [D] \geq 0$. Then $\mathcal{O}_S(D) = \mathcal{O}_S(\lfloor D \rfloor)$. In particular, $\mathcal{O}_S(D)$ is a line bundle.*

Definition 14. *Let S be a surface. A $\mathbb{Q}$-divisor P on S is **nef**, if $P \cdot C \geq 0$ for every effective divisor C on S. We say, that D is **pseudo-effective**, if $D \cdot P \geq 0$ for every nef divisor P on S.*

Proposition 15. *Let S be a surface and let P be a nef $\mathbb{Q}$-divisor on S. Then $P^2 \geq 0$.*

Proof. (We follow the proof in [Bad01], Thm. 1.25, pp. 11–12.)

Suppose that $P^2 < 0$. Let H be an ample divisor on S. Since $H^2 > 0$, there exists a positive real number $\epsilon_0 \in \mathbb{R}$ such that $(P + \alpha H)^2 \leq 0$ for all rational numbers $\alpha \in [0, \epsilon_0]$ and such that $(P + \alpha H)^2 > 0$ for all rational numbers $\alpha > \epsilon_0$. Since $(P + \alpha H) \cdot C > 0$ for all irreducible curves C and all $\alpha > 0$, the Nakai–Moishezon criterion ([Har77], Thm. V.1.10, p. 365) implies that $P + \alpha H$ is ample for all rational $\alpha > \epsilon_0$. In particular, we have $(P + \alpha H) \cdot P \geq 0$ in this case, because P is nef and some multiple of an ample divisor is linearly equivalent to an effective divisor. By density of $\mathbb{Q}$ in $\mathbb{R}$, we conclude $P^2 + \epsilon_0 P \cdot H \geq 0$. Therefore, there exists a positive rational number $\alpha_0 < \epsilon_0$ such that $P^2 + \alpha_0 P \cdot H \geq 0$ (if ϵ_0 is rational, take $\alpha_0 = \epsilon_0$, otherwise $P^2 + \epsilon_0 P \cdot H > 0$ by irrationality, in which case α_0 exists by density of $\mathbb{Q}$ in $\mathbb{R}$). However, this implies $(P + \alpha_0 H)^2 \geq P \cdot H + H^2 > 0$, a contradiction to $\alpha_0 < \epsilon_0$. □

Proposition 16. *Let D be an integral divisor on a surface S such that $\kappa(D) \geq 1$. Then D is pseudo-effective. Moreover, for any divisor D' on S and n sufficiently large: $h^0(\mathcal{O}_S(D' - nD)) = 0$.*

Proof. Let C be a nef divisor on S. Since $\kappa(D) \geq 1$, there is a positive integer m such that $h^0(\mathcal{O}_S(mD)) > 1$. Therefore mD is linearly equivalent to some non-zero effective divisor. Thus $C \cdot D \geq 0$, because C is nef. This shows that D is pseudo-effective.

To prove the second statement we consider some ample divisor H. Then $mD \cdot H > 0$ implies $D \cdot H > 0$, so for $n \gg 0$ we get $(D' - nD) \cdot H < 0$. In particular $D' - nD$ is not linearly equivalent to an effective divisor, i.e. $h^0(\mathcal{O}_S(D' - nD)) = 0$. □

Theorem 17 (Fujita's vanishing theorem)**.** *Let S be a surface, let D be an arbitrary divisor on S and let H be an ample divisor on S. Then there exists an integer n_0 such that $h^i(nH + D + P) = 0$ for all $i > 0$, $n \geq n_0$ and any nef divisor P on S.*

Proof. [Fuj81], Thm. 5.1, p. 367. □

Proposition 18. *Let S be a surface and let P be a nef divisor on S. Then $h^0(nP) = (P^2/2)n^2 + O(n)$, i.e. $h^0(nP) - (P^2/2)n^2$ is bounded by a linear polynomial in n.*

Proof. (We follow the proof in [Laz04], Cor. 1.4.41, p. 69.)

If $\kappa(P) \leq 1$ and $P^2 = 0$, then this follows from Theorem 7. Now we suppose, that $\kappa(P) = 2$ or $P^2 > 0$ (we have $P^2 \geq 0$ by Proposition 15). We claim, that in both cases $h^2(nP) = 0$ for $n \gg 0$.

By Serre duality, $h^2(nP) = h^0(K_S - nP)$ for all integers $n > 0$. In the case $P^2 > 0$ we get $(K_S - nP) \cdot P < 0$ for $n \gg 0$. Since P is nef, this means, that $K_S - nP$ cannot be linearly equivalent to an effective divisor, i.e. $h^0(K_S - nP) = 0$. In the other case $\kappa(P) = 2$ and therefore $h^0(K_S - nP) = 0$ for $n \gg 0$ by Proposition 16. This proves the claim.

The Riemann–Roch theorem on surfaces now shows that

$$h^0(nP) - h^1(nP) = \frac{1}{2} nP \cdot (nP - K) + 1 + p_a(S) = \frac{P^2}{2} n^2 + O(n),$$

so it suffices to show that $h^1(nP)$ is bounded by a linear polynomial in n.

Let H be a very ample divisor on S. By Theorem 17 we can replace H by a sufficiently large multiple such that $h^1(H + nP) = h^2(H + nP) = 0$ for all $n \geq 0$. The long exact cohomology sequence associated to

$$0 \to \mathcal{O}_S(nP) \to \mathcal{O}_S(nP + H) \to \mathcal{O}_H(nP + H) \to 0$$

shows that $h^1(S, \mathcal{O}_S(nP)) \leq h^0(H, \mathcal{O}_H(nP+H))$ and $h^1(H, \mathcal{O}_H(nP+H)) = h^2(S, \mathcal{O}_S(nP)) = 0$. But $h^0(H, \mathcal{O}_H(nP + H)) = \chi(H, \mathcal{O}_H(nP + H))$ is a linear polynomial in n by Proposition 5. □

Proposition 19. *Let P be an integral nef divisor on a surface S. Then $\kappa(P) = 2$ if and only if $P^2 > 0$.*

Proof. This follows from Proposition 18 and Theorem 7. □

2.3 Zariski decomposition

We now state the Zariski decomposition theorem.

Theorem 20. *Let S be a surface, let D be a pseudo-effective $\mathbb{Q}$-divisor on S. Then there exists a unique decomposition*

$$D = P + N$$

with $\mathbb{Q}$-divisors P and N such that:

(i) *P is nef and N is effective.*

(ii) *If $C_1, \ldots, C_r$ are the components of N, then $(P \cdot C_i) = 0$ for every i and the $r \times r$-matrix with entries $(C_i \cdot C_j)$ is negative definite.*

Conversely, if such a decomposition exists, then D is pseudo-effective. We call $D = P + N$ the ***Zariski decomposition*** *of D.*

Proof. A nice presentation can be found in [Bad01], Thm. 14.14, p. 220. □

An easy consequence of the properties stated above is:

Proposition 21. *Let S be a surface and let D be a pseudo-effective divisor on S and let $D = P+N$ be its Zariski decomposition. Then $H^0(S, \mathcal{O}_S(nD)) = H^0(S, \mathcal{O}_S(nP)) = H^0(S, \mathcal{O}_S(\lfloor nP \rfloor))$ for all positive integers n.*

Proof. (We follow the proof in [Bad01], Lemma 14.15, p. 222)

The Zariski decomposition of nD is just $nD = nP + nN$, so it suffices to show $H^0(S, \mathcal{O}_S(D)) = H^0(S, \mathcal{O}_S(P))$. One inclusion is trivial, because N is effective. For the other inclusion, let $f \in H^0(S, \mathcal{O}_S(D))$, i.e. $(f) + D \geq 0$. We write $(f) + D = D_1 + D_2$ with effective divisors D_1 and D_2 such that Supp $D_1 \subset$ Supp (N) and D_2 has no common component with N. In particular, $D_2 \cdot C_i \geq 0$, where C_i are the components of N. In order to show $(f) + P \geq 0$, it suffices to prove $D_1 - N \geq 0$. We have

$$(D_1 - N) \cdot C_i \leq ((f) + D - N) \cdot C_i = P \cdot C_i = 0.$$

We write $D_1 - N = A - B$ for effective divisors A and B without common components. Then $A \cdot B = 0$ and $(A - B) \cdot C_i = (D_1 - N) \cdot C_i \leq 0$. Since B is effective, this implies $-B^2 = (A - B) \cdot B \leq 0$. But the intersection matrix of N is negative definite, so $B = 0$. This concludes the proof. □

Corollary 22. *Let S be a surface. Let D be an integral divisor on S such that $\kappa(D) \geq 1$ and let $D = P + N$ be its Zariski decomposition. Let s be a positive integer such that sP is an integral divisor. Then there exist polynomials $q_1, q_2 \in \mathbb{Q}[x]$ of degree ≤ 2 such that for every sufficiently large integer $n = as + b$, $a \in \mathbb{N}$, $0 \leq b < s$:*

(i) $h^0(\mathcal{O}_S(nD)) = h^0(\mathcal{O}_S(asP + bD))$.

(ii) $h^1(\mathcal{O}_S(nD)) = h^1(\mathcal{O}_S(asP + bD)) + q_1(n) + q_2(b)$.

(iii) $h^2(\mathcal{O}_S(nD)) = h^2(\mathcal{O}_S(asP + bD)) = 0$.

Proof. (We follow the proof in [Cut93], Prop. 12, p. 538.)

The Zariski decomposition of $D' := asP + bD$ is just $D' = nP + bN$, so $H^0(S, \mathcal{O}_S(nD)) = H^0(S, \mathcal{O}_S(nP)) = H^0(S, \mathcal{O}_S(D'))$ by Proposition 21. This shows (i).

Statement (iii) follows from Serre duality: For all integers n we have: $h^2(\mathcal{O}_S(nD)) = h^0(\mathcal{O}_S(K_S - nD))$ and $h^2(\mathcal{O}_S(asP + bD)) = h^0(\mathcal{O}_S(K_S - asP - bD))$. But Proposition 21 implies $\kappa(P) = \kappa(D) \geq 1$, so the claim follows from Proposition 16.

It remains to show (ii). By (i) and (iii) it suffices to relate the Euler characteristic χ of nD and $D' := asP + bD$. We note that $asN = nD - D'$ is an effective integral divisor, so we can consider the standard exact sequence of asN. After tensoring with $\mathcal{O}_S(nD)$ we get:

$$0 \to \mathcal{O}_S(D') \to \mathcal{O}_S(nD) \to \mathcal{O}_{asN}(nD) \to 0.$$

By additivity of the Euler characteristic and Proposition 5 this implies:

$$\chi(\mathcal{O}_S(nD)) - \chi(\mathcal{O}_S(D')) = \chi(\mathcal{O}_{asN}(nD)) = asN \cdot nD + 1 - p_a(asN).$$

Now $p_a(asN) = 1 + asN \cdot (asN + K_S)/2$. So for n sufficiently large we obtain:

$$\begin{aligned}
&h^1(\mathcal{O}_S(nD)) - h^1(\mathcal{O}_S(D')) = \chi(\mathcal{O}_S(nD)) - \chi(\mathcal{O}_S(D')) \\
&= (n-b)nN \cdot D - \frac{(n-b)^2}{2} N^2 - \frac{n-b}{2} N \cdot K \\
&= \frac{n^2}{2} N^2 - \frac{2}{2} N \cdot K + \frac{b^2}{2} N^2 + \frac{b}{2} N \cdot K.
\end{aligned}$$

Here we used $N \cdot D = N^2$ (by properties of the Zariski decomposition). $\square$

3 Group schemes

We introduce the notion of a group scheme and state existence theorems for the Picard scheme.

3.1 Representable functors

Definition 23. *Let $\mathcal{C}$ be a category. A contravariant functor $F : \mathcal{C} \to$ (Sets) from $\mathcal{C}$ to the category of sets is* ***representable****, if there exists an object $A \in \mathcal{C}$ such that F and $\mathrm{Hom}(\cdot, A)$ are naturally isomorphic as functors from $\mathcal{C}$ to (Sets).*

Lemma 24 (Yoneda lemma)**.** *Let $\mathcal{C}$ be a category. Let $F : \mathcal{C} \to$ (Sets) be a contravariant functor and let A be an object in $\mathcal{C}$. Then the natural transformations from $\mathrm{Hom}(\cdot, A)$ to F are in bijection with elements in $F(A)$. Explicitly, a natural transformation $\varphi : \mathrm{Hom}(\cdot, A) \to F$ is identified with the element $\varphi_A(\mathrm{id})$.*

Proof. The proof is straightforward. $\square$

In particular, a representable functor $F : \mathcal{C} \to$ (Sets) gives an object A in $\mathcal{C}$ and an element $u \in F(A)$. One easily checks, that they satisfy the following universal property:

Definition 25. *Let $\mathcal{C}$ be a category and let $F : \mathcal{C} \to (\mathrm{Sets})$ be a contravariant functor. A* ***universal element*** *of F is a pair (A, u), where A is an object in $\mathcal{C}$ and $u \in F(A)$ with the following property: If (A', u') is another pair, then there exists a unique morphism $f : A' \to A$ such that $F(f) : F(A) \to F(A')$ maps u to u'.*

Proposition 26. *Let $\mathcal{C}$ be a category and let $F : \mathcal{C} \to (\mathrm{Sets})$ be a contravariant functor. Then F is representable if and only if there exists a universal element of F. Universal elements of F are the final objects in the category of pairs (A, u) (defined in the obvious way) and are therefore unique up to unique isomorphism.*

Proof. Again, the proof is easy. □

Lemma 27 (Yoneda embedding)**.** *Let $\mathcal{C}$ be a category. The embedding $\mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, (Sets)]$ given by $A \mapsto \mathrm{Hom}(\cdot, A)$ is fully faithful.*

Proof. Essentially, this follows from applying the Yoneda lemma to a functor $F = \mathrm{Hom}(\cdot, A)$. □

Definition 28. *A* ***group scheme*** *over a field $\Bbbk$ is a contravariant functor $G : (\mathrm{Sch}/\Bbbk) \to (\mathrm{Groups})$ such that considered as a functor to the category of sets (via composition with a forgetful functor), G is representable.*

By Proposition 26, a group scheme G has a universal element (G_0, u), where G_0 is a scheme over $\Bbbk$ and $u \in G(G_0)$ corresponds to $\mathrm{id} \in \mathrm{Hom}(G_0, G_0)$. Often we will denote G_0 by G as well and for a $\Bbbk$-scheme T we will identify $G(T)$ with the morphisms over $\Bbbk$ from T to G_0. The group structure on $G(T)$ induces functorial morphisms $m_T : G(T) \times G(T) \to G(T)$, $i_T : G(T) \to G(T)$ and $e_T : \mathrm{Spec}\ \Bbbk \to G(T)$ which behave with respect to their composition like multiplication, inversion and the neutral element of a group. By the Yoneda embedding (Lemma 27), this induces morphisms $m : G_0 \times_k G_0 \to G_0$, $i : G_0 \to G_0$ and $e : \mathrm{Spec}\ k \to G_0$ behaving in the same way. We refer to [Vis05], §2.2, pp. 18–19 for details.

Remark 29. *Let G be a group scheme over $\Bbbk$, locally of finite type. Then the group $G(\Bbbk)$ corresponds to $\Bbbk$-morphisms from $\mathrm{Spec}\ \Bbbk$ to $G = G_0$, which in turn can be identified with the closed points of $G = G_0$. We also call these points the* ***$\Bbbk$-valued points*** *of G.*

Definition 30. *A group scheme G over $\Bbbk$ is an* ***algebraic group****, if the representing scheme G_0 is a variety over $\Bbbk$.*

Lemma 31. *Let G be a connected commutative algebraic group over an algebraically closed field $\Bbbk$ of characteristic zero. If the cyclic subgroup $\langle x \rangle$ is dense in G for some $x \in G$, then any infinite subset of $\langle x \rangle$ is dense in G as well.*

Proof. [Cut93], Thm. 7, p. 534. □

3.2 Picard scheme

Let X be a scheme over $\mathbb{k}$. We want to introduce a group scheme resembling the Picard group on X. However, the relative Picard functor $(\mathrm{Sch}/\mathbb{k}) \to (\mathrm{Groups})$, $T \mapsto \mathrm{Pic}(X \times_{\mathbb{k}} T)$ is in general not a group scheme. We can see this in the following way: If we let U vary over the open subsets of a scheme T, then it is easy to show that for a group scheme G the groups $G(U)$ form a sheaf on T. But

$$U \mapsto \mathrm{Pic}(X \times U) = H^1(X \times_{\mathbb{k}} U, \mathcal{O}^*_{X\times U}) = H^1(X \times U, \mathcal{O}^*_{X\times T}|_{X\times U})$$

is in general only a presheaf. Its associated sheaf is $R^1 f_{T*}\mathcal{O}^*_{X\times T}$, where $f_T : X \times T \to T$ is the projection (see [Har77], Prop. III.8.1, p. 250). This motivates the following definition.

Definition 32. *Let X be a scheme over $\mathbb{k}$. We define the contravariant functor $\mathrm{Pic}_X : (\mathrm{Sch}/\mathbb{k}) \to (\mathrm{Groups})$ by $\mathrm{Pic}_X(T) := H^0(T, R^1 f_{T*}\mathcal{O}^*_{X\times T})$.*

Remark 33. *From the above discussion we get that there is a natural transformation from the functor $\mathrm{Pic}(X \times \cdot)$ to the functor Pic_X, given by sheafification. Note, that this induces a natural isomorphism $\mathrm{Pic}(X) \to \mathrm{Pic}_X(\mathbb{k})$, because no sheafification is necessary over $\mathrm{Spec}\ \mathbb{k}$.*

Theorem 34. *Let X be a proper scheme over an algebraically closed field. Then Pic_X is a representable functor and the representing scheme Pic_X is locally of finite type. If the field has characteristic zero, then Pic_X is reduced.*

Proof. [Mur64], II.15, p. 42. □

Theorem 35. *Let X be a proper scheme over an algebraically closed field. Then there exists a subgroup scheme $\mathrm{Pic}^\tau_X \subset \mathrm{Pic}_X$ which is of finite type. The natural isomorphism $\mathrm{Pic}(X) \to \mathrm{Pic}_X(\mathbb{k})$ induces an isomorphism $\mathrm{Pic}^\tau(X) \cong \mathrm{Pic}^\tau_X(\mathbb{k})$*

Proof. [Kle05], Prop. 9.6.12, p. 296. □

4 The main theorem

4.1 Preliminary results

Here we state without proofs some results needed in the proof of the main theorem.

Theorem 36. *Let S be a surface and let D be a divisor on S such that $\kappa(D) = 0$ or $\kappa(D) = 1$. Then the $\mathbb{k}$-algebra $R(S, D) = \bigoplus_{n\geq 0} H^0(S, O_S(nD))$ is finitely generated.*

Proof. [Zar62], Prop. 11.5, p. 610. □

Theorem 37. *Let S be a surface and let $D \geq 0$ be an effective divisor on S. Then there exists a polynomial $p \in \mathbb{Q}[x]$ of degree ≤ 2 and a bounded function $\lambda : \mathbb{N} \to \mathbb{Q}$ such that $h^0(nD) = p(n) + \lambda(n)$ for all $n \gg 0$.*

Proof. [Zar62], § 12, p. 611. □

Proposition 38. *Let $R = \bigoplus_{n \geq 0} R_n$ be a finitely generated graded $\Bbbk$-algebra of transcendence degree ≤ 3. Then there exists an integer $s > 0$ and polynomials $p_0, \dots, p_{s-1} \in \mathbb{Q}[x]$ of degree ≤ 2, such that for all sufficiently large integers $n = as + b$, $a \in \mathbb{N}$, $0 \leq b < s$: $\dim_{\Bbbk} R_n = p_b(n)$.*

Proof. [Eis95], Ex. 12.12, p. 280. □

Proposition 39. *Let S be a surface and let D be a nef divisor on S such that $\kappa(D) \geq 0$. Let E be an irreducible curve on S such that $E \subset \operatorname{Fix} |nD|$ for infinitely many integers $n > 0$. Then $D \cdot E = 0$.*

Proof. [Zar62], Thm. 9.1, p. 596. □

Proposition 40. *Let S be a surface and let D be a divisor on S such that the linear system $|D|$ has no fixed component. Then the linear system $|nD|$ is base-point-free for all $n \gg 0$.*

Proof. [Bad01], Thm. 9.17, p. 133. □

Proposition 41. *Let S be a surface over $\Bbbk$ and let D be a divisor on S such that the linear system $|D|$ is base-point-free. Then $R(S, D)$ is a finitely generated $\Bbbk$-algebra.*

Proof. [Bad01], Thm. 9.2, p. 125. □

4.2 Reduction to a problem on curves

Proposition 42. *Let S be a surface and let P be a nef divisor on S such that $\kappa(P) = 2$. Let $D_0, \dots, D_{s-1}$ be arbitrary divisors on S. Then there exists an effective divisor C on S such that:*

(i) The restriction map $H^1(S, \mathcal{O}_S(nP + D_i)) \to H^1(C, \mathcal{O}_C(nP + D_i))$ is an isomorphism for all $n \gg 0$, $0 \leq i < s$.

(ii) $C' \cdot P = 0$ for each component C' of C.

(iii) If $C' \cdot P = 0$ for some irreducible curve C' on S, then C' is a component of C.

Proof. (We follow the proof in [Cut93], Prop. 13, pp. 538–539.)

We proceed in several steps:

Step 1: There exists an ample divisor H on S such that $h^1(H+nP+D_i) = h^2(H + nP + D_i) = 0$ for all $j > 0$, $0 \leq i < s$ and $n \geq 0$.

Choose some ample divisor H. By Theorem 17 we can replace H by some multiple which has the property stated above.
Step 2: There exists some $m > 0$ such that $h^0(\mathcal{O}_S(mP - H)) > 0$.

Proposition 16 implies that $h^0(\mathcal{O}_S(K + H - mP)) = 0$ for $m \gg 0$. So $h^2(\mathcal{O}_S(mP - H)) = 0$ for $m \gg 0$ by Serre duality. So the Riemann–Roch theorem on surfaces shows:

$$h^0(\mathcal{O}_S(mP - H)) \geq 1/2(mP - H \cdot mP - H - K) + 1 + p_a(S),$$

which is > 0 for m sufficiently large, because $(P^2) > 0$ by Proposition 19.
Step 3: There exists an effective divisor C_0 on S satisfying (i) and (iii).

By Step 2 there exists an effective divisor C_0 linearly equivalent to $mP - H$. Let $0 \leq i < s$. We consider the standard exact sequence of C_0 tensored with $\mathcal{O}_S(nP + D_i)$:

$$0 \to \mathcal{O}_S(nP + D_i - C_0) \to \mathcal{O}_S(nP + D_i) \to \mathcal{O}_{C_0}(nP + D_i) \to 0.$$

Since $\mathcal{O}_S(nP + D_i - C_0) = \mathcal{O}_S((n-m)P + D_i + H)$ and by the choice of H we get for n sufficiently large: $h^1(S, nP + D_i - C_0) = h^2(S, nP + D_i - C_0) = 0$. Therefore the long exact cohomology sequence corresponding to the short exact sequence from above implies that the restriction map $H^1(S, nP + D_i) \to H^1(C_0, nP + D_i)$ is an isomorphism. To show (iii), let C' be an irreducible curve on S such that $C' \cdot P = 0$. Then $C' \cdot C_0 = C' \cdot (mP - H) = -C' \cdot H < 0$ and since C_0 is effective, C' must be a component of C_0.
Step 4: There exists an effective divisor C on S satisfying (i), (ii) and (iii).

For each component C' of C_0 we have $(C' \cdot P) \geq 0$ because P is nef. So we can decompose $C_0 = C + C_1$ into effective divisors C and C_1, such that $(C' \cdot P) = 0$ for each component C' of C and $(C' \cdot P) > 0$ for each component C' of C_1. If $C_1 = 0$, we are done by Step 3. Otherwise $\deg_{C_1}(\mathcal{O}_{C_1}(P)) = C_1 \cdot P > 0$, so $\mathcal{O}_{C_1}(P)$ is ample on C'. We consider the short exact sequence of $\mathcal{O}_S$-modules corresponding to the decomposition $C_0 = C + C_1$ by Proposition 3, tensored with $\mathcal{O}_S(nP + D_i)$:

$$0 \to \mathcal{O}_{C_1}(nP + D_i - C) \to \mathcal{O}_{C_0}(nP + D_i) \to \mathcal{O}_C(nP + D_i) \to 0.$$

Now $\mathcal{O}_{C_1}(P)$ is ample, so by Proposition 4 we get $h^1(C_1, nP + D_i - C) = 0$ for $n \gg 0$. Considering the long exact cohomology sequence of the short exact sequence from above this implies $h^1(\mathcal{O}_{C_0}(nP + D_i)) = h^1(\mathcal{O}_C(nP + D_i))$ (applying Proposition 4 once more). By step 3 we are done. □

4.3 A periodicity result on curves

Proposition 43. *Let X be a proper scheme over an algebraically closed field k of characteristic zero. Let $\mathcal{M}$ and $\mathcal{N}$ be line bundles on X and assume that $\mathcal{M}$ is numerically trivial. Then $n \mapsto h^j(X, \mathcal{M}^{\otimes n} \otimes \mathcal{N})$ is a periodic function in n for all $j > 0$, $n \gg 0$.*

Proof. (We follow the proof in [Cut93], Thm. 8, pp. 540–541.)

We proceed in several steps:

Step 1: The function $\mathrm{Pic}^\tau_X(\Bbbk) \to \mathbb{N}$, $\mathcal{L} \mapsto h^1(X, \mathcal{L} \otimes \mathcal{N})$ is upper semicontinuous.

Since the characteristic of $\Bbbk$ is zero, Pic_X is reduced by Proposition 34 and Pic^τ_X is an algebraic group over $\Bbbk$. Let $s \in \mathrm{Pic}_X(\mathrm{Pic}_X)$ be the universal element of Pic_X. Let $s' \in \mathrm{Pic}_X(\mathrm{Pic}^\tau_X)$ be the image of s under the map $\mathrm{Pic}_X(\mathrm{Pic}_X) \to \mathrm{Pic}_X(\mathrm{Pic}^\tau_X)$ induced by the inclusion of Pic^τ_X into Pic_X. Since the natural transformation $\mathrm{Pic}(X \times \cdot) \to \mathrm{Pic}_X(\cdot)$ is given by sheafification (see the discussion in 33), we can cover Pic^τ_X by open subsets U_i such that there exists $\mathcal{L}_i \in \mathrm{Pic}(X \times U_i)$, that map to $s'|_{U_i}$ under $\mathrm{Pic}(X \times U_i) \to \mathrm{Pic}_X(U_i)$. Let $\varphi : \Bbbk \to U_i$ correspond to an element $y \in U_i(\Bbbk) \subset \mathrm{Pic}_X(\Bbbk)$. On Spec $\Bbbk$, the map $\mathrm{Pic}(X) \to \mathrm{Pic}_X(\Bbbk)$ is an isomorphism (by remark 33) which restricts to an isomorphism $\mathrm{Pic}^\tau(X) \to \mathrm{Pic}^\tau_X(\Bbbk)$. By naturality of the transformation $\mathrm{Pic}(X \times \cdot) \to \mathrm{Pic}_X(\cdot)$, the line bundle $\mathcal{L}_y$ corresponding to y under this isomorphism is $(\mathrm{id} \times \varphi)^* \mathcal{L}_i$, where $\mathrm{id} \times \varphi : X = X \times \mathrm{Spec}\ \Bbbk \to X \times U_i$. Now with respect to $g_i : U_i \times X \to X$ there exists an isomorphism $\psi_i : X \to (U_i \times X)_y$ such that $\psi_i^*(\mathcal{L}_i \otimes g_i^* \mathcal{N})_y = \mathcal{L}_y \otimes \mathcal{N}$ (obtain ψ_i by the morphism $\mathrm{id} \times \varphi : X \to X \times U_i$ and the structure map $X \to \mathrm{Spec}\ \Bbbk$. So, by the semicontinuity theorem we obtain that

$$y \approx \mathcal{L}_y \mapsto h^j(X, \mathcal{L}_y \otimes \mathcal{N}) = h^j((U_i \times X)_y, (\mathcal{L}_i \otimes g_i^* \mathcal{N})_y)$$

is upper semicontinuous on $U_i(\Bbbk)$. Semicontinuity is a local property, so this concludes Step 1.

Step 2: Let G be the closure in Pic^τ_X of the subgroup of $\mathrm{Pic}^\tau_X(\Bbbk) = \mathrm{Pic}^\tau(X)$ generated by $\mathcal{M}$ (with the induced reduced subscheme structure). Then G is a commutative algebraic group. Let $G_0, G_1, \ldots, G_{s-1}$ be the connected components of G. We may assume, that they are numerated such that for each $0 \leq i < s$, the component G_i contains $\mathcal{M}^{\otimes as+i}$ (note, that the group of connected components is G/G_0, if G_0 is the connnected component of the neutral element, and is therefore generated by the component of $\mathcal{M}$). Note, that $G_0(\Bbbk)$ is the closure of the cyclic subgroup $\langle \mathcal{M}^s \rangle$.

For $0 \leq i < s$ and $k \geq 0$ let

$$A^i_k := \{\mathcal{M}^{rs} \mid h^j(X, \mathcal{M}^{\otimes rs+i} \otimes \mathcal{N}) \geq k\} \subset \langle \mathcal{M}^s \rangle \subset G_0(\Bbbk).$$

By Lemma 31, A^i_k is a finite set or dense in G_0. We know by Step 1 that A^i_k is relatively closed in $\langle \mathcal{M}^s \rangle$, so we conclude that A^i_k is a finite set or $A^i_k = \langle \mathcal{M}^s \rangle$. For each i let

$$\lambda(i) := \max\{k \geq 0 \mid A^i_k = \langle \mathcal{M}^s \rangle\}.$$

Then $h^j(X, \mathcal{M}^{\otimes rs+i} \otimes \mathcal{N}) = \lambda(i)$ for $r \gg 0$. In particular, $h^j(X, \mathcal{M}^{\otimes n} \otimes \mathcal{N})$ is periodic in n for $n \gg 0$. $\square$

Corollary 44. *Let C be an effective divisor on a surface S over an (algebraically closed) field of characteristic zero. Let D be a divisor on S such that $(C' \cdot D) = 0$ for each component C' of C. Then for any divisor D' on S, $n \mapsto h^1(C, \mathcal{O}_C(nD + D'))$ is periodic in n for n sufficiently large.*

4.4 The case of characteristic 0

Theorem 45. *Let S be a nonsingular projective variety of dimension 2 over a field $\Bbbk$ of characteristic zero. Let D be a divisor on S. Then there exists an integer $s > 0$ and polynomials $p_0, \ldots, p_{s-1} \in \mathbb{Q}[x]$ of degree ≤ 2, such that for all sufficiently large integers $n = as + b$, $a \in \mathbb{N}$, $0 \leq b < s$: $h^0(nD) = p_b(n)$. Moreover, if $\kappa(D) \neq 1$ or if D is effective, then the polynomials p_i only differ in their constant terms.*

Corollary 46. *Let S be a nonsingular projective variety of dimension 2 over a field $\Bbbk$ of characteristic zero. Let D be a divisor on S satisfying $\kappa(D) \neq 1$ or $D \geq 0$. Then there exists a polynomial $p \in \mathbb{Q}[x]$ of degree ≤ 2 and a periodic function $\lambda : \mathbb{N} \to \mathbb{Q}$, such that:*

$$h^0(\mathcal{O}_S(nD)) = p(n) + \lambda(n).$$

Proof. Let $p_0, \ldots, p_{s-1}$ be as in the theorem above. They only differ in their constant terms. So we can define $\lambda(i) := p_i - p_0$ for $0 \leq i < s$ and extend λ to an s-periodic function. Then for sufficiently large $n = as + b$, $a \in \mathbb{N}$, $0 \leq b < s$ we get

$$h^0(\mathcal{O}_S(nD)) = p_b(n) = p_0(n) + \lambda(b) = p_0(n) + \lambda(n).$$

□

Proof of Theorem 45. (We follow the proof in [Cut93], Thm. 4, p. 542.)

We consider the possible cases for $\kappa(D)$:

Case 1: $\kappa(D) = -\infty$

Then $h^0(\mathcal{O}_S(nD)) = 0$ for all $n > 0$, so this case is trivial.

Case 2: $\kappa(D) = 0$

At first we note that in this case $h^0(\mathcal{O}_S(nD)) \leq 1$ for all $n > 0$: If $x, y \in A_n := H^0(S, \mathcal{O}_S(nD))$ for some $n > 0$, then tr.deg $R(S, D) = 1$ implies that $f(x, y) = 0$ for some non-zero polynomial $f \in \Bbbk[x_1, x_2]$. We can assume that f is homogeneous. Then the fact that $\Bbbk$ is algebraically closed shows that x, y are linearly dependent. This shows $\dim_{\Bbbk} A_n \leq 1$.

Now let $s := \min\{n > 0 \mid A_n \neq 0\}$ and let $0 \neq f \in A_s$. For any $a > 0$, f^a is a non-zero element of A_{as}, so $\dim_{\Bbbk} A_{as} = 1$. We now show $A_{as+b} = 0$ for all $a > 0$, $0 < b < s$, which proves the claim. Suppose there is a non-zero element $g \in A_m$ for some $m = as + b$, $0 < b < s$. Then g^s and f^m are both elements of A_{sm} and therefore linearly dependent. Thus $m(f) = s(g)$ as principal divisors. This implies $(g/f^a) + bD = b/m(g) + bD =$

$b/m((g) + mD) \geq 0$, because $g \in A_m = H^0(S, \mathcal{O}_S(mD))$. So g/f^a is a nonzero element of $H^0(S, \mathcal{O}_S(bD))$, which contradicts the minimality of a.

Case 3: $\kappa(D) = 1$

By Theorem 36, $R(S, D) = \bigoplus_{n \geq 0} H^0(S, O_S(nD))$ is finitely generated. Therefore, we can apply Proposition 38. Moreover, Theorem 37 implies that the polynomials given by Proposition 38 can only differ in their constant terms in case D is an effective divisor.

Case 4: $\kappa(D) = 2$

Then D is pseudo-effective by Proposition 16, so we can consider its Zariski decomposition $D = P + N$. Let $s > 0$ be such that sP is an integral divisor. By Proposition 21 we have $\kappa(sP) = \kappa(sD) = 2$. Applying Proposition 42 to the divisors $D_i := iD$ $(0 \leq i < s)$ and combining this with Corollary 44 we know that for each $0 \leq b < s$ the function $a \mapsto h^1(\mathcal{O}_S(asP + bD))$ is periodic for $a \gg 0$. Therefore the function $n \mapsto h^1(\mathcal{O}_S(asP + bD))$, where $n = as + b$, $a \geq 0$, $0 \leq b < s$, is periodic in n with period rs (r some positive integer) for $n \gg 0$. By corollary 22 for $n = as + b$ suficiently large:

$$\begin{aligned} h^0(\mathcal{O}_S(nD)) &= \chi(\mathcal{O}_S(nD)) + h^1(nD) \\ &= \chi(\mathcal{O}_S(nD)) + h^1(\mathcal{O}_S(asP + bD)) + q_1(n) + q_2(b) \end{aligned}$$

for polynomials $q_1, q_2 \in \mathbb{Q}[x]$ of degree ≤ 2. Since $n \mapsto \chi(S, nD)$ is also given by a polynomial (for n suficiently large) this proves the claim. $\square$

4.5 The case of finite fields

Theorem 47. *Let S be a nonsingular projective variety of dimension 2 over a field $\Bbbk$ that is an algebraic extension of $\mathbb{F}_p$. Let D be a divisor on S. Then there exists an integer $s > 0$ and polynomials $p_0, \ldots, p_{s-1} \in \mathbb{Q}[x]$ of degree ≤ 2, such that for all sufficiently large integers $n = as + b$, $a \in \mathbb{N}$, $0 \leq b < s$:*

$$h^0(nD) = p_b(n).$$

Moreover, if D is effective, then the polynomials p_i only differ in their constant terms, i.e.

$$h^0(nD) = p(n) + \lambda(n)$$

for a polynomial $p \in \mathbb{Q}[x]$ and a periodic function $\lambda : \mathbb{N} \to \mathbb{Q}$.

Proof. (We follow the proof in [Cut93], Thm. 3, pp. 544–545.)

By considering $S_{\bar{\Bbbk}} := S \times_{\mathrm{Spec}\, \Bbbk} \mathrm{Spec}\, \bar{\Bbbk}$ we may assume, that $\Bbbk = \bar{\mathbb{F}}_p$.

We show that $R(S, D) = \bigoplus_{n \geq 0} H^0(S, \mathcal{O}_S(nD))$ is finitely generated. Then the statement follows from Proposition 38 and Theorem 37. The case $\kappa(D) = -\infty$ is trivial. If $\kappa(D) = 0$ or $\kappa(D) = 1$, then the finite generation follows from Theorem 36 (the case $\kappa(D) = 0$ could more easily be treated as in the proof of Theorem 45).

So it remains to consider the case $\kappa(D) = 2$. Then D is pseudo-effective by Proposition 16, so we can consider its Zariski decomposition $D = P + N$. Let $s > 0$ be such that sP is an integral divisor. By Proposition 21 we have $\kappa(sP) = \kappa(sD) = 2$. We apply Proposition 42 to the nef divisor sP (and $D_i = 0$). Let C be the effective divisor given by the Proposition. Since C is a projective variety over $\bar{\mathbb{F}}_p$, the group $\mathrm{Pic}^0(C)$ is a torsion group. So the fact that $\mathcal{O}_C(sP)$ is numerically trivial, implies that we can replace s by some multiple such that $sP|_C$ is linearly equivalent to the trivial divisor $0 \in \mathrm{Pic}(C)$. Thus, the trivial divisor on C corresponds to a section $f \in H^0(C, \mathcal{O}_C(sP))$ such that $(f) + sP|_C = 0$. By the long exact cohomology sequence associated to

$$0 \to \mathcal{O}_S(asP - C) \to \mathcal{O}_S(asP) \to \mathcal{O}_C(asP) \to 0$$

(for an integer $a > 0$) and by property (i) of Proposition 42 we obtain, that after replacing s by a suitable multiple the restriction map

$$H^0(S, \mathcal{O}_S(sP)) \to H^0(C, \mathcal{O}_C(sP))$$

is surjective. In particular, f lifts to a section $f' \in H^0(S, \mathcal{O}_S(sP))$. Then $D' := sP + (f')$ is in the linear system $|sP|$ and $\mathrm{Supp}\, D' \cap \mathrm{Supp}\, C = \emptyset$, because D' restricts to the trivial divisor on C. This implies that $\mathrm{Bs}\, |sP| \cap \mathrm{Supp}\, C = \emptyset$. By property (iii) of Proposition 42, we see that Proposition 39 implies that, after replacing s by a sufficiently large multiple, $|sP|$ has no fixed component. By Proposition 40, $|sP|$ is base-point-free for a larger multiple of s. This implies, by Proposition 41, that $R(S, sP)$ is finitely generated. We conclude, that the Veronese subring $R(S, D)^{(s)} = R(S, sD) = R(S, sP)$ (by Proposition 21) is finitely generated and $R(S, D)$ is an integral extension of $R(S, sD)$, so $R(S, D)$ is finitely generated. □

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