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Abstract

This thesis develops various topics in combinatorial algebraic geometry with computational aspects and is centered around toric and tropical methods.

In Chapter 1, we study injective morphisms from complex projective varieties X to projective spaces of small dimension. Based on connectedness theorems, we prove that the ambient dimension needs to be at least $2 \dim X$ for all injections given by a linear subsystem of a strict power of a line bundle. We showcase different techniques for constructing injections $X \to \mathbb{P}^{2 \dim X}$ to correct and improve results by Dufresne and Jeffries [DJ18]. Connections to separating invariants and the geometry of partially symmetric tensors are discussed.

Chapter 2 revolves around toric degenerations of unirational varieties in families based on the theory of Khovanskii bases. We develop algorithms based on computational tropical geometry to classify degenerations arising naturally from a parametric description of the family. A specific instance of interest are families of cubic surfaces and their Cox rings. We study an open problem by Sturmfels and Xu [SX10] regarding the classification of their degenerations.

In Chapter 3, we present an algorithm for computing zero-dimensional tropical varieties using projections. Our main tools are fast unimodular transforms of lexicographical Gröbner bases. We prove that our algorithm requires only a polynomial number of arithmetic operations if given a Gröbner basis, and we demonstrate that our implementation compares favorably to other existing implementations. Applying it to the computation of general positive-dimensional tropical varieties, we argue that the complexity for calculating tropical links is dominated by the complexity of the Gröbner walk.

Chapter 4 introduces the notion of tropical defects, certificates that a system of polynomial equations is not a tropical basis, and provides two algorithms for finding them in affine spaces of complementary dimension to the zero set. We use these techniques to solve open problems regarding del Pezzo surfaces of degree three and realizability of valuated gaussoids on four elements.

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Introduction

Systems of polynomial equations and the study of their solution sets ("algebraic varieties") lie at the heart of algebraic geometry. Combinatorial techniques often facilitate and deepen their understanding. Two major fields have emerged in this area:

- *Toric geometry* studies a special class of algebraic varieties which are fully described in terms of an interplay of discrete and convex polyhedral data.
- *Tropical geometry* revolves around combinatorial shadows of embedded varieties which can be understood as encoding the limit behavior under suitable compactifications and deformations.

With the increasingly important role of computations in the sciences and as an exploratory tool in mathematics, algebraic techniques become more and more relevant, as they are well suited for symbolic computations. In this context of computational algebraic geometry, toric and tropical varieties play a central role, as they are computationally more feasible due to their combinatorial nature. Of particular interest is linking these two approaches: Toric varieties often define a tractable subclass for a theoretical question or computational problem, while for a non-toric variety, its tropicalization can give insights how to degenerate it to a toric variety while preserving essential invariants.

The present thesis lies in this scope of combinatorial algebraic geometry: on the interface between toric and tropical algebra with a view towards an improved understanding of algebraic varieties in general. The thesis is structured in four parts dedicated to topics that, while interrelated, are of interest independently of each other. They revolve around the following matters:

(1) We introduce a *dimensionality reduction problem* which we investigate with a particular emphasis on toric varieties. Those form a rich class of examples exhibiting general phenomena and their combinatorial structure facilitates the study.
(2) As a general tool for passing to a toric setting, we use computational methods in tropical geometry to systematically find *toric degenerations in families of parameterized varieties.*
(3) With this motivation, we develop algorithms revolving around the *computation of tropical varieties* improving existing computational methods.

(4) Tropical varieties often obtain a more explicit combinatorial descriptions from *tropical bases*. We provide computational methods to study the failure of polynomial systems to form tropical bases, which can be seen as finding discrepancies between algebraic and purely combinatorial objects.

These four aspects feature diverse connections between toric, tropical and classical algebraic geometry. Most of their treatment here have in a modified presentation appeared in the form of articles, in part written in collaborations with other authors. At the current stage, two of these articles are published [GRS19] or accepted for publication [DGW20], while the remaining two are still undergoing review [Gör19; GRZ19]. The personal contributions of the author of this thesis to the presented results from multiple-authored papers are laid out in detail below in a "Statement of contributions" after this introduction.

In the following, we describe each of the four topics in more detail.

Dimensionality reduction in projective geometry

For a system of polynomial equations

$$(\star)\qquad \begin{aligned} f_1(x_1, x_2, \dots, x_n) &= 0, \\ f_2(x_1, x_2, \dots, x_n) &= 0, \\ &\vdots \\ f_m(x_1, x_2, \dots, x_n) &= 0, \end{aligned}$$

its set of complex solutions $X \subseteq \mathbb{C}^n$ is an affine algebraic variety. Its intrinsic dimension $\dim(X)$ can be much smaller than the ambient dimension n, yet computational methods, such as numerical homotopy algorithms, typically scale significantly in terms of the number of variables. Moreover, varieties of low codimension $n - \dim(X)$ often allow a better structural understanding, e.g. when studying relations among the defining equations and, more generally, free resolutions.

The following natural *dimensionality reduction problem* arises:

Problem. *Given* $(\star)$*, find a polynomial change of variables*

$$y_1 = g_1(x_1, x_2, \dots, x_n), \qquad \dots \qquad , y_k = g_k(x_1, x_2, \dots, x_n)$$

with minimal $k \leq n$ *such that every solution of* $(\star)$ *is uniquely determined by its values for* $y_1, \dots, y_k$*.*

Geometrically, this corresponds to searching for a polynomial map $\mathbb{C}^n \to \mathbb{C}^k$ whose restriction to X is injective, i.e., to find a low-dimensional ambient space into which X injects.

We note that it would be algebraically more natural in the category of affine varieties to ask for an *embedding* of X into a low-dimensional affine space instead of an injective morphism, as it would induce an

isomorphism of X onto its image. However, this would be significantly more restrictive: X can have singular points where the tangent space has a much larger dimension than $\dim(X)$, preventing a low-dimensional embedding. On the other hand, it is always possible to embed a dense open subset of X into $\mathbb{C}^{\dim(X)+1}$, but this does not cover all solutions of the polynomial system at once, introducing boundary cases. In between these two extremes lies the above-mentioned study of injections of algebraic varieties which we treat in Chapter 1.

For motivation, we mention a strong connection to algebraic invariants: To the action of a linear algebraic group $G \subseteq \mathrm{GL}(V)$ acting on a finite-dimensional vector space V, the ring $\mathbb{C}[V]^G$ of invariant polynomial functions is finitely-generated under the technical assumption that the group is reductive. Geometrically, one associates a quotient variety $V/\!\!/G \subseteq \mathbb{C}^n$ by picking generators $p_1, \ldots, p_n \in \mathbb{C}[V]^G$ and considering the image of the polynomial map $V \to \mathbb{C}^n$ they induce. Here, the polynomial system $(\star)$ is given by letting $f_1, \ldots, f_m$ be the polynomial relations among $p_1, \ldots, p_n$. Points in $V/\!\!/G$ correspond essentially to orbits of the group action.

Unfortunately, for explicitly given G, it can be very hard and computationally challenging to find invariants generating the ring $\mathbb{C}[V]^G$, and the minimal number of generators is often large. From this, the field of *separating invariants* has emerged, studying sets of invariants that may fail to generate the invariant ring without diminishing the potential to distinguish orbits. Geometrically, this corresponds precisely to studying injections $V/\!\!/G \to \mathbb{C}^k$, and separating invariants have established as an important practical and computational tool in applications of invariant theory.

Going beyond invariant theory, we initiate a systematic study of injections of algebraic varieties into low-dimensional spaces as a general dimensionality reduction technique. On the technical side, however, instead of the affine setting described above, we focus on *projective varieties*, i.e., common zero loci of homogeneous polynomials inside complex projective spaces. The passage to projective geometry is a classical compactification approach in algebraic geometry, corresponding to adding solutions "at infinity" to the polynomial system $(\star)$. In this framework, the natural dimensionality reduction problem is as follows:

Question. *Given a projective variety X, what is the smallest k such that there exists an injective morphism $X \to \mathbb{P}^k$?*

Projective varieties exhibit global phenomena absent from the affine case which can be used as an organizing principle: Morphisms to projective spaces can be expressed in terms of line bundles and their linear systems, providing a natural way to arrange the study into two parts.

(a) For which line bundles on X do their global sections define injective morphisms to projective spaces?
(b) For each line bundle as in (a), what is the smallest number of global sections defining an injection?

We answer (a) for toric varieties by giving an explicit combinatorial criterion in terms of lattice polytopes. For some toric examples, we explicitly construct injections into low-dimensional spaces, providing upper bounds for (b). We also prove lower bounds for (b) depending on the line bundle, in the context of arbitrary projective varieties.

An important source of examples arises from the theory of *tensors*, which play a fundamental role in modern data science. A tensor of format $n_1 \times n_2 \times \cdots \times n_r$ is an element in $\mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2} \otimes \cdots \otimes \mathbb{C}^{n_r}$ and can be understood as a multidimensional array of numbers. Expressing a given tensor as a sum of decomposable (rank 1) tensors $v_1 \otimes \cdots \otimes v_r$ is a core problem in the subject. Geometrically, the set of decomposable tensors forms a projective variety of dimension $n_1 + n_2 + \cdots + n_r - r$ inside a projective space of much larger dimension $n_1 n_2 \cdots n_r - 1$. It is of interest to find a linear map to a lower-dimensional ambient space under which decomposable tensors are still uniquely identified.

In algebraic language, this translates to our problem (b) for the product of projective spaces $X = \mathbb{P}^{n_1-1} \times \mathbb{P}^{n_2-1} \times \cdots \times \mathbb{P}^{n_r-1}$ and the line bundle defining its Segre-embedding. For other line bundles on products of projective spaces, (b) corresponds to an analogous question for tensors with (partial) symmetries. Independently, dimensionality reduction for this specific case has also received particular attention from the theory of separating invariants [DJ18]. With this motivation, products of projective spaces form one of our primary explicit examples and we improve the known results.

Toric degenerations in families

In many circumstances, toric varieties can be studied more easily than arbitrary algebraic varieties due to their combinatorial nature. While this makes toric geometry useful as a testing ground for conjectures or to understand general phenomena, it is equally important to raise the question how much of the toric story carries over to algebraic geometry in a larger generality. A central tool in this context is the notion of *toric degenerations*: For a given variety, one tries to deform it to a toric variety while preserving essential invariants.

The basic idea is to construct a family of varieties containing a non-toric variety of interest as well as a toric variety, and to impose a technical condition guaranteeing that important properties remain invariant throughout the family — typically, this is the algebraic notion of flatness of the family. In most contexts, the family arises as the solution set of a system of polynomial equations depending on one additional parameter that can be varied.

A common computational technique for obtaining toric degenerations starting from the defining equations of a variety $X \subseteq \mathbb{C}^n$ is a *Gröbner deformation*, which has a simple geometric description: For a weight vector $w \in \mathbb{Z}^n$, the multiplicative group $\mathbb{C}^*$ acts on the ambient space $\mathbb{C}^n$ by $t \cdot (x_1, x_2, \ldots, x_n) = (t^{w_1}x_1, t^{w_2}x_2, \ldots, t^{w_n}x_n)$; one considers the image of X under this action and takes the limit as $t \to 0$. This construction describes a flat family and the defining equations of the limit object, the *special fiber*, can be computed with Gröbner basis algorithms fundamental to symbolic computation.

Depending on the weights w, the special fiber in a Gröbner deformation may or may not be a toric variety. Finding suitable weights can be aided by tropical geometry: The tropical variety $\mathrm{Trop}(X)$ is a union of rational convex polyhedral cones in $\mathbb{R}^n$ whose integral points are precisely those weight vectors w for which the special fiber is not contained in the coordinate hyperplanes of $\mathbb{C}^n$. Interior points of the maximal-dimensional cones in $\mathrm{Trop}(X)$ are reasonable candidates for weight vectors describing toric degenerations.

The systematic use of tropical geometry for the study of toric degenerations based on Gröbner deformations play a central role in the modern theory of Khovanskii bases [KM19]. Despite recent advances in computational tropical geometry (to which we contribute in Chapter 3), this approach is in general computationally challenging, and, even more crucially, the existence of a Gröbner degeneration to a toric variety is not always guaranteed. We study an approach to toric degenerations that may in special cases overcome these difficulties and that also gives a different viewpoint on the subject.

Not always are algebraic varieties given explicitly in terms of their defining polynomial equations, on which the computational approach of studying Gröbner degenerations is based. Instead, important classes of algebraic varieties are only described *parametrically*, as the closure of the image of a polynomial map $\varphi\colon \mathbb{C}^m \to \mathbb{C}^n$. Such *unirational varieties* show up for example in the context of geometric invariant theory as quotients $V /\!\!/ G$ described in the previous section. Unirational varieties often come in families, with the defining polynomial map φ depending on several non-zero parameter values $(\theta_1, \ldots, \theta_k)$ that can vary over another variety $S \subseteq (\mathbb{C}^*)^k$.

We explore toric degenerations in this setting of *families of unirational varieties* which are given parametrically by a polynomial map $\Phi\colon \mathbb{C}^m \times S \to \mathbb{C}^n \times S$. We exhibit that even if one is primarily interested in a specific member $X \subseteq \mathbb{C}^n$ of the family, it is useful to make use of the parameter space S in constructing toric degenerations. The approach we consider is to vary the parameters $(\theta_1, \ldots, \theta_k)$ along an infinitesimal curve in S approaching the coordinate hyperplanes of $\mathbb{C}^k$, while at the same time suitably rescaling the coordinates of the ambient space $\mathbb{C}^n$ as for Gröbner deformations. In the limit process, we

obtain a variety which, under fortunate circumstances, is described by a *monomial map* and therefore a toric variety.

The main task in this approach is to choose a suitable infinitesimal curve in the parameter space whose choice guarantees a toric limit. We organize this search systematically with the help of computational tropical geometry by subdividing the tropical variety associated to S. For a single member $X \subseteq \mathbb{C}^n$ of the family, one may obtain toric degenerations going beyond Gröbner deformations of the individual variety.

We apply this degeneration technique to the example of *total coordinate spaces of smooth cubic surfaces.* Total coordinate spaces are the geometric realizations of Cox rings [ADHL15] which are intrinsic coordinate rings of projective varieties independent of the ambient space. Toric degenerations of total coordinate spaces are of special interest as they induce toric degenerations with respect to all embeddings into projective spaces.

Smooth cubic surfaces can be seen as blow-ups of the projective plane in six points, and these points can be varied in a parameter space S which can be interpreted as the Grassmannian variety $\mathrm{Gr}(3,6)$ parameterizing three-dimensional subspaces of $\mathbb{C}^6$. Cox rings of cubic surfaces allow an invariant-theoretic characterization and this expresses the variation of total coordinate spaces parametrically as a family of unirational varieties. Applying the developed techniques to this example, we compute suitable subdivisions of the tropical Grassmannian $\mathrm{Trop}(\mathrm{Gr}(3,6))$ to address an open classification problem on toric degenerations in [SX10].

Computing zero-dimensional tropical varieties

Tropical geometry studies combinatorial shadows encoding limiting behaviour of solution sets $X \subseteq \mathbb{C}^n$ to polynomial systems of equations

$$f_1(x_1, x_2, \ldots, x_n) = 0, \qquad \ldots \qquad , f_m(x_1, x_2, \ldots, x_n) = 0.$$

The structure of the polynomial equations imposes restrictions on the orders of magnitude of the values for $x_1, \ldots, x_n$ in a solution. As an example, the single equation $x_1^7 - x_2^2 = 0$ has as large integral solutions only pairs (a_1, a_2) for which the decimal expansion of a_2 has about 3.5-times as many digits as that of a_1. For polynomial systems with more equations and more terms per equation, such an estimation of possible orders of magnitude is significantly more sophisticated and is studied systematically by tropical geometry.

We use a common convention in tropical geometry to focus on *infinitesimal* orders of magnitude. This is formalized as follows: Instead of considering complex solutions $(a_1, a_2, \ldots, a_n) \in \mathbb{C}^n$ to the polynomial system of equations, one studies solutions where each entry a_i is a *formal power series* $\sum_{k \geq 0} \lambda_k t^k$, or, more generally, a *formal Puiseux*

series $\sum_{k\in\frac{1}{r}\mathbb{Z},\,k\geq s}\lambda_k t^k$. A measure for the infinitesimal order of magnitude of such series expressions is their valuation, i.e., the smallest k with $\lambda_k \neq 0$. To a non-zero solution $(a_1, a_2, \ldots, a_n)$ in formal Puiseux series expressions, one associates the vector of valuations $(k_1, k_2, \ldots, k_n)$. These vectors form a set in $\mathbb{Q}^n$ whose closure in $\mathbb{R}^n$ is called the *tropical variety* of X. This notion easily generalizes to polynomial systems in which the equations already involve formal series expressions.

An equivalent interpretation of tropical varieties can be given in terms of the Gröbner deformations described in the previous section: The Gröbner deformation of $X \subseteq \mathbb{C}^n$ associated to a weight vector $w \in \mathbb{Z}^n$ has as limiting object a special fiber which remains unchanged under rescaling w by a positive integer. This allows to generalize the notion of the special fiber to *rational* weights $w \in \mathbb{Q}^n$. Only if the weights are well-balanced, the special fiber is not contained in the coordinate hyperplanes of $\mathbb{C}^n$. The tropical variety $\mathrm{Trop}(X)$ is the closure of the set of these rational weight vectors.

Additional to its use in the theory of toric degenerations as briefly described in the previous section, tropical geometry often allows insights to the structure of algebraic varieties, as the dimension, the degree and similar invariants are reflected in their combinatorial counterpart.

Tropical varieties are polyhedral complexes. Their computation from the defining equations of algebraic varieties is a challenging algorithmic problem. Modern algorithms [BJSST07] rely on a traversal of the complex: Roughly, they determine a suitable starting point on the tropical variety, then compute the maximal cone it is contained in. At the boundary of the cone, one determines the directions to which the neighboring cones lie and repeats the process with points found in these directions.

While the computation of a surrounding cone is based on Gröbner basis algorithms, the passage to neighboring cones can again be performed by computing several *zero-dimensional* tropical varieties, i.e., tropical varieties which consist of finitely many points.

Due to this central role in the general computation of tropical varieties, we investigate algorithmic aspects in the zero-dimensional case. We develop a new algorithm for computing finite tropical varieties based on reductions to the univariate setting via suitable projection techniques. Our approach allows us to compare the complexity of the different steps in the computation of arbitrary tropical varieties.

Tropical defects of polynomial equations

The simplest instance of a multivariate system of polynomial equations is the case of only one defining equation $f(x_1, x_2, \ldots, x_n) = 0$. In this case, the tropical variety of the solution set is called a *tropical*

hypersurface $\mathrm{Trop}(f)$ and can be easily determined from the expression of the defining polynomial.

In fact, if $M \subseteq \mathbb{N}^n$ is the set of exponent vectors $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ of the monomials $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ appearing in f, then $\mathrm{Trop}(f)$ can be seen as the set of all linear functions $\mathbb{R}^n \to \mathbb{R}$ whose minimum among M is not unique. This characterization makes the computation of tropical hypersurfaces and the algorithmic decision of containment $w \overset{?}{\in} \mathrm{Trop}(f)$ particularly simple.

For arbitrary varieties X given by the vanishing of several polynomials $f_1, f_2, \ldots, f_m$, it is conceivable that the tropical variety $\mathrm{Trop}(X)$ would be the intersection of the tropical hypersurfaces $\mathrm{Trop}(f_i)$. This is however only true in special situations or after replacing the given defining equations $f_1, f_2, \ldots, f_m$ by a suitable algebraically equivalent description, a *tropical basis*.

The question whether a given system of polynomial equations is a tropical basis of the underlying variety is fundamental to tropical geometry. The knowledge of a tropical basis to a polynomial system can aid the understanding of the tropical variety and its computation. In particular, containment tests $w \overset{?}{\in} \mathrm{Trop}(X)$ become algorithmically simple, while in general they require potentially harder Gröbner basis computations.

The intersection of finitely many tropical hypersurfaces can often be given a combinatorial description, purely based on the structure of the polynomial equations. Examples include valuated matroids and valuated gaussoids which form important classes of combinatorial objects. On the other hand, tropical varieties describe combinatorial shadows of algebraic solutions. In the case of a tropical basis, these two notions agree. In general, however, not every "combinatorial solution" arises from an algebraic solution to the polynomial system. These so-called *tropical defects* often reflect a combinatorial notion of non-realizabilty central to the understanding of discrete objects.

The computation of a tropical basis or its certification is a very challenging algorithmic task. We focus our attention to finding tropical defects of a given system of polynomial equations. The algorithms we develop try to methodologically identify them. While they give in all generality only a heuristic approach, they can be useful for identifying particular discrepancies between combinatorial and algebraic descriptions.

We apply our algorithms to tropical varieties arising from families of Cox rings of cubic surfaces and to the setting of valuated gaussoids. In both cases, we discover tropical defects, disproving conjectures in [RSS16] and [BDKS19].

Statement of contributions

Parts of this thesis revolve around results that arose from joint work with other researchers. Despite the inherent difficulty of attributing the mathematical work in collaborations to the individual authors, the following describes to the extent possible my personal contributions to the results presented in this thesis.

1.) Chapter 1 is based on the single-authored article [Gör19].

2.) Chapter 2 is based on the joint work [DGW20] with Maria Donten-Bury and Milena Wrobel.
   - §2.1 is entirely original work in this thesis, going beyond the scope of [DGW20].
   - In §2.2, I develop the approach from [DGW20] within a more general framework introduced in §2.1. My main contribution to the presented techniques is the formulation of Algorithm 2.2.3 and its proof of correctness.
   - In §2.3, I implemented and carried out the computations that classify the moneric classes in Theorem 2.3.2.
   - I worked out §2.4.

3.) Chapter 3 is based on the joint work [GRZ19] with Yue Ren and Leon Zhang.
   - In §3.2, I proved Lemma 3.2.3 and designed Algorithm 3.2.4.
   - Algorithms 3.3.1 and 3.3.2 arose from continuous discussions of the three authors. I particularly contributed to their formulation in a framework that allows for the runtime optimizations in §3.4. Moreover, I worked out their correctness proofs.
   - I developed, implemented and experimented with the runtime optimizations described in §3.4 that make our code competitive.
   - I devised the complexity analysis described in Theorem 3.6.6.

4.) Chapter 4 is based on the joint work [GRS19] with Yue Ren and Jeff Sommars.
   - I contributed to the conception of Algorithms 4.1.9 and 4.1.13 and worked out their correctness proofs.
   - For the application in §4.2, I provided the theoretical background and the necessary pre-processing of computational input data.

Paul Görlach

CHAPTER 1

Injection dimensions of projective varieties

The Whitney embedding theorem asserts that every n-dimensional real smooth manifold can be embedded into $\mathbb{R}^{2n}$. The analogous question in algebraic geometry whether every n-dimensional complex projective variety admits a closed embedding into $2n$-dimensional complex projective space has a negative answer. However, when we relax the requirement on the morphism to $\mathbb{P}^{2n}$ from being a closed embedding to being injective (on the level of points), this is an open problem:

Question 1.1. *Does every n-dimensional irreducible complex projective variety X admit an injective morphism $X \xrightarrow{\text{inj}} \mathbb{P}^{2n}$?*

In general, we define the **injection dimension** $\gamma(X) \in \mathbb{N}$ of a complex projective variety X to be the smallest dimension of a projective space to which there exists an injective morphism from X. In a more refined setting, it is desirable to study the injection dimension $\gamma(X, \mathscr{L})$ with respect to a fixed line bundle $\mathscr{L}$ on X, where we restrict to injective morphisms $X \xrightarrow{\text{inj}} \mathbb{P}^s$ given by a linear subsystem of $|\mathscr{L}|$.

Already in the case of curves, the study of injection dimensions is interesting and largely open. Restricting to very ample line bundles, injections to $\mathbb{P}^2$ are given by cuspidal projections of embeddings of the curve. Recent progress in this area has been made in [BV18], especially in the context of space curves lying on irreducible quadrics. More classical work dates back to [Pie81], where it was proved that general canonical curves of genus 4 do not admit cuspidal projections. This describes examples of curves C with $\gamma(C, \omega_C) = 3 > 2 \dim C$, showing that refining Question 1.1 to injection dimensions with respect to all very ample line bundles cannot have an affirmative answer in general.

In general, line bundles giving rise to injective morphisms form a class lying between ample and very ample line bundles. For toric varieties, we give a combinatorial characterization of this class:

Theorem 1.2 (Theorem 1.1.9)**.** *Let $(X, \mathscr{L})$ be the polarized toric variety corresponding to a full-dimensional lattice polytope $P \subseteq \mathbb{R}^n$. The complete linear system $|\mathscr{L}|$ separates points if and only if*

$$(kr \cdot P) \cap (k\mathbb{Z})^n = (kr \cdot (P \cap \mathbb{Z}^n)) \cap (k\mathbb{Z})^n \quad \textit{for all } k \gg 0, r \geq 1,$$

where we denote $m \cdot Z := \underbrace{Z + \ldots + Z}_{m \text{ times}}$ for $m \in \mathbb{N}$, $Z \subseteq \mathbb{R}^n$.

This chapter is based on the article [Gör19] by the author of this thesis.

For all complex projective varieties X, a classical projection argument shows that $\gamma(X, \mathscr{L}) \leq 2 \dim X + 1$ for all line bundles $\mathscr{L}$ giving rise to injections. With techniques inspired by work on separating invariants [DJ15; DJ18; Rei16; Rei18], we prove that there is very little room for improvement for line bundles admitting a root of some order.

Theorem 1.3 (Theorem 1.3.6)**.** *Let X be a complex projective variety and let $\mathscr{L}$ be a line bundle on X. Then $\gamma(X, \mathscr{L}^{\otimes k}) \geq 2 \dim X$ for all $k \geq 2$, i.e., every injection $X \xrightarrow{\text{inj}} \mathbb{P}^s$ given by a linear subsystem of $|\mathscr{L}^{\otimes k}|$ satisfies $s \geq 2 \dim X$.*

This theorem vastly generalizes previous work in [DJ18, §5], whose arguments amount to proving the above result for the special case $X = \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_r}$ and $\mathscr{L} = \mathcal{O}(1, \ldots, 1)$.[1]

We further show that:

- Question 1.1 can in general not be improved upon: For every $n \geq 3$, there exist n-dimensional irreducible projective varieties that cannot be injectively mapped to $\mathbb{P}^{2n-1}$. See Example 1.3.7 and Example 1.3.13.
- Question 1.1 has an affirmative answer for $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^n$ and for weighted projective spaces of the form $X = \mathbb{P}(1, q_1, \ldots, q_n)$ with $\operatorname{lcm}\{q_i, q_j\} = \operatorname{lcm}\{q_1, \ldots, q_n\}$ for all $i \neq j$. We provide injections into small ambient spaces in Corollary 1.4.11 and Theorem 1.4.5.

In the setting of normal varieties with singularities, we provide an extension of Theorem 1.3 by giving a similar bound when the assumption of divisibility in the Picard group is replaced by divisibility in the class group, see Theorem 1.3.8. Applied to weighted projective spaces, this gives rise to the following result:

Theorem 1.4 (Corollary 1.3.9 and Theorem 1.4.5)**.** *Consider a weighted projective space $\mathbb{P}(q_0, \ldots, q_n)$ with $\gcd(q_0, \ldots, \widehat{q_i}, \ldots, q_n) = 1$ for all i, and let $\ell \geq 2$ be minimal such that $\operatorname{lcm}(q_{i_1}, \ldots, q_{i_\ell}) = \operatorname{lcm}(q_0, \ldots, q_n)$ for all $i_1, \ldots, i_\ell$ distinct. Let $\mathscr{L}$ be the ample line bundle generating the Picard group. Then*

$$\gamma(\mathbb{P}(q_0, \ldots, q_n), \mathscr{L}^{\otimes k}) \geq \begin{cases} n + \ell - 2 & \text{if } k = 1, \\ 2n & \text{if } k \geq 2. \end{cases}$$

For $\ell \in \{2, 3\}$ and $q_0 = 1$, equality holds.

This generalizes the classification of injection dimensions for (non-weighted) projective spaces carried out in [DJ15] and has analogues in the theory of separating invariants for actions of finite cyclic groups

[1] The result claimed in [DJ18, §5] is more general, but unfortunately incorrect. Proposition 1.4.2 gives a counterexample; see also the discussion at the beginning of Section 1.3.

[Duf08; DJ15]. From Theorem 1.4, we deduce that for the weighted projective space $\mathbb{P}(1, 6, 10, 15)$, the smallest injection dimension cannot be attained via linear projections starting from any embedding as a subvariety of projective space, see Example 1.3.10.

We then focus on products of projective spaces: $X = \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_r}$. This case is of particular importance from the theory of tensors, as it can be interpreted as a dimensionality reduction problem for decomposable tensors with partial symmetries. Moreover, injection dimensions of products of projective spaces can be seen as a measure for the discrepancy between rank 2 and border rank 2 tensors. We briefly expand on this connection in Section 1.2.2.

On the other hand, injections of products of projective spaces have received special attention from the theory of separating invariants. In [DJ18], techniques from local cohomology were employed to bound their injection dimensions as follows:

$$\gamma(\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_r}) \geq 2(\textstyle\sum_{i=1}^r n_i) - 2\min\{n_1, \ldots, n_r\} + 1.$$

We give a geometric argument for the following improved bound:

Theorem 1.5 (Corollary 1.3.5)**.** *For all $n_1, \ldots, n_r \geq 1$, we have*

$$\gamma(\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_r}) \geq 2(\textstyle\sum_{i=1}^r n_i) - \min\{n_1, \ldots, n_r\}.$$

Moreover, we use the example of products of projective spaces to showcase several techniques for producing explicit injective morphisms into twice-dimensional projective spaces in Section 1.4.

1.1. Separating polytopes and secant avoidance

In this section, we start out by establishing basic notions and gathering general observations on injective morphisms from arbitrary projective varieties to projective spaces and their relation to secant loci. We then focus on the setting of toric varieties, in which case we give a combinatorial criterion for complete linear systems to separate points.

1.1.1. General observations

First, we fix some conventions for the entire chapter: Throughout, we work over the base field $\mathbb{C}$ and consider complex varieties, not assumed to be irreducible in general. For a finite-dimensional vector space V, we denote by $\mathbb{C}[V^*] := \operatorname{Sym}^\bullet V^*$ the graded ring of polynomial functions on V and by $\mathbb{P}(V) := \operatorname{Proj} \mathbb{C}[V^*]$ the projective space parameterizing one-dimensional subspaces of V. For $v \in V \setminus \{0\}$, the corresponding point in $\mathbb{P}(V)$ is denoted $[v]$. The term subvariety (or subscheme, point etc.) refers to a *closed* subvariety (subscheme, point etc.), unless mentioned otherwise.

We recall that a choice of global sections $f_0, \ldots, f_s \in H^0(X, \mathscr{L})$ of a line bundle $\mathscr{L}$ on a variety X determines a rational map to a

projective space $X \dashrightarrow \mathbb{P}^s$. In a coordinate-free manner, it is the composition of the natural evaluation $\varphi_{\mathscr{L}} \colon X \dashrightarrow \mathbb{P}(H^0(X, \mathscr{L})^*)$ with the projection $\mathbb{P}(H^0(X, \mathscr{L})^*) \dashrightarrow \mathbb{P}(V^*)$, where V is the subspace of $H^0(X, \mathscr{L})$ spanned by $f_0, \ldots, f_s$. In general, for any non-zero subspace $V \subseteq H^0(X, \mathscr{L})$, the composition $X \dashrightarrow \mathbb{P}(H^0(X, \mathscr{L})^*) \dashrightarrow \mathbb{P}(V^*)$ is a well-defined rational map, which we denote by φ_V. In particular, $\varphi_{\mathscr{L}} = \varphi_{H^0(X,\mathscr{L})}$ in our notations.

Definition 1.1.1 (Injection dimension)**.** Let X be a projective variety and $\mathscr{L}$ a line bundle on X. The **injection dimension of X with respect to $\mathscr{L}$**, denoted $\gamma(X, \mathscr{L})$, is defined as the smallest dimension of a projective space into which X can be injected by global sections of $\mathscr{L}$. Formally,

$$\gamma(X, \mathscr{L}) := \inf \Big\{ \dim V - 1 \ \mid \ V \subseteq H^0(X, \mathscr{L}) \text{ non-zero subspace such that } \varphi_V \colon X \dashrightarrow \mathbb{P}(V^*) \text{ is an injective morphism} \Big\}.$$

We define the **injection dimension of** X as

$$\begin{aligned} \gamma(X) &:= \min\{\gamma(X, \mathscr{L}) \mid \mathscr{L} \text{ line bundle on } X\} \\ &= \min\{s \in \mathbb{N} \mid \text{there exists } X \xrightarrow{\text{inj}} \mathbb{P}^s\}. \end{aligned}$$

Except for the case of projective spaces, the injection dimension is strictly larger than the dimension of the variety, as we note as an easy consequence of Zariski's Main Theorem:

Lemma 1.1.2. *Let X be a projective variety not isomorphic to a projective space. Then $\gamma(X) \geq \dim X + 1$.*

PROOF. Let $n := \dim X$ and assume that there is an injective morphism $\varphi \colon X \xrightarrow{\text{inj}} \mathbb{P}^n$. Since φ is proper and finite, the restriction of φ to any n-dimensional irreducible component of X has an n-dimensional image and is therefore surjective. By injectivity of φ, X must be irreducible and $\varphi \colon X \to \mathbb{P}^n$ is bijective. Being a finite surjective morphism of degree 1, the morphism φ is birational. Then normality of projective space implies that φ is an isomorphism by Zariski's Main Theorem (as e.g. in [Vak17, Exercise 29.6.D]). $\square$

Remark 1.1.3. More generally, the proof of Lemma 1.1.2 shows that the image of an injection $\varphi \colon X \xrightarrow{\text{inj}} \mathbb{P}^n$ is a normal variety if and only if X is normal and φ is an isomorphism.

The lower bound in Lemma 1.1.2 can be attained in surprisingly large classes of projective varieties. In [LW83, Corollary 4.2], an explicit injection of n-dimensional complete intersections into $\mathbb{P}^{n+1}$ was constructed under the assumption that the defining equations have pairwise relatively prime degrees. However, there are also n-dimensional complete intersections whose injection dimension is at least $n + 2$, see [LW83, Corollary 4.4].

Throughout our study of injection dimensions, the following elementary connection between injections in projective and affine settings will repeatedly come up:

Lemma 1.1.4. *Let $S = \bigoplus_{d\geq 0} S_d$ be a finitely generated graded algebra over $S_0 = \mathbb{C}$ and let $V \subseteq S_1$ be a subspace. The rational map $\varphi_V \colon \operatorname{Proj} S \dashrightarrow \mathbb{P}(V^*)$ is an injective morphism if and only if the morphism of affine cones $\widehat{\varphi}_V \colon \operatorname{Spec} S \to V^*$ is injective.*

Proof. The graded ring homomorphism $\operatorname{Sym}^\bullet V \hookrightarrow \operatorname{Sym}^\bullet S_1 \to S$ induces the rational map $\varphi_V \colon \operatorname{Proj} S \dashrightarrow \operatorname{Proj} \operatorname{Sym}^\bullet V = \mathbb{P}(V^*)$ as well as the morphism of affine cones $\widehat{\varphi}_V \colon \operatorname{Spec} S \to \operatorname{Spec} \operatorname{Sym}^\bullet V = V^*$. Note that φ_V is a morphism if and only if $\widehat{\varphi}_V^{-1}(0)$ consists only of the closed point $o \in \operatorname{Spec} S$ corresponding to the maximal ideal $S_{\geq 1}$. In that case, we have the commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} S \setminus \{o\} \arrow[r, "\widehat{\varphi}_V|_{\operatorname{Spec} S\setminus\{o\}}"'] \arrow[d, "/\mathbb{C}^*"] & V^* \setminus \{0\} \arrow[d, "/\mathbb{C}^*"] \\
\operatorname{Proj} S \arrow[r, "\varphi_V"] & \mathbb{P}(V^*),
\end{tikzcd}$$

where the vertical morphisms are geometric quotients for the $\mathbb{C}^*$-actions induced by the gradings of S and $\operatorname{Sym}^\bullet V$. Since $\operatorname{Sym}^\bullet V \to S$ is degree-preserving, $\widehat{\varphi}_V$ is $\mathbb{C}^*$-equivariant, hence $\varphi_V \colon \operatorname{Proj} S \dashrightarrow \mathbb{P}(V^*)$ is an injective morphism if and only if $\widehat{\varphi}_V$ is injective. $\square$

If $\mathscr{L}$ is a very ample line bundle, i.e., if $\varphi_{\mathscr{L}} \colon X \dashrightarrow \mathbb{P}(H^0(X, \mathscr{L})^*)$ is a closed embedding, then $\gamma(X, \mathscr{L}) \leq h^0(X, \mathscr{L}) - 1 < \infty$. Conversely, if $\gamma(X, \mathscr{L}) < \infty$, then $\mathscr{L}$ is a globally generated ample line bundle, since $\mathscr{L}$ is the pullback of $\mathcal{O}(1)$ under the injective (hence finite) morphism $\varphi_{\mathscr{L}}$. Therefore, we have the implications

$$\begin{aligned} \mathscr{L} \text{ very ample} \quad &\Rightarrow \quad \gamma(X, \mathscr{L}) < \infty \\ &\Rightarrow \quad \mathscr{L} \text{ ample and globally generated.} \end{aligned} \tag{1.1}$$

Note that the argument for the second implication also shows that non-projective complete varieties cannot be injected to projective spaces due to the lack of ample line bundles. In other words, a complete variety admits an injective morphism to a projective space if and only if it admits an embedding into a projective space.

The reverse implications of (1.1) are not true, as the following examples show.

Example 1.1.5. Consider the three-dimensional weighted projective space $X = \mathbb{P}(1, 6, 10, 15)$. The morphism

$$\begin{aligned} \mathbb{P}(1, 6, 10, 15) &\to \mathbb{P}^4, \\ [x_0 : x_1 : x_2 : x_3] &\mapsto [x_0^{30} : x_0^{24} x_1 : x_0^{20} x_2 + x_1^5 : x_0^{15} x_3 + x_2^3 : x_3^2] \end{aligned}$$

is defined by global sections of $\mathscr{L} = \mathcal{O}(30)$ and it is injective, as we will confirm in Theorem 1.4.5 below. Hence, $\gamma(X, \mathscr{L}) = 4$. On the other hand, we observe that the line bundle $\mathscr{L}$ is not very ample: The polarized toric variety $(\mathbb{P}(1, 6, 10, 15), \mathscr{L})$ corresponds to the lattice polytope $P := \operatorname{conv}(0, 5e_1, 3e_2, 2e_3) \subseteq \mathbb{R}^3$. Note that the semigroup $S := \mathbb{N}(P \cap \mathbb{Z}^3 - 5e_1)$ is not saturated in $\mathbb{Z}^3$, because

$$(-6, 2, 1) = \frac{1}{2}((-2, 1, 0) + (-5, 3, 0) + (-5, 0, 2)) \in \left(\frac{1}{2}S \cap \mathbb{Z}^3\right) \setminus S.$$

By [CLS11, Proposition 6.1.10], this shows that $\mathscr{L}$ is not very ample. In fact, we discuss in Example 1.3.10 that $\gamma(X, \mathscr{L}^{\otimes k}) \geq 6$ for all $k \geq 2$, highlighting that to attain the smallest possible injection dimension, one cannot restrict to very ample line bundles only. Theorem 1.1.9 below will shed more light on this example.

Example 1.1.6. Let X be an elliptic curve and $p \in X$. Then $\mathscr{L} = \mathcal{O}_X(2p)$ determines a double cover $\varphi_{\mathscr{L}} \colon X \to \mathbb{P}(H^0(X, \mathscr{L})^*) \cong \mathbb{P}^1$. The non-injectivity of this morphism implies $\gamma(X, \mathscr{L}) = \infty$. On the other hand, $\mathscr{L}$ is globally generated and ample.

1.1.2. Toric injections

In the following, we give a combinatorial description of line bundles on toric varieties with finite injection dimension, giving a complete picture on (1.1) in the toric setting. For more results on injective morphisms preserving toric structures from a viewpoint of separating invariants for representations of tori, we refer to [DJ18, §6].

Lemma 1.1.7. *Let $S \subseteq T \subseteq \mathbb{Z}^n$ be affine monoids spanning the same lattice and the same convex cone in $\mathbb{R}^n$. The induced morphism $\operatorname{Spec} \mathbb{C}[T] \to \operatorname{Spec} \mathbb{C}[S]$ is injective if and only if for every $m \in T$, we have $k \cdot m \in S$ for all $k \gg 0$.*

PROOF. It is enough to prove the claim for $T = S + \mathbb{N}m$ with $m \in T \setminus S$. The assumption that $\mathbb{Z}S = \mathbb{Z}T$ and $\operatorname{cone}(S) = \operatorname{cone}(T)$ ensures that the morphism $\psi \colon \operatorname{Spec} \mathbb{C}[T] \to \operatorname{Spec} \mathbb{C}[S]$ is finite, i.e., that the set

$$K := \{k \in \mathbb{Z}_{>0} \mid k \cdot m \in S\}$$

is non-empty. Denote by $r \in \mathbb{Z}_{>0}$ the smallest integer with $K \subseteq r\mathbb{Z}$. We need to show that ψ is injective if and only if $r = 1$.

If $r = 1$, then $k, k+1 \in K$ for some positive integer k. Hence, in $\mathbb{C}[T]_{\chi^m}$, we have $\chi^m = \chi^{(k+1)m}/\chi^{km}$, showing that the morphism ψ restricts to an isomorphism of open subsets

$$\operatorname{Spec} \mathbb{C}[T] \supseteq D(\chi^m) \xrightarrow{\cong} D(\chi^{km}) \subseteq \operatorname{Spec} \mathbb{C}[S].$$

On the other hand, ψ also maps the complement $V(\chi^m) \subseteq \operatorname{Spec} \mathbb{C}[T]$ bijectively onto the closed set $V(\chi^{km}) \subseteq \operatorname{Spec} \mathbb{C}[S]$. We conclude that the morphism ψ is a bijection of sets.

Conversely, consider the minimal face F of the rational polyhedral cone $\operatorname{cone}(S) = \operatorname{cone}(T) \subseteq \mathbb{R}^n$ containing m. The semigroup homomorphism

$$S \to \mathbb{C}, \qquad \mu \mapsto \begin{cases} 1 & \text{if } \mu \in F, \\ 0 & \text{otherwise,} \end{cases}$$

corresponds to a closed point of $\operatorname{Spec} \mathbb{C}[S]$ over which the fiber of ψ consists of r distinct points. Hence, ψ can only be injective if $r = 1$. $\square$

In studying the geometry of polarized projective toric varieties, combinatorial notions for lattice polytopes, such as normal and very ample polytopes play a central role. We introduce a new notion for lattice polytopes:

Definition 1.1.8. We call a full-dimensional lattice polytope $P \subseteq \mathbb{R}^n$ **separating** if and only if for all $k \gg 0$, the following holds:

$$(kr \cdot P) \cap (k\mathbb{Z})^n = (kr \cdot (P \cap \mathbb{Z}^n)) \cap (k\mathbb{Z})^n \qquad \text{for all } r \geq 1,$$

where we denote $m \cdot Z := \underbrace{Z + \ldots + Z}_{m \text{ times}}$ for $m \in \mathbb{N}$, $Z \subseteq \mathbb{R}^n$.

The notion of separating polytopes relates to injective morphisms.

Theorem 1.1.9. *Let $(X, \mathscr{L})$ be the polarized toric variety corresponding to a full-dimensional lattice polytope $P \subseteq \mathbb{R}^n$. The complete linear system $|\mathscr{L}|$ induces an injective morphism $X \xrightarrow{\text{inj}} \mathbb{P}(H^0(X, \mathscr{L})^*)$ if and only if P is separating.*

Proof. Denote by $C(P)$ the convex cone over $P \times \{1\}$ in $\mathbb{R}^{n+1}$ and consider the affine monoid $T := C(P) \cap \mathbb{Z}^{n+1}$. By Lemma 1.1.4, the rational map $\varphi_{\mathscr{L}} \colon X \dashrightarrow \mathbb{P}(H^0(X, \mathscr{L})^*)$ is an injective morphism if and only if $\psi \colon \operatorname{Spec} \mathbb{C}[T] \to \operatorname{Spec} \mathbb{C}[S]$ is injective, where S is the submonoid of T generated by $(P \cap \mathbb{Z}^n) \times \{1\}$. By Lemma 1.1.7, this is equivalent to

$$\{k \cdot m \mid m \in T\} \subseteq S \qquad \text{for all } k \gg 0.$$

Note that $S = \bigcup_{r \geq 0} r \cdot (P \cap \mathbb{Z}^n) \times \{r\}$ and $T = \bigcup_{r \geq 0} (r \cdot P) \times \{r\} \cap \mathbb{Z}^{n+1}$, so this inclusion can also be written as

$$(kr \cdot P) \cap (k\mathbb{Z})^n \subseteq (kr \cdot (P \cap \mathbb{Z}^n)) \cap (k\mathbb{Z})^n \qquad \text{for all } r \geq 1.$$

The reverse inclusion always holds, so we conclude the claimed equivalence. $\square$

We note that the geometrically trivial fact that embeddings are injective translates to the following combinatorial implication that could also be checked from the definitions:

Corollary 1.1.10. *Very ample lattice polytopes $P \subseteq \mathbb{R}^n$ are separating.*

Note that Example 1.1.5 provides a counterexample to the converse of Corollary 1.1.10. Moreover, not every full-dimensional lattice polytope is separating:

Example 1.1.11. For $n \geq 3$, the lattice polytope

$$P := \operatorname{conv}(0, e_1, \ldots, e_{n-1}, e_1 + e_2 + \ldots + e_{n-1} + ne_n) \subseteq \mathbb{R}^n$$

is (with respect to the lattice $\mathbb{Z}^n \subseteq \mathbb{R}^n$) not separating: For $r := n$, the all-one vector lies in $r \cdot P$, hence $(k, \ldots, k) \in (kr \cdot P) \cap (k\mathbb{Z})^n$ for all $k \geq 1$. However, the last coordinate of every point in $P \cap \mathbb{Z}^n$ is divisible by n, so $(kr \cdot (P \cap \mathbb{Z}^n)) \cap (k\mathbb{Z})^n$ can only contain $(k, \ldots, k)$ if n divides k. We revisit this example in Example 1.3.7, showing that the corresponding toric variety does not inject into $\mathbb{P}^{2n-1}$.

1.1.3. Secant avoidance

Injection dimensions are closely tied to the behavior of secant loci, as we point out next. In fact, this is an instance of the relation between higher secant loci of varieties and the study of k-regular maps, see e.g. [BJJM19]. In that context, injective maps to low-dimensional ambient spaces appear under the name "2-regular maps" and are an important first case of interest.

Definition 1.1.12. Let Y be a subvariety of $\mathbb{P}(V)$, where V is a finite-dimensional vector space. The **secant locus** of Y in $\mathbb{P}(V)$, denoted $\sigma_2^\circ(Y)$ is the set

$$\sigma_2^\circ(Y) := \bigcup_{p,q \in Y} \langle p, q \rangle \subseteq \mathbb{P}(V),$$

where $\langle p, q \rangle \subseteq \mathbb{P}(V)$ denotes the linear subspace spanned by the points p and q. Its closure in $\mathbb{P}(V)$ is the **secant variety** of $Y \subseteq \mathbb{P}^m$ and is denoted $\sigma_2(Y)$.

Injection dimensions can be given a straightforward reinterpretation in terms of the smallest codimension of a linear space avoiding a secant locus, based on the following classical observation:

Lemma 1.1.13. *Let $W \subseteq V$ be finite-dimensional vector spaces, let $Y \subseteq \mathbb{P}(V)$ be a subvariety and consider the linear space $L := \mathbb{P}(W)$. The rational map $\pi \colon \mathbb{P}(V) \dashrightarrow \mathbb{P}(V/W)$ (i.e., the projection from L) restricts to an injective morphism $\pi|_Y \colon Y \xrightarrow{\text{inj}} \mathbb{P}(V/W)$ if and only if $L \cap \sigma_2^\circ(Y) = \emptyset$.*

PROOF. The projection from L is a well-defined morphism on Y if and only if $Y \cap L = \emptyset$. Let $y_1 \neq y_2 \in \mathbb{P}(V) \setminus L$. Then $\pi(y_1) = \pi(y_2)$ if and only if there are representatives $z_1, z_2 \in V \setminus \{0\}$ with $y_i = [z_i] \in \mathbb{P}(V)$ such that $z_1 - z_2 \in W$. But

$$\{[z_1 - z_2] \in \mathbb{P}(V) \mid z_i \in V \setminus \{0\},\ [z_i] = y_i\} = \langle y_1, y_2 \rangle.$$

This shows that π is well-defined and injective on Y if and only if $\mathbb{P}(W) \cap \sigma_2^\circ(Y) = \emptyset$. $\square$

Proposition 1.1.14. *Let X be a projective variety and let $\mathscr{L}$ be a line bundle on X with $\gamma(X,\mathscr{L}) < \infty$. Let Y be the image of the morphism $\varphi_{\mathscr{L}}\colon X \xrightarrow{\text{inj}} \mathbb{P}(H^0(X,\mathscr{L})^*)$. Then*

$$\gamma(X,\mathscr{L}) = \min\{\operatorname{codim} L - 1 \mid L \subset \mathbb{P}(H^0(X,\mathscr{L})^*) \text{ linear subspace with } L \cap \sigma_2^\circ(Y) = \emptyset\}.$$

Proof. Let $V \subseteq H^0(X,\mathscr{L})$ be a non-zero subspace and consider $W := \ker(H^0(X,\mathscr{L})^* \twoheadrightarrow V^*)$. The rational map φ_V is the projection $\pi\colon \mathbb{P}(H^0(X,\mathscr{L})^*) \dashrightarrow \mathbb{P}(H^0(X,\mathscr{L})^*/W) \cong \mathbb{P}(V^*)$. By Lemma 1.1.13, it is an injective morphism on Y if and only if $W \cap \sigma_2^\circ(Y) = \emptyset$. With the observation that $\dim V = \operatorname{codim} W$, this proves the claim. □

In particular, this gives the following folklore result which, contrary to the study of closed embeddings, does not require smoothness of the variety.

Corollary 1.1.15. *Let X be a projective variety. For any line bundle $\mathscr{L}$ with $\gamma(X,\mathscr{L}) < \infty$, we have $\gamma(X,\mathscr{L}) \leq 2\dim X + 1$.*

Proof. Note that $Y := \varphi_{\mathscr{L}}(X)$ has the same dimension as X, since $\varphi_{\mathscr{L}}$ is injective. Since $\dim \sigma_2(Y) \leq 2\dim Y + 1 = 2\dim X + 1$, a general linear subspace of $\mathbb{P}(H^0(X,\mathscr{L})^*)$ of codimension $2\dim X + 2$ does not meet $\sigma_2(Y)$, and in particular it avoids the secant locus of Y. The bound then follows from Proposition 1.1.14. □

A brave generalization of Question 1.1 would be to ask whether $\gamma(X,\mathscr{L}) \leq 2\dim X$ holds for every very ample line bundle $\mathscr{L}$. This is not the case: As shown in [Pie81], a general canonical curve $X \subseteq \mathbb{P}^3$ of genus 4 does not admit a cuspidal projection to $\mathbb{P}^2$. By Lemma 1.1.13, this means that $\gamma(X,\omega_X) = 3 > 2\dim X$. On the other hand, it is conjectured in [DJ18, Conjecture 4.9] that for products of projective spaces, one does have $\gamma(\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_r}, \mathcal{O}(d_1,\ldots,d_r)) \leq 2(\sum_{i=1}^r n_i)$ for all $d_1,\ldots,d_r > 0$. A very simple case illustrating Proposition 1.1.14 is the following example.

Example 1.1.16. Let $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and consider the very ample line bundle $\mathscr{L} = \mathcal{O}(1,1,1)$, corresponding to the Segre embedding

$$\varphi_{\mathscr{L}}\colon \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 = X \hookrightarrow \mathbb{P}(H^0(X,\mathscr{L})^*) \cong \mathbb{P}^7,$$
$$[x_0 : x_1] \times [y_0 : y_1] \times [z_0 : z_1] \mapsto [x_i y_j z_k \mid i,j,k \in \{0,1\}]$$

Here, the secant variety of $\varphi(X)$ fills the entire 7-dimensional ambient space, but the secant locus $\sigma_2^\circ(\varphi(X))$ does not. For example, one can check that

$$[(x_0y_0z_1)^* + (x_0y_1z_0)^* + (x_1y_0z_0)^*] \in \mathbb{P}(H^0(X,\mathscr{L})^*) \setminus \sigma_2^\circ(\varphi_{\mathscr{L}}(X)).$$

Hence, $\gamma(X, \mathscr{L}) \leq 6$ by Proposition 1.1.14 and projecting from this point defines an injection of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^6$. Explicitly,

$$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\text{inj}} \mathbb{P}^6,$$

$$[x_0 : x_1] \times [y_0 : y_1] \times [z_0 : z_1] \mapsto [x_0y_0z_0 : x_0y_0z_1 - x_0y_1z_0 : x_0y_0z_1 - x_1y_0z_0 : x_0y_1z_1 : x_1y_0z_1 : x_1y_1z_1].$$

In fact, one can check that $\mathbb{P}^7 \backslash \sigma_2^\circ(\varphi(X))$ does not contain any line, so by Proposition 1.1.14 there cannot exist an injection of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^5$ given by multilinear forms, showing $\gamma(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1,1)) = 6$. We generalize this example in Corollary 1.4.11, constructing an injective morphism $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^m \xrightarrow{\text{inj}} \mathbb{P}^{2(m+2)}$ for all $m \geq 1$.

By Proposition 1.1.14, the injection dimension $\gamma(X, \mathscr{L})$ is determined by the largest-dimensional linear space avoiding the secant locus of $\varphi_{\mathscr{L}}(X)$. A natural question is whether "largest-dimensional" can be replaced by "maximal with respect to inclusion":

Question 1.1.17. *Let X be a projective variety, $\mathscr{L}$ a line bundle on X and $V \subseteq H^0(X, \mathscr{L})$ a subspace such that φ_V is a closed embedding. Does there exist a subspace $W \subseteq V$ of dimension $\gamma(X, \mathscr{L}) + 1$ such that φ_W is an injective morphism?*

This question was raised in [DJ18] and it was observed that a positive answer to it would be a significant step towards proving that $\gamma(X, \mathscr{L}) \leq 2 \dim X$ whenever X is smooth and $\mathscr{L}$ is a line bundle with $\sigma_2^\circ(\varphi_{\mathscr{L}}(X)) \neq \sigma_2(\varphi_{\mathscr{L}}(X))$. However, the following example gives a negative answer to Question 1.1.17.

Example 1.1.18. Let $X = \mathbb{P}^1$, $\mathscr{L} = \mathcal{O}_{\mathbb{P}^1}(5)$. The subspace V of $H^0(X, \mathscr{L})$ spanned by $f_0 := x_0^5$, $f_1 := x_0^4x_1 + x_0^3x_1^2$, $f_2 := x_0^2x_1^3 + x_0x_1^4$ and $f_3 := x_1^5$ defines a closed embedding

$$\varphi_V \colon \mathbb{P}^1 = X \hookrightarrow \mathbb{P}(V^*) \cong \mathbb{P}^3, \qquad [x_0 : x_1] \mapsto [f_0 : f_1 : f_2 : f_3],$$

describing a rational quintic space curve $C \subseteq \mathbb{P}^3$. One can algorithmically confirm that every point in $\mathbb{P}^3$ lies on a secant line of C, i.e., $\sigma_2^\circ(C) = \mathbb{P}^3$. By Lemma 1.1.13, this implies that for any $W \subsetneq V$, the projection $\mathbb{P}(V^*) \dashrightarrow \mathbb{P}(W^*)$ cannot be injective on C, hence φ_W is not injective. Geometrically, this means that C does not admit a cuspidal projection. On the other hand, $\gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(5)) = 2$ because of the injective morphism

$$\mathbb{P}^1 \xrightarrow{\text{inj}} \mathbb{P}^2, \qquad [x_0 : x_1] \mapsto [x_0^5 : x_0^4x_1 : x_1^5].$$

1.2. Separating invariants and partially symmetric tensors

In this section, we highlight close interactions between injection dimensions and the theory of separating invariants. Moreover, we relate the

case of products of projective spaces to the identifiability of decomposable partially symmetric tensors under linear quotient operations.

1.2.1. Graded separating invariants

Classical Invariant Theory revolves around the problem of describing generators (and their relations) of invariant rings for group actions on vector spaces or, more generally, on varieties. However, generating sets of invariant rings tend to be very large (possibly infinite) and hard to explicitly construct. The study of *rational invariants*, i.e., generators for quotient fields of invariant rings, is often simpler to carry out [CS07; HK07], but in some applications describing only the generic behavior of the group action can be insufficient. An intermediate approach between these two extremes is the more recent field of study of *separating invariants* [DK15, §2.4], [Kem09], which maintain the full geometric information about orbit separation while remedying many of the complications of complete generating sets of invariants [Dom07; DKW08; NS09]. Separating invariants are of major importance for applications, as for example in the recent work [CCH19]; see [DK15, §5] for an overview of possible applications.

Here, we highlight the close connection between separating invariants and injection dimensions of projective varieties. We focus on the most classical situation: separating invariants for linear actions of reductive algebraic groups on finite-dimensional vector spaces.

Definition 1.2.1. A **separating set of invariants** for a finite-dimensional representation V of a group G is a set of invariant polynomials $F \subseteq \mathbb{C}[V^*]^G$ such that for all points $v, w \in V$ the following equivalence holds:

$$f(v) = f(w) \text{ for all } f \in F \quad \Leftrightarrow \quad f(v) = f(w) \text{ for all } f \in \mathbb{C}[V^*]^G.$$

Equivalently, in the case of a reductive algebraic group G, a finite set of invariants $F = \{f_1, \dots, f_s\} \subseteq \mathbb{C}[V^*]^G$ is separating if and only if the morphism $V /\!\!/ G = \operatorname{Spec} \mathbb{C}[V^*]^G \to \mathbb{A}^s$ given by $(f_1, \dots, f_s)$ is injective. This means that in the *affine setting*, there is an immediate translation between injective morphisms to affine spaces and separating sets of invariants, whenever the coordinate ring of an affine variety has a description as an invariant ring — the difference being rather a change of language.

In this chapter, we look at the *projective setting*: We study injective morphisms from projective varieties to projective spaces. Here, the corresponding translation to the world of separating invariants is more subtle and we dedicate this section to carefully working it out in detail.

Often, small separating sets of invariants are obtained in two steps: (1) identify a large separating set, (2) form a smaller separating set by taking suitable linear combinations. This is closely related to injections

of projective varieties in the situation that the separating set in (1) consists of homogeneous polynomials satisfying homogeneous relations.

Definition 1.2.2. Let V be a finite-dimensional representation of a group G. We call a finite separating set of invariants $F = \{f_1, \dots, f_s\} \subseteq \mathbb{C}[V^*]^G$ **graded** if each $f_i \in \mathbb{C}[V^*]^G$ is a homogeneous polynomial and its ideal of relations

$$\ker\Big(\mathbb{C}[z_1, \dots, z_s] \to \mathbb{C}[V^*]^G, \quad z_i \mapsto f_i\Big) \subseteq \mathbb{C}[z_1, \dots, z_s]$$

is homogeneous.

Equivalently, a finite separating set $F = \{f_1, \dots, f_s\}$ of homogeneous polynomials is graded if and only if the separating algebra $S := \mathbb{C}[F]$ can be given a grading $S = \bigoplus_{d \geq 0} S_d$ with $F \subseteq S_1$. We want to emphasize that this grading, induced by the homomorphism $\mathbb{C}[z_1, \dots, z_s] \twoheadrightarrow \mathbb{C}[F]$, does typically *not* agree with the natural grading of $\mathbb{C}[F]$ as a graded subalgebra of the polynomial ring $\mathbb{C}[V^*]$. See Example 1.2.3 below.

For convenience, we formulated Definition 1.2.2 for finite separating sets, but note that this is not a restriction: A set $F \subseteq \mathbb{C}[V^*]^G$ spanning a finite-dimensional vector space $\langle F \rangle \subseteq \mathbb{C}[V^*]^G$ is a separating set if and only if a basis of $\langle F \rangle$ is.

Example 1.2.3. The action of $\mathbb{Z}_6 = \{\xi \in \mathbb{C}^* \mid \xi^6 = 1\}$ on $V = \mathbb{C}^4$ given by

$$\mathbb{Z}_6 \to \mathrm{GL}(5, \mathbb{C}), \qquad \xi \mapsto \mathrm{diag}(\xi^2, \xi^2, \xi^3, \xi^3)$$

has an invariant ring $\mathbb{C}[V^*]^{\mathbb{Z}_6}$ generated by the 7 invariant homogeneous polynomials

$$F := \{f_i := x_1^i x_2^{3-i},\ g_j := x_3^j x_4^{2-j} \mid i \in \{0,1,2,3\},\ j \in \{0,1,2\}\}.$$

Their ideal of relations is homogeneous, generated by four quadratic binomials, so F is a graded separating set of invariants. Note that there are two different gradings on $\mathbb{C}[F]$: With respect to the grading induced from $\mathbb{C}[V^*]$, we have $\deg f_i = 3$ and $\deg g_j = 2$. On the other hand, with respect to the grading induced by $\mathbb{C}[z_1, \dots, z_7] \twoheadrightarrow \mathbb{C}[F]$, every element of F is homogeneous of degree 1. A separating set of smallest cardinality obtained by linear combinations from elements in F is $E := \{f_0, f_1, f_2 + f_3, g_0, g_1, g_2\}$, see Example 1.3.11.

Proposition 1.2.4. *Let V be a finite-dimensional representation of a reductive algebraic group G. Let $F = \{f_1, \dots, f_m\} \subseteq \mathbb{C}[V^*]^G$ be a graded separating set and $\mathfrak{a}_F := \ker(\mathbb{C}[z_1, \dots, z_m] \twoheadrightarrow \mathbb{C}[F])$ its ideal of relations. Let $s \leq m-1$ be minimal such that the projective variety $V(\mathfrak{a}_F) \subseteq \mathbb{P}^{m-1}$ can be injected to $\mathbb{P}^s$ by a linear projection $\mathbb{P}^{m-1} \dashrightarrow \mathbb{P}^s$. Then*

$$s = \min\{|E| - 1 \mid E \subseteq \langle F \rangle \text{ separating set}\}.$$

In particular, a lower bound for the size of a separating set obtained by linear combinations of $f_1, \dots, f_m$ is given by $1+\gamma(\operatorname{Proj}\mathbb{C}[F], \mathcal{O}(1))$, where we equip $\mathbb{C}[F]$ with the grading induced by $\mathbb{C}[z_1, \dots, z_m] \twoheadrightarrow \mathbb{C}[F]$.

Proof. Replacing F with a linearly independent subset, we reduce to the case that F is a basis for the subspace of $\mathbb{C}[V^*]^G$ it spans. Let $E = \{g_1, \dots, g_k\} \subseteq \langle F \rangle$ be a finite set of linear combinations of $f_1, \dots, f_m$ with $\dim \langle E \rangle = k$. Consider the morphism

$$\psi\colon V/\!\!/G = \operatorname{Spec}\mathbb{C}[V^*]^G \to \operatorname{Spec}\mathbb{C}[F] \to \operatorname{Spec}\mathbb{C}[E] \xhookrightarrow{(g_1,\dots,g_k)} \mathbb{A}^k$$

and note that E is a separating set if and only if ψ is injective on the set-theoretic image of the quotient morphism $V \to V/\!\!/G$. For reductive groups, the latter quotient morphism is surjective, so E is separating if and only if ψ is injective.

Since F is a separating set, this means in particular that the morphism $\operatorname{Spec}\mathbb{C}[V^*]^G \to \operatorname{Spec}\mathbb{C}[F]$ is injective. By [DK15, Theorem 2.4.6], the assumption that F consists of homogeneous polynomials (with respect to the grading of $\mathbb{C}[V^*]$) implies that this morphism is the normalization of $\operatorname{Spec}\mathbb{C}[F]$ and hence also surjective. In particular, ψ is injective if and only if $\widehat{\varphi}_{\langle E \rangle}\colon \operatorname{Spec}\mathbb{C}[F] \xrightarrow{(g_1,\dots,g_k)} \mathbb{A}^k$ is injective.

Considering the grading $\mathbb{C}[F] = \bigoplus_{d \geq 0} S_d$ induced by the homomorphism $\mathbb{C}[z_1, \dots, z_m] \twoheadrightarrow \mathbb{C}[F]$, we have $\operatorname{Proj}\mathbb{C}[F] \cong V(\mathfrak{a}_F) \subseteq \mathbb{P}^{m-1}$ and $E \subseteq S_1$. Then the above $\widehat{\varphi}_{\langle E \rangle}$ is the morphism of affine cones over the rational map $\varphi_{\langle E \rangle}\colon \operatorname{Proj}\mathbb{C}[F] \dashrightarrow \mathbb{P}(\langle E \rangle^*) = \mathbb{P}^{k-1}$ given by $g_1, \dots, g_k \in S_1 \subseteq H^0(\operatorname{Proj}\mathbb{C}[F], \mathcal{O}(1))$. By Lemma 1.1.4, injectivity of $\widehat{\varphi}_{\langle E \rangle}$ is equivalent to $\varphi_{\langle E \rangle}$ being an injective morphism. To sum up, a linearly independent set $E \subseteq \langle F \rangle$ is separating if and only if

$$\varphi_{\langle E \rangle}\colon \operatorname{Proj}\mathbb{C}[F] \cong V(\mathfrak{a}_F) \subseteq \mathbb{P}^{m-1} = \mathbb{P}(\langle F \rangle^*) \dashrightarrow \mathbb{P}(\langle E \rangle^*)$$

is an injective morphism, proving the claim. □

A classical fact about separating invariants is that there always exists a separating set of size $2\dim\mathbb{C}[V^*]^G + 1$, see [Duf08, Proposition 5.1.1]. Note that in the presence of a graded separating set, this bound can be improved by one, combining Proposition 1.2.4 and Corollary 1.1.15. An example of a representation not admitting a graded separating set is $\mathbb{Z}_3 \to \mathrm{GL}(2, \mathbb{C})$, $\xi \mapsto \operatorname{diag}(\xi, \xi^2)$: Its invariant ring $\mathbb{C}[x_1, x_2]^{\mathbb{Z}_3} = \mathbb{C}[x_1^3, x_1x_2, x_2^3]$ contains no homogeneous polynomials separating orbits and satisfying homogeneous relations.

Proposition 1.2.5. *Let V be a finite-dimensional representation of a reductive algebraic group G. Assume that the invariant ring $S := \mathbb{C}[V^*]^G$ can be given a grading $S = \bigoplus_{d \geq 0} S_d$ with $S_0 = \mathbb{C}$ such that S_1 is a separating set. Then, with respect to this grading,*

$$\gamma(\operatorname{Proj} S, \mathcal{O}(1)) = \min\{|F| - 1 \mid F \subseteq S_1 \textit{ finite separating set}\}.$$

As before, we remark that the grading of $S = \mathbb{C}[V^*]^G$ in Proposition 1.2.5 need not agree with the grading induced from the polynomial ring $\mathbb{C}[V^*]$. Moreover, note that Proposition 1.2.5 does not assume S_1 to be spanned by homogeneous polynomials with respect to the $\mathbb{C}[V^*]$-grading (in which case Proposition 1.2.5 is a consequence of Proposition 1.2.4).

PROOF OF PROPOSITION 1.2.5. Since G is reductive, the invariant ring $S = \mathbb{C}[V^*]^G$ is a finitely generated $\mathbb{C}$-algebra and, in particular, S_1 is a finite-dimensional vector space. Moreover, the reductivity of G implies that a subset $F \subseteq S_1$ is separating if and only if the morphism $\widehat{\varphi}_{\langle F\rangle} \colon \operatorname{Spec} S \to \langle F\rangle^*$ is injective. By Lemma 1.1.4, this is equivalent to $\varphi_{\langle F\rangle} \colon \operatorname{Proj} S \dashrightarrow \mathbb{P}(\langle F\rangle^*)$ being an injective morphism.

By assumption, this is the case for the separating set $F = S_1$, so in particular the rational map $\varphi_{S_1} \colon \operatorname{Proj} S \dashrightarrow \mathbb{P}(S_1^*)$ is a morphism. This means that the vanishing set of the homogeneous ideal $(S_1) \subseteq S$ in $\operatorname{Proj} S$ is empty, hence the coherent sheaf $\mathcal{O}_{\operatorname{Proj} S}(1)$ is a line bundle. Its global sections are $H^0(\operatorname{Proj} S, \mathcal{O}(1)) = S_1$, since $S = \mathbb{C}[V^*]^G$ is a normal ring, see [DK15, Proposition 2.4.4]. In particular, every rational map from $\operatorname{Proj} S$ to a projective space given by a linear subsystem of $|\mathcal{O}(1)|$ is of the form $\varphi_{\langle F\rangle}$ for some $F \subseteq S_1$. Then the previous observation proves the claim. □

Remark 1.2.6. Proposition 1.2.5 and its proof generalize verbatim to the setting that the finite-dimensional representation V of G is replaced by the action of a reductive algebraic group G on a normal irreducible affine variety $X = \operatorname{Spec} R$, with the obvious generalization of Definition 1.2.1 to this case, as in [DK15, Definition 2.4.1].

Proposition 1.2.4 and Proposition 1.2.5 show that graded separating sets have an interpretation as injective morphisms of projective varieties to projective spaces by subsystems of a fixed line bundle. Conversely, one can often interpret injections of a given projective variety as separating sets with respect to a suitable invariant ring. We highlight this in the setting of normal toric varieties:

Theorem 1.2.7. *Let $\mathscr{L}$ be an ample line bundle on a normal projective toric variety X. There is a finite-dimensional representation V of a diagonalizable group G and a grading $S = \bigoplus_{d\geq 0} S_d$ of the invariant ring $S := \mathbb{C}[V^*]^G$ such that*

$$\gamma(X, \mathscr{L}) = \inf\{|F| - 1 \mid F \subseteq S_1 \textit{ finite separating set of invariants}\}.$$

PROOF. Let $X = X_\Sigma$ be the normal toric variety associated to a fan of rational polyhedral cones Σ. Its Cox ring $\operatorname{Cox}(X_\Sigma)$ is the polynomial ring $\mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$ and it is graded by the class group $\operatorname{Cl}(X)$, which is a finitely generated abelian group, see [CLS11, §5.2]. If $\mathscr{L} \in \operatorname{Pic}(X)$ corresponds to $\alpha \in \operatorname{Cl}(X)$ under the inclusion $\operatorname{Pic}(X) \hookrightarrow \operatorname{Cl}(X)$,

then the α-graded piece of the Cox ring is $\mathrm{Cox}(X_\Sigma)_\alpha = H^0(X, \mathscr{L})$ by [CLS11, Proposition 5.4.1].

The grading of the Cox ring by the class group corresponds to a linear action of the character group $G_0 := \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Cl}(X), \mathbb{C}^*)$ on the vector space $V := \mathbb{C}^{\Sigma(1)} = \operatorname{Spec} \mathrm{Cox}(X_\Sigma)$, and the graded pieces of $\mathrm{Cox}(X_\Sigma)$ are the eigenspaces for the induced action of G_0 on the coordinate ring $\mathbb{C}[V^*] = \mathrm{Cox}(X_\Sigma)$. Since $\mathrm{Cl}(X)$ is a finitely generated abelian group, G_0 is a diagonalizable group (i.e., the product of a torus and a finite abelian group).

Consider the subgroup $G := \{\xi \in G_0 \mid \xi(\alpha) = 1\}$, which is again diagonalizable. Then a homogeneous element $f \in \mathrm{Cox}(X_\Sigma)_\beta$ is invariant under the action of G if and only if $\xi(\beta) = 1$ for all $\xi \in G$. By [Spr98, Exercise 3.2.10.(4)], this is only the case when β lies in the subgroup of $\mathrm{Cl}(X)$ generated by α. This means

$$\mathbb{C}[V^*]^G = \bigoplus_{d\in\mathbb{Z}} \mathrm{Cox}(X_\Sigma)_{d\alpha} \cong \bigoplus_{d\in\mathbb{Z}} H^0(X, \mathscr{L}^{\otimes d}) = \bigoplus_{d\in\mathbb{N}} H^0(X, \mathscr{L}^{\otimes d}),$$

where the last equality follows from the fact that $H^0(X, \mathscr{L}^{\otimes d}) = 0$ for all $d < 0$, since $\mathscr{L}$ is ample. Defining $S_d := H^0(X, \mathscr{L}^{\otimes d})$, this describes a grading $S = \bigoplus_{d\geq 0} S_d$ on the invariant ring $S := \mathbb{C}[V^*]^G$ such that $S_1 = H^0(X, \mathscr{L})$. Then the claim follows from Proposition 1.2.5, using that $X \cong \operatorname{Proj} S$. Note that $\gamma(X, \mathscr{L}) = \infty$ holds if and only if $S_1 = H^0(X, \mathscr{L})$ is not a separating set. □

Example 1.2.8. We exemplify Theorem 1.2.7 in the case of a weighted projective space $X = \mathbb{P}(q_0, \ldots, q_n)$. Its Cox ring is the polynomial ring $\mathbb{C}[x_0, \ldots, x_n]$ as a $\mathbb{Z}$-graded ring with $\deg(x_i) = q_i$. This grading corresponds to the action of the group $G_0 = \mathbb{C}^*$ on $V = \mathbb{C}^{n+1}$ given by $\mathbb{C}^* \to \mathrm{GL}(n+1, \mathbb{C})$, $t \mapsto \mathrm{diag}(t^{q_0}, \ldots, t^{q_n})$. Every ample line bundle on X is of the form $\mathscr{L} \cong \mathcal{O}(k)$ with $k > 0$ divisible by all q_i. Its section ring $\bigoplus_{d\in\mathbb{N}} H^0(X, \mathscr{L}^{\otimes d})$ is the k-th Veronese subring $\bigoplus_{d\geq 0} \mathbb{C}[x_0, \ldots, x_n]_{dk}$, which is the invariant ring for the action of the subgroup $G := \{\xi \in \mathbb{C}^* \mid \xi^k = 1\} \subseteq G_0$ on V.

Hence, $\gamma(\mathbb{P}(q_0, \ldots, q_n), \mathcal{O}(k)) + 1$ is the smallest size of a separating set of invariants $F \subseteq \mathbb{C}[x_0, \ldots, x_n]^{\mathbb{Z}_k}$ for the representation

$$\mathbb{Z}_k \to \mathrm{GL}(n+1, \mathbb{C}), \quad \xi \mapsto \mathrm{diag}(\xi^{q_0}, \ldots, \xi^{q_n}),$$

such that the polynomials in F are homogeneous of $(q_0, \ldots, q_n)$-weighted degree k.

1.2.2. Segre–Veronese varieties and tensors

A major source of examples in the constructive parts of this chapter are products of projective spaces. Since every ample line bundle on $X = \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_r}$ is very ample, (1.1) makes clear which line bundles give rise to injections: For $(d_1, \ldots, d_r) \in \mathbb{Z}^r$ we have

$$\gamma(X, \mathcal{O}(d_1, \ldots, d_r)) < \infty \qquad \Leftrightarrow \qquad d_1, \ldots, d_r > 0.$$

For $\mathscr{L} = \mathcal{O}(d_1, \ldots, d_r)$ with $d_i \geq 1$, the vector space $H^0(X, \mathscr{L})$ is identified with the set of multihomogeneous forms of degree $(d_1, \ldots, d_r)$:

$$H^0(X, \mathscr{L}) = \bigotimes_{i=1}^{r} \operatorname{Sym}^{d_i}(\mathbb{C}^{n_i+1})^*.$$

Then the closed embedding $\varphi_{\mathscr{L}} \colon X \hookrightarrow \mathbb{P}(H^0(X, \mathscr{L})^*)$ is the natural morphism

$$\begin{aligned} \varphi_{\mathscr{L}} \colon \mathbb{P}(\mathbb{C}^{n_1+1}) \times \ldots \times \mathbb{P}(\mathbb{C}^{n_r+1}) &\hookrightarrow \mathbb{P}\left(\bigotimes_{i=1}^{r} \operatorname{Sym}^{d_i} \mathbb{C}^{n_i+1}\right), \\ [v_1] \times \ldots \times [v_r] &\mapsto [v_1^{d_1} v_2^{d_2} \ldots v_r^{d_r}]. \end{aligned}$$

Its image $Y := \operatorname{im}(\varphi_{\mathscr{L}})$ is a subvariety of $\mathbb{P}(\bigotimes_{i=1}^{r} \operatorname{Sym}^{d_i} \mathbb{C}^{n_i+1})$ called the **Segre–Veronese variety** of type $(n_1, \ldots, n_r; d_1, \ldots, d_r)$. The space $\mathbb{P}(\bigotimes_{i=1}^{r} \operatorname{Sym}^{d_i} \mathbb{C}^{n_i+1})$ consists of partially symmetric tensors up to scaling, and the subvariety Y consists of the *decomposable* (or *rank 1*) partially symmetric tensors. Its secant locus $\sigma_2^\circ(Y)$ is the set of partially symmetric tensors (up to scaling) of *rank at most* 2, while the secant variety $\sigma_2(Y)$ corresponds to the notion of *border rank at most* 2. We refer the reader to [MS20, §9] for a brief introduction to varieties of tensors, their ranks and secant loci. For in-depth background on the theory of (partially symmetric) tensors and their importance in applications, see [Lan12] . In this language, Proposition 1.1.14 gives the following reinterpretation of $\gamma(\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_r}, \mathcal{O}(d_1, \ldots, d_r))$:

Corollary 1.2.9. *Let $X = \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_r}$ and $\mathscr{L} := \mathcal{O}(d_1, \ldots, d_r)$ for $n_i, d_i \geq 1$. Then*

$$\gamma(X, \mathscr{L}) = \min\Big\{ \operatorname{codim} L - 1 \;\mid\; L \subseteq \bigotimes_{i=1}^{r} \operatorname{Sym}^{d_i} \mathbb{C}^{n_i+1} \text{ subspace not containing any non-zero partially symmetric tensor of rank} \leq 2 \Big\}.$$

In other words, finding a low-dimensional injection of $\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_r}$ by polynomials of multidegree $(d_1, \ldots, d_r)$ is equivalent to the search of a high-dimensional subspace $L \subseteq \bigotimes_{i=1}^{r} \operatorname{Sym}^{d_i} \mathbb{C}^{n_i+1}$ such that decomposable partially symmetric tensors stay identifiable under the quotient $\operatorname{Sym}^{d_i} \mathbb{C}^{n_i+1} \twoheadrightarrow \operatorname{Sym}^{d_i} \mathbb{C}^{n_i+1}/L$ (in the sense that any decomposable tensors can be uniquely reconstructed from its image under the quotient operation).

By Theorem 1.2.7, this question also has a formulation in terms of separating invariants. We work this out carefully here, since an incorrect description in the literature gave rise to wrong lower bounds on injection dimensions [DJ18]. We comment on this unfortunate flaw in the literature and its correction at the beginning of Section 1.3.

The Cox ring of $X = \prod_{i=1}^r \mathbb{P}^{n_i}$ is the polynomial ring $\mathbb{C}[V^*]$, where $V := \bigoplus_{i=1}^k \mathbb{C}^{n_i+1}$. Explicitly, denoting the coordinates on $\mathbb{C}^{n_i+1}$ by $x_{i0}, x_{i1}, \ldots, x_{in_i}$, this is

$$\mathrm{Cox}(\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_r}) = \mathbb{C}[x_{ij} \mid i \in \{1, \ldots, r\}, j \in \{0, 1, \ldots, n_i\}],$$

equipped with a $\mathbb{Z}^r$-grading given by $\deg x_{ij} = e_i \in \mathbb{Z}^r$. This grading corresponds to the action of $G_0 = (\mathbb{C}^*)^r$ on $V := \bigoplus_{i=1}^r \mathbb{C}^{n_i+1}$ given by

$$(t_1, \ldots, t_r) \cdot (v_1, \ldots, v_r) := (t_1 v_1, \ldots, t_r v_r) \quad \text{for all } v = (v_1, \ldots, v_r) \in V.$$

Every ample line bundle on X is of the form $\mathscr{L} \cong \mathcal{O}(d_1, \ldots, d_r)$ with $d_i > 0$. Its section ring $\bigoplus_{k \in \mathbb{N}} H^0(X, \mathscr{L}^{\otimes k})$ is the homogeneous coordinate ring of the Segre–Veronese variety of type $(n_1, \ldots, n_r; d_1, \ldots, d_r)$. It is the invariant ring for the action on V of the subgroup

$$G := \{(t_1, \ldots, t_r) \in (\mathbb{C}^*)^r \mid t_1^{d_1} \cdots t_r^{d_r} = 1\} \subseteq G_0,$$

which is isomorphic to $(\mathbb{C}^*)^{r-1} \times \mathbb{Z}_k$, where $k := \gcd\{d_1, \ldots, d_r\}$. Explicitly, consider the action of $(\mathbb{C}^*)^{r-1} \times \mathbb{Z}_k$ on V given by

$$((t_1, \ldots, t_{r-1}) \times \xi) \times e_{ij} \mapsto \begin{cases} t_i^{d_1 \cdots \widehat{d_i} \cdots d_r} \, \xi \, e_{ij} & \text{if } i \leq r-1 \\ (t_1 \cdots t_{r-1})^{-d_1 \cdots d_{r-1}} \, \xi \, e_{ij} & \text{if } i = r. \end{cases}$$

Its invariant ring is generated by all the monomials of the multidegree $(d_1, \ldots, d_r)$. A separating set of invariants consisting of s linear combinations of these corresponds to an injection of $\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_r} \xrightarrow{\text{inj}} \mathbb{P}^{s-1}$ given by global sections of $\mathcal{O}(d_1, \ldots, d_r)$.

1.3. Obstructions to low-dimensional injections

In this section, we provide lower bounds on injection dimensions due to topological obstructions: The first approach (Proposition 1.3.3) exploits that in projective spaces any two subvarieties of complementary dimension must intersect, while this is not necessarily the case for arbitrary projective varieties — this discrepancy leads to lower bounds on injection dimensions, irrespective of the choice of a line bundle. We use this simple argument to improve previously known lower bounds on $\gamma(\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_r})$.

Secondly, a more sophisticated argument based on (dis-)connectedness properties for orbits of linear spaces under a finite group action bounds injection dimensions for line bundles which admit a root of some order (Theorem 1.3.6) or are divisible in the class group (Theorem 1.3.8). We apply this to construct n-dimensional irreducible varieties of injection dimension $\geq 2n$ and comment on injection dimensions of weighted projective spaces.

Previous work from the perspective of separating invariants [Duf08], [DJ15] established that $\gamma(\mathbb{P}^n, \mathcal{O}(d)) = 2n$ for all $d \geq 2$, whereas of course $\gamma(\mathbb{P}^n, \mathcal{O}(1)) = n$. There, the (slightly stronger) question of

injecting the affine cones over Veronese varieties into affine spaces is studied with techniques from local cohomology in order to obtain lower bounds.

With a similar approach, the article [DJ18] claims to prove for products of projective spaces $\mathbb{P}^{n_1}\times\ldots\times\mathbb{P}^{n_r}$ that the injection dimension with respect to a very ample line bundle $\mathcal{O}(d_1,\ldots,d_r)$ is bounded below by $2(\sum_{i=1}^r n_i)$, provided that not all d_i are equal to 1. Unfortunately, this is wrong; we provide an explicit counterexample in Proposition 1.4.2 showing $\gamma(\mathbb{P}^1\times\mathbb{P}^1,\mathcal{O}(1,2)) = 3$. The flaw in [DJ18] is an incorrect description of a group action identifying the homogeneous coordinate ring of a Segre–Veronese variety with an invariant ring in the case that not all d_i are equal. Correcting this with the description given in Section 1.2.2, a straightforward adaption of [DJ18, Proof of Theorem 5.4] shows $\gamma(\mathbb{P}^{n_1}\times\ldots\times\mathbb{P}^{n_r},\mathcal{O}(d_1,\ldots,d_r)) \geq 2(\sum_{i=1}^r n_i)$ whenever $\gcd\{d_1,\ldots,d_r\} > 1$.

Theorem 1.3.6 and Theorem 1.3.8 generalize these results from (products of) projective spaces to arbitrary projective varieties.

1.3.1. Existence of fibrations

We start out with an elementary observation giving lower bounds on injection dimensions:

Lemma 1.3.1. *Let X be a projective variety and let $Y,Z\subseteq X$ be disjoint closed subsets. Then*

$$\gamma(X) \geq \dim Y + \dim Z + 1.$$

PROOF. For any injection $\varphi\colon X \xrightarrow{\text{inj}} \mathbb{P}^s$, we have $\varphi(Y)\cap\varphi(Z) = \varphi(Y\cap Z) = \emptyset$ and $\dim\varphi(Y) = \dim Y$, $\dim\varphi(Z) = \dim Z$. But two subvarieties of $\mathbb{P}^s$ can only be disjoint if their dimensions sum to at most $s-1$, hence $s \geq \dim Y + \dim Z + 1$. □

Example 1.3.2. Let $L\subseteq\mathbb{P}^n$ be a linear subspace of codimension 2 and consider the blow-up of $\mathbb{P}^n$ along L. Then the strict transforms of two distinct hyperplanes containing L are disjoint effective divisors on $\operatorname{Bl}_L\mathbb{P}^n$. Hence, $\gamma(\operatorname{Bl}_L\mathbb{P}^n)\geq 2n-1$ by Lemma 1.3.1.

A particular consequence of Lemma 1.3.1 is that the existence of fibrations $X\to S$ is an obstruction to low-dimensional injections of X:

Proposition 1.3.3. *Let $X\to S$ be a surjective morphism of irreducible projective varieties with* $\dim S\geq 1$. *Then*

$$\gamma(X) \geq 2\dim X - \dim S.$$

PROOF. We can find disjoint irreducible subvarieties $Y_0,Z_0\subseteq S$ with $\dim Y_0 + \dim Z_0 = \dim S - 1$. Let $Y,Z\subseteq X$ be the fibers of

$X \to S$ over them. Note that $Y \cap Z = \emptyset$, so Lemma 1.3.1 implies

$$\begin{aligned}\gamma(X) &\geq \dim Y + \dim Z + 1\\ &\geq (\dim X - \dim S + \dim Y_0) + (\dim X - \dim S + \dim Z_0) + 1\\ &\geq 2\dim X - \dim S.\end{aligned}$$

□

Example 1.3.4. An n-dimensional projective bundle over a curve has injection dimension at least $2n - 1$.

In [DJ18, Proposition 5.6], the following general bound for injection dimensions of products of projective spaces was derived with techniques from local cohomology:

$$\gamma(\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_r}) \geq 2\Big(\sum_{i=1}^{r} n_i\Big) - 2\min\{n_1, \ldots, n_r\} + 1.$$

Our basic geometric observations improve this bound as follows:

Corollary 1.3.5. *For all $n_1, \ldots, n_r \geq 1$, we have*

$$\gamma(\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_r}) \geq 2\Big(\sum_{i=1}^{r} n_i\Big) - \min\{n_1, \ldots, n_r\}.$$

Proof. This follows from applying Proposition 1.3.3 to the projections $\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_r} \to \mathbb{P}^{n_i}$ for $i = 1, \ldots, r$. □

1.3.2. Divisibility in the Picard/class group

Our main lower bound for injection dimension with respect to fixed line bundles follows. It is inspired by and vastly generalizes previous work for (products of) projective spaces in [DJ15; DJ18].

Theorem 1.3.6. *Let X be a projective variety and let $\mathscr{L}$ be a line bundle on X. Then $\gamma(X, \mathscr{L}^{\otimes k}) \geq 2\dim X$ for all $k \geq 2$.*

Proof. By restricting to a top-dimensional component, we may assume that X is irreducible. Fix $k \geq 2$. We may assume that $\gamma(X, \mathscr{L}^{\otimes k}) < \infty$, since the claim is otherwise trivial. Then the line bundle $\mathscr{L}$ is ample by (1.1), so by [Laz04, Example 1.2.22], its section ring $R := \bigoplus_{d \geq 0} H^0(X, \mathscr{L}^{\otimes d})$ is a finitely generated graded algebra over $R_0 = H^0(X, \mathcal{O}_X) = \mathbb{C}$, and we have $\operatorname{Proj} R \cong X$. The $\mathbb{C}$-algebra $S := R \otimes_{\mathbb{C}} R$ is then also finitely generated, and it inherits a grading $S = \bigoplus_{d \geq 0} S_d$ with graded pieces $S_d = \bigoplus_{i=0}^{d} R_i \otimes_{\mathbb{C}} R_{d-i}$.

The Veronese subalgebra $R^{(k)} := \bigoplus_{d \geq 0} R_{kd}$ is the section ring of the line bundle $\mathscr{L}^{\otimes k}$ and we have $\operatorname{Proj} R^{(k)} \cong \operatorname{Proj} R \cong X$. Note that $R^{(k)}$ is the invariant ring under the degree-preserving action of the cyclic group $\mathbb{Z}_k = \{\xi \in \mathbb{C}^* \mid \xi^k = 1\}$ on R given by

$$\mathbb{Z}_k \times R_d \to R_d, \qquad (\xi, f) \mapsto \xi^d f.$$

Let $V \subseteq H^0(X, \mathscr{L}^{\otimes k}) = R_k$ be a non-zero subspace such that $\varphi_V \colon X \xrightarrow{\text{inj}} \mathbb{P}(V^*)$ is an injective morphism. We aim to show that

$\dim \mathbb{P}(V^*) \geq 2 \dim X$. We consider the following commutative diagram:

$$\begin{tikzcd}
 & \operatorname{Spec} R^{(k)} \otimes_{\mathbb{C}} R^{(k)} \arrow[dl, dashed] \arrow[dr, "\hat{\varphi}_V \times \hat{\varphi}_V"'] & & \operatorname{Spec} R \otimes_{\mathbb{C}} R \arrow[ll, "\pi", "/(\mathbb{Z}_k \times \mathbb{Z}_k)"'] \arrow[dl, "\hat{\psi}"] \arrow[dr, dashed] & \\
\operatorname{Proj} R^{(k)} \times \operatorname{Proj} R^{(k)} \arrow[dr, "\varphi_V \times \varphi_V"'] & & V^* \times V^* \arrow[dl, dashed] \arrow[dr, dashed] & & \operatorname{Proj} R \otimes_{\mathbb{C}} R \arrow[dl, "\psi"] \\
 & \mathbb{P}(V^*) \times \mathbb{P}(V^*) & & \mathbb{P}(V^* \times V^*). &
\end{tikzcd}$$

Injectivity of φ_V means that the preimage of the diagonal inside $\mathbb{P}(V^*) \times \mathbb{P}(V^*)$ under $\varphi_V \times \varphi_V$ is set-theoretically the diagonal in $X \times X$, i.e.,

$$((\varphi_V \times \varphi_V)^{-1} \Delta_{\mathbb{P}(V^*) \times \mathbb{P}(V^*)})^{\mathrm{red}} = \Delta_{\operatorname{Proj} R^{(k)} \times \operatorname{Proj} R^{(k)}}.$$

On the level of affine cones, the morphism $\hat{\varphi}_V \colon \operatorname{Spec} R^{(k)} \to V^*$ is injective by Lemma 1.1.4, so this equality lifts to

$$((\hat{\varphi}_V \times \hat{\varphi}_V)^{-1} \Delta_{V^* \times V^*})^{\mathrm{red}} = \Delta_{\operatorname{Spec} R^{(k)} \times \operatorname{Spec} R^{(k)}}.$$

The morphism $\pi \colon \operatorname{Spec} R \otimes_{\mathbb{C}} R \to \operatorname{Spec} R^{(k)} \otimes_{\mathbb{C}} R^{(k)}$ is the geometric quotient for the action of $\mathbb{Z}_k \times \mathbb{Z}_k$ on $\operatorname{Spec} S = \operatorname{Spec} R \times \operatorname{Spec} R$, so the preimage of the diagonal under $\hat{\psi} := (\hat{\varphi}_V \times \hat{\varphi}_V) \circ \pi$ is the $\mathbb{Z}_k \times \mathbb{Z}_k$-orbit of the diagonal in $\operatorname{Spec} R \times \operatorname{Spec} R$:

$$(\hat{\psi}^{-1} \Delta_{V^* \times V^*})^{\mathrm{red}} = (\mathbb{Z}_k \times \mathbb{Z}_k) \cdot \Delta_{\operatorname{Spec} R \times \operatorname{Spec} R}.$$

Under projectivization of $V^* \times V^*$, the diagonal $\Delta_{V^* \times V^*}$ becomes a linear subspace $L \subseteq \mathbb{P}(V^* \times V^*)$ of dimension $\dim \mathbb{P}(V^*)$. Then the previous equality of sets becomes

$$(\psi^{-1} L)^{\mathrm{red}} = (\mathbb{Z}_k \times \mathbb{Z}_k) \cdot Y = \bigcup_{\xi \in \mathbb{Z}_k} (1 \times \xi) \cdot Y,$$

where $Y := V(f \otimes 1 - 1 \otimes f \mid f \in R) \subseteq \operatorname{Proj} R \otimes_{\mathbb{C}} R$.

We claim that this is in fact a disjoint union, so that $\psi^{-1}(L)$ has $k \geq 2$ connected components. Then, by [Laz04, Theorem 3.3.3], this disconnectedness forces

$$\operatorname{codim}_{\mathbb{P}(V^* \times V^*)} L \geq \dim \operatorname{im} \psi.$$

Note that $\operatorname{codim}_{\mathbb{P}(V^* \times V^*)} L = \dim \mathbb{P}(V^*) + 1$. On the other hand,

$$\begin{aligned} \dim \operatorname{im} \psi = \dim \operatorname{im} \hat{\psi} - 1 &= \dim \operatorname{im}(\hat{\varphi}_V \times \hat{\varphi}_V) - 1 \\ &= \dim \operatorname{im}(\varphi_V \times \varphi_V) + 1 = 2 \dim X + 1, \end{aligned}$$

where the last equality holds by injectivity of φ_V. We conclude that $\mathbb{P}(V^*) \geq 2 \dim X$.

It remains to prove that $(1 \times \xi) \cdot Y$ and $(1 \times \xi') \cdot Y$ are disjoint for $\xi \neq \xi' \in \mathbb{Z}_k$. For this, we may assume that $\xi' = 1$. Note that $(1 \times \xi) \cdot Y = V(f \otimes 1 - 1 \otimes \xi^d f \mid f \in R_d, d \geq 0)$. Therefore,

$$Y \cap (1 \times \xi) \cdot Y = Y \cap V(f \otimes 1,\ 1 \otimes f \ \mid \ f \in R_d,\ d \geq 0 \text{ with } \xi^d \neq 1).$$

In particular, we have

(1.2) $Y \cap (1 \times \xi) \cdot Y \subseteq V(R_d \otimes 1 + 1 \otimes R_d)$ for all $d \geq 0$ s.t. $(d, k) = 1$.

For $d \gg 0$, the line bundle $\mathscr{L}^{\otimes d}$ is globally generated, so the vanishing locus of $R_d = H^0(X, \mathscr{L}^{\otimes d})$ inside $\operatorname{Proj} R \cong X$ is empty. This means that for $d \gg 0$, we have $\sqrt{(R_d)} \supseteq R_{\geq 1}$, which implies

$$\sqrt{(R_d \otimes 1 + 1 \otimes R_d)} \supseteq (R_{\geq 1} \otimes 1 + 1 \otimes R_{\geq 1}) = (R \otimes_{\mathbb{C}} R)_{\geq 1}.$$

Then (1.2) shows $Y \cap (1 \times \xi) \cdot Y = \emptyset$, proving the claim. □

We use Theorem 1.3.6 to give an example indicating that we cannot do better than what we ask for in Question 1.1, even as the dimension of the varieties increase:

Example 1.3.7. Let $(X, \mathscr{L})$ be the polarized normal toric variety of dimension $n \geq 3$ corresponding to the full-dimensional lattice polytope

$$P := \operatorname{conv}(0, e_1, \ldots, e_{n-1}, e_1 + e_2 + \ldots + e_{n-1} + ne_n) \subseteq \mathbb{R}^n.$$

Then $\operatorname{Pic}(X) \cong \mathbb{Z}$ is generated by $\mathscr{L}$, and the complete linear system $|\mathscr{L}|$ determines a non-injective finite morphism $\varphi_{\mathscr{L}} \colon X \to \mathbb{P}^n$. In particular, $\gamma(X, \mathscr{L}^{\otimes k}) = \infty$ for all $k \leq 1$. For $k \geq 2$, Theorem 1.3.6 shows that $\gamma(X, \mathscr{L}^{\otimes k}) \geq 2 \dim X$. In particular, $\gamma(X) \geq 2 \dim X$, i.e., X cannot be injected to $\mathbb{P}^s$ for $s < 2 \dim X$. For smooth examples with the same property see Example 1.3.13.

In the case of normal varieties, Theorem 1.3.6 can be sharpened for singular situations, replacing the assumption on divisibility in the Picard group by divisibility in the class group. Then we obtain the following bound:

Theorem 1.3.8. *Let X be a normal projective variety, let D be a Weil divisor on X, let $k \geq 2$ and assume that kD is Cartier. Then*

$$\gamma(X, \mathcal{O}_X(kD)) \geq 2 \dim X - \delta,$$

where

$$\delta := \min\{1 + \dim \Big(\bigcap_{q \nmid m} \operatorname{Bs} |mD| \Big) \mid q \text{ prime power dividing } k\},$$

using the convention $\dim \emptyset = -1$.

PROOF. We proceed as in the proof of Theorem 1.3.6, but replace the graded $\mathbb{C}$-algebra $\bigoplus_{d \geq 0} H^0(X, \mathscr{L}^{\otimes d})$ by $R := \bigoplus_{d \geq 0} H^0(X, \mathcal{O}_X(dD))$. As before, we only need to consider the case that X is irreducible and that $\gamma(X, \mathcal{O}_X(kD)) < \infty$ (in particular, D is ample).

Consider a non-zero subspace $V \subseteq H^0(X, \mathcal{O}_X(kD)) = R_k$ inducing an injective morphism $X \xrightarrow{\text{inj}} \mathbb{P}(V^*)$ with $\dim \mathbb{P}(V^*) \leq 2 \dim X$. With the same notations as in the previous proof, this injection gives rise

to a morphism $\psi\colon \operatorname{Proj} R \otimes_{\mathbb{C}} R \to \mathbb{P}(V^* \times V^*)$ and a linear subspace $L \subseteq \mathbb{P}(V^* \times V^*)$ of dimension $\dim \mathbb{P}(V^*)$ such that

$$(\psi^{-1} L)^{\mathrm{red}} = \bigcup_{\xi \in \mathbb{Z}_k} (1 \times \xi) \cdot Y,$$

where $Y := V(f \otimes 1 - 1 \otimes f \mid f \in R) \subseteq \operatorname{Proj} R \otimes_{\mathbb{C}} R$.

In this setting, it remains no longer true that $(1 \times \xi) \cdot Y$ and $(1 \times \xi') \cdot Y$ are disjoint for $\xi \neq \xi' \in \mathbb{Z}_k$. Instead, we show that

$$\dim \Big((1 \times \xi) \cdot Y \cap (1 \times \xi') \cdot Y \Big) = d_{\operatorname{ord}(\xi \xi'^{-1})}, \tag{1.3}$$

where for each $r \in \mathbb{N}$, we denote by d_r the dimension of the closed set $B_r := \bigcap_{r \nmid m} \operatorname{Bs} |mD| \subseteq X$. Denoting

$$\delta := \min\{1 + d_q \mid q \text{ prime power dividing } k\},$$

we then show that (1.3) implies that

$$\bigcup_{\xi \in \mathbb{Z}_k} (1 \times \xi) \cdot Y \text{ is not connected in dimension } \delta. \tag{1.4}$$

But on the other hand, it follows from classical connectedness theorems, in particular [FOV99, Lemma 3.2.2], that the preimage of the linear subspace L under the finite morphism $\psi\colon \operatorname{Proj} R \otimes_{\mathbb{C}} R \to \mathbb{P}(V^* \times V^*)$ is connected in dimension

$$\dim \operatorname{Proj} R \otimes_{\mathbb{C}} R - \dim L - 2 = 2 \dim X - \dim \mathbb{P}(V^*) - 1,$$

so we deduce that $\dim \mathbb{P}(V^*) \geq 2 \dim X - \delta$.

It remains to prove (1.3) and (1.4). For (1.3), we may restrict to the case $\xi' = 1$ and we denote $r := \operatorname{ord}(\xi)$. Since $(1 \times \xi) \cdot Y$ is the vanishing set of all $f \otimes 1 - 1 \otimes \xi^m f$ for $f \in R_m, m \geq 0$, we have

$$Y \cap (1 \times \xi) \cdot Y = Y \cap V(R_m \otimes_{\mathbb{C}} 1 + 1 \otimes_{\mathbb{C}} R_m \mid m \geq 0 \text{ with } \operatorname{ord}(\xi) \nmid m).$$

Consider the commutative diagram

$$\begin{tikzcd}
 & \operatorname{Spec} R \otimes_{\mathbb{C}} R \arrow[dl, dashed] \arrow[d, "\hat{\psi}"] \arrow[dr, dashed] & \\
\operatorname{Proj} R \times \operatorname{Proj} R \arrow[d, "\varphi_V \times \varphi_V"'] & V^* \times V^* \arrow[dl, dashed] \arrow[dr, dashed] & \operatorname{Proj} R \otimes_{\mathbb{C}} R \arrow[d, "\psi"] \\
\mathbb{P}(V^*) \times \mathbb{P}(V^*) & & \mathbb{P}(V^* \times V^*).
\end{tikzcd}$$

Note that in $\operatorname{Proj} R \cong X$, we have $V(R_m) = \operatorname{Bs} |mD|$. Hence, the affine cone over $Y \cap (1 \times \xi) \cdot Y$ in $\operatorname{Spec} R \otimes_{\mathbb{C}} R$ is the diagonal $\Delta_{\hat{B}_r \times \hat{B}_r} \subseteq \hat{B}_r \times \hat{B}_r$, where $\hat{B}_r \subseteq \operatorname{Spec} R$ is the affine cone over $B_r \subseteq \operatorname{Proj} R$. In particular, $\dim Y \cap (1 \times \xi) \cdot Y = d_r$, proving (1.3).

In order to show (1.4), let $q = p^\ell$ be a prime power dividing k. Let ζ be a primitive k-th root of unity. For $i \in \{1, \ldots, p\}$ consider the set

$Z_i := \bigcup_{j=1}^{k/p}(1 \times \zeta^{jp+i}) \cdot Y \subseteq \operatorname{Proj} R \otimes_{\mathbb{C}} R$. Then we have the following equalities of sets:

$$\bigcup_{\xi \in \mathbb{Z}_k} (1 \times \xi) \cdot Y = \bigcup_{i=1}^{p} Z_i \qquad \text{and}$$

$$Z_{i_1} \cap Z_{i_2} = \bigcup_{j_1=0}^{k/p} \bigcup_{j_2=0}^{k/p} \Big((1 \times \zeta^{j_1 p + i_1}) \cdot Y \cap (1 \times \zeta^{j_2 p + i_2}) \cdot Y \Big).$$

Note that for $i_1 \neq i_2 \in \{1, \dots, p\}$, the order of $\zeta^{(j_1 - j_2)p + (i_1 - i_2)}$ is divisible by q. Since $d_{rq} \leq d_q$ for all $r \geq 1$, we conclude from (1.3) that $\dim Z_{i_1} \cap Z_{i_2} \leq d_q$ for all $i_1 \neq i_2$. Hence, $\bigcup_{\xi \in \mathbb{Z}_k} (1 \times \xi) \cdot Y$ is not connected in dimension $d_q + 1$. This establishes (1.4) and concludes the proof. □

Our situation in the proof of Theorem 1.3.8 is very close to the setting in [Rei18], where small separating sets of invariants for finite group actions on affine varieties are investigated. In fact, we remark that Theorem 1.3.8 could also be obtained as a consequence of [Rei18, Theorem 4.5], if we additionally assumed that the section ring $R = \bigoplus_{d \geq 0} H^0(X, \mathcal{O}_X(dD))$ is integrally closed.

As an example, we apply Theorem 1.3.8 to weighted projective spaces. With the link between injection dimensions of weighted projective spaces and separating invariants of finite cyclic groups (see Example 1.2.8), the following bound on injection dimensions could also be proved on the basis of [DJ15, Theorem 3.4]. For the purpose of illustration, we choose to base our proof on Theorem 1.3.8.

Corollary 1.3.9. *Consider a weighted projective space $\mathbb{P}(q_0, \dots, q_n)$ that is well-formed, i.e.,* $\gcd(q_0, \dots, \widehat{q_i}, \dots, q_n) = 1$ *for all i. Let $\ell \geq 2$ be minimal such that*

$$\operatorname{lcm}(q_{i_1}, \dots, q_{i_\ell}) = \operatorname{lcm}(q_0, \dots, q_n) \qquad \textit{holds for all } i_1, \dots, i_\ell \textit{ distinct.}$$

Let $\mathscr{L}$ be the ample line bundle generating the Picard group. Then

$$\gamma(\mathbb{P}(q_0, \dots, q_n), \mathscr{L}^{\otimes k}) \geq \begin{cases} n + \ell - 2 & \textit{if } k = 1, \\ 2n & \textit{if } k \geq 2. \end{cases}$$

Proof. The weighted projective space $X := \mathbb{P}(q_0, \dots, q_n)$ is $\operatorname{Proj} R$, where $R = \mathbb{C}[x_0, \dots, x_n]$ is graded via $\deg(x_i) = q_i$. The twisting sheaf $\mathcal{O}_{\operatorname{Proj} R}(1)$ is the reflexive sheaf $\mathcal{O}_X(D)$ on X corresponding to a Weil divisor D generating the class group. For all $m \in \mathbb{Z}$, we have $H^0(X, \mathcal{O}_X(mD)) = R_m$. The ample line bundle generating $\operatorname{Pic}(X)$ is $\mathscr{L} = \mathcal{O}_{\operatorname{Proj} R}(a) = \mathcal{O}_X(aD)$, where $a := \operatorname{lcm}\{q_0, \dots, q_n\}$.

Theorem 1.3.6 shows that $\gamma(X, \mathscr{L}^{\otimes k}) \geq 2n$ for all $k \geq 2$. We are therefore only concerned with establishing a lower bound for $\gamma(X, \mathscr{L})$ based on Theorem 1.3.8. We may assume that not all weights q_i are equal to 1 (otherwise, $X = \mathbb{P}^n$ and the claim $\gamma(\mathbb{P}^n, \mathcal{O}(1)) \geq n$ is trivial).

Then $a \geq 2$ and, by minimality of ℓ, there exists a prime power p^r dividing a and weights $q_{i_1}, \dots, q_{i_{\ell-1}}$ not divisible by p^r. Note that

$$\bigcap_{p^r \nmid m} \operatorname{Bs} |mD| = V(R_m \mid p^r \nmid m) \subseteq V(x_{i_1}, \dots, x_{i_{\ell-1}}).$$

We conclude with Theorem 1.3.8 that

$$\gamma(X, \mathcal{O}_X(aD)) \geq 2n - 1 - \dim \Big(\bigcap_{p^r \nmid m} \operatorname{Bs} |mD| \Big) \geq n + \ell - 2. \qquad \square$$

Example 1.3.10. The weighted projective space $\mathbb{P}(1, 6, 10, 15)$ injects to $\mathbb{P}^4$ (see Example 1.1.5 and Theorem 1.4.5), but not by linear projections from any embedding. This follows from Theorem 1.3.6 because the ample line bundle $\mathcal{O}(30)$ generating the Picard group is not very ample (see Example 1.1.5), whence all very ample line bundles admit a root of some order in $\operatorname{Pic}(X) \cong \mathbb{Z}$.

The case $\ell = 2$ in Corollary 1.3.9 is the case of (non-weighted) projective spaces, for which explicit constructions in the context of separating invariants [Duf08, Proposition 5.2.2] show that the above bound is sharp. In the next case, $\ell = 3$, we show in Theorem 1.4.5 that the bound in Corollary 1.3.9 remains sharp when one of the weights is 1. However, the next example shows that the latter assumption cannot be dropped.

Example 1.3.11. Corollary 1.3.9 shows that $\gamma(\mathbb{P}(2, 2, 3, 3), \mathcal{O}(6)) \geq 4$. We show that actually $\gamma(\mathbb{P}(2, 2, 3, 3), \mathcal{O}(6)) = 5$ holds. Indeed, note that the global sections of $\mathcal{O}(6)$ are just $\operatorname{Sym}^3(\mathbb{C}^2)^* \oplus \operatorname{Sym}^2(\mathbb{C}^2)^*$ and the secant locus of $Y := \operatorname{im}(\varphi_{\mathcal{O}(6)}) \subseteq \mathbb{P}(\operatorname{Sym}^3 \mathbb{C}^2 \oplus \operatorname{Sym}^2 \mathbb{C}^2)$ is

$$\begin{aligned} \sigma_2^\circ(Y) = \{[v \oplus w] \mid (v = 0 \text{ or } [v] \in \sigma_2^\circ(C_3)) \text{ and} \\ (w = 0 \text{ or } [w] \in \sigma_2^\circ(C_2))\}, \end{aligned}$$

where $C_d \subseteq \mathbb{P} \operatorname{Sym}^d \mathbb{C}^2$ for $d \in \{2, 3\}$ denotes the d-th rational normal curve. The secant locus of the plane conic C_2 fills its ambient space, while the secant locus of the twisted cubic curve C_3 consists of all points that cannot be written as $[v_1 v_2^2] \in \mathbb{P}(\operatorname{Sym}^3 \mathbb{C}^2)$ with $\{v_1, v_2\}$ a basis of $\mathbb{C}^2$. Hence, a linear subspace $L \subseteq \mathbb{P}(\operatorname{Sym}^3 \mathbb{C}^2 \oplus \operatorname{Sym}^2 \mathbb{C}^2)$ does not meet $\sigma_2^\circ(Y)$ if and only if it does not meet the center of the projection $\pi_1 \colon \mathbb{P}(\operatorname{Sym}^3 \mathbb{C}^2 \oplus \operatorname{Sym}^2 \mathbb{C}^2) \dashrightarrow \mathbb{P}(\operatorname{Sym}^3 \mathbb{C}^2)$ and $\pi_1(L) \cap \sigma_2^\circ(C_3) = \emptyset$. Every line in $\mathbb{P}(\operatorname{Sym}^3 \mathbb{C}^2)$ intersects $\sigma_2^\circ(C_3)$ by Proposition 1.1.14 and Remark 1.1.3, so $\pi_1(L)$ and hence L needs to be a point. In fact, L can be chosen to be the point $[v_1 v_2^2 \oplus 0] \notin \sigma_2^\circ(Y)$ (for some basis $\{v_1, v_2\}$ of $\mathbb{C}^2$). By Proposition 1.1.14, this shows $\gamma(\mathbb{P}(2, 2, 3, 3), \mathcal{O}(6)) = 5$.

Example 1.3.11 generalizes easily to show that an n-dimensional weighed projective space of the form $\mathbb{P}(2, \dots, 2, d, d)$ with $n, d \geq 3$ and d odd cannot be injected to $\mathbb{P}^{2n-2}$, while the bound from Corollary 1.3.9 only established that injections to $\mathbb{P}^{2n-3}$ are impossible. In fact, we always expect the following:

Conjecture 1.3.12. *Let $d, e \geq 2$ be relatively prime and let $r, s \geq 2$. Then*

$$\gamma(\mathbb{P}(\underbrace{d, \ldots, d}_{r}, \underbrace{e, \ldots, e}_{s})) = 2(r + s - 1) - 1.$$

From $\gamma(\mathbb{P}^{r-1}, \mathcal{O}(e)) = 2(r-1)$ and $\gamma(\mathbb{P}^{s-1}, \mathcal{O}(d)) = 2(s-1)$, see [Duf08, Proposition 5.2.2], the existence of an injection

$$\mathbb{P}(d, \ldots, d, e, \ldots, e) \xrightarrow{\text{inj}} \mathbb{P}^{2(r+s-1)-1}$$

is clear. The result in question in Conjecture 1.3.12 is therefore the lower bound improving the one obtained from Corollary 1.3.9.

We finish this section by applying our lower bounds on injection dimensions also to a non-toric example.

Example 1.3.13. Let $n \geq 3$ and let $q_0, \ldots, q_{n+1} \geq 2$ be pairwise relatively prime. Let $X \subseteq \mathbb{P}(q_0, \ldots, q_{n+1})$ be a general hypersurface of weighted degree $d := q_0 q_1 \cdots q_{n+1}$. Then X is an n-dimensional connected smooth projective variety with $\gamma(X) \geq 2n$. Indeed, [RS06, Theorem 1] implies that the restriction homomorphism of class groups $\mathrm{Cl}(\mathbb{P}(q_0, \ldots, q_n)) \to \mathrm{Cl}(X)$ is an isomorphism. Since $\mathbb{P}(q_0, \ldots, q_n)$ has only finitely many singular points, X is smooth, so we conclude that $\mathrm{Pic}(X) = \mathrm{Cl}(X)$ is generated by the restriction of the reflexive sheaf $\mathcal{O}(1)$ on $\mathbb{P}(q_0, \ldots, q_{n+1})$ to X. On the other hand, by generality of X, the homomorphism $H^0(\mathbb{P}(q_0, \ldots, q_n), \mathcal{O}(1)) \to H^0(X, \mathcal{O}(1)|_X)$ is surjective, so the line bundle $\mathcal{O}(1)|_X$ has no global sections. In particular, it cannot give rise to injective morphisms. Every other line bundle is a power of $\mathcal{O}(1)|_X$, so Theorem 1.3.6 implies that $\gamma(X) \geq 2 \dim X$.

1.4. Explicit constructions of injections

In this section, we give three specific approaches for explicitly constructing injective morphisms $X \to \mathbb{P}^{2 \dim X}$ with a focus on products of projective spaces and weighted projective spaces.

1.4.1. Constructions from tangential varieties

The following approach is helpful for producing injections of the r-fold product $\mathbb{P}^m \times \ldots \times \mathbb{P}^m$ for $m \geq 1$: Consider $d = (d_1, \ldots, d_r) \in \mathbb{Z}^r_{>0}$ and let $D := \sum_{i=1}^r d_i$. The image of the morphism

$$\psi_{m,d} \colon \mathbb{P}(\mathbb{C}^{m+1}) \times \ldots \times \mathbb{P}(\mathbb{C}^{m+1}) \to \mathbb{P}(\mathrm{Sym}^D \mathbb{C}^{m+1}),$$
$$[v_1] \times \ldots \times [v_r] \mapsto [v_1^{d_1} \cdots v_r^{d_r}]$$

is the **Chow–Veronese variety** of type (m, d). In the case $m = 1$, this is also called the **coincident root locus** of type d, and its understanding is of major interest from the viewpoint of practical applications, see for example [LS16]. We observe that injections of these varieties for suitable d give rise to injections of products of projective spaces:

Lemma 1.4.1. *Let $r, m \in \mathbb{Z}_{>0}$ and let $d \in \mathbb{Z}^r_{>0}$ be such that*

$$\sum_{i \in I} d_i \neq \sum_{j \in J} d_j \qquad \text{for all } I, J \subseteq [r] \text{ with } I \cap J = \emptyset. \tag{1.5}$$

Then $\psi_{m,d}$ is injective. In particular, if $Y \subseteq \mathbb{P}(\operatorname{Sym}^D \mathbb{C}^{m+1})$ denotes the Chow–Veronese variety of type (m, d), then

$$\gamma(\mathbb{P}^m \times \ldots \times \mathbb{P}^m, \mathcal{O}(kd)) \leq \gamma(Y, \mathcal{O}_Y(k)) \qquad \text{for all } k > 0.$$

PROOF. Let $p \in Y$, then $p = [z]$ for $z \in \operatorname{Sym}^D \mathbb{C}^{m+1} \setminus \{0\}$ of the form $z = v_1^{d_1} \cdots v_r^{d_r}$ for some $v_i \in \mathbb{C}^{m+1} \setminus \{0\}$. To show injectivity of $\psi_{m,d}$, we need to see that each v_i is uniquely determined up to scaling. But $\operatorname{Sym}^\bullet \mathbb{C}^{m+1} \cong \mathbb{C}[x_0, \ldots, x_m]$ is a unique factorization domain, so z uniquely determines the set $\{[v_1], \ldots, [v_r]\} \subseteq \mathbb{P}(\mathbb{C}^{m+1})$ as the linear factors of z up to scaling. Moreover, for each i, the non-zero factor $v_i \in \mathbb{C}^{m+1}$ appears in z with multiplicity $m_i := \sum_{j \in J_i} d_j$, where $J_i := \{j \mid [v_j] = [v_i]\}$. By assumption (1.5), the integer m_i uniquely determines the set J_i. Therefore, each v_i is uniquely determined up to scaling from z. Hence, $\psi_{m,d}$ is injective. The second claim follows from $\psi_{m,d}^*(\mathcal{O}_Y(1)) = \mathcal{O}(d)$. □

In the case $r = 2$, $d = (1, k-1)$ with $k \geq 3$, the Chow–Veronese variety of type (m, d) is the tangential variety of the k-th Veronese variety $\nu_k(\mathbb{P}^m) \subseteq \mathbb{P}(\operatorname{Sym}^k \mathbb{C}^{m+1})$. As a simple consequence of Lemma 1.4.1, we construct two explicit injections from the cases in which this tangential variety has a secant variety of small dimension (as classified in [CGG02] and [AV18]).

Proposition 1.4.2. *The following two morphisms are injective:*

$$\begin{aligned} \mathbb{P}^1 \times \mathbb{P}^1 &\to \mathbb{P}^3, \\ [x_0 : x_1] \times [y_0 : y_1] &\mapsto [x_0 y_0^2 : x_1 y_0^2 + 2x_0 y_0 y_1 : 2x_1 y_0 y_1 + x_0 y_1^2 : x_1 y_1^2], \end{aligned}$$

$$\begin{aligned} \mathbb{P}^2 \times \mathbb{P}^2 &\to \mathbb{P}^8, \\ [x_0 : x_1 : x_2] \times [y_0 : y_1 : y_2] &\mapsto [x_0 y_0^2 : x_0 y_1^2 + 2x_1 y_0 y_1 : x_1 y_0^2 - x_2 y_1^2 + 2x_0 y_0 y_1 - 2x_1 y_1 y_2 : \\ &\qquad x_1 y_1^2 : x_1 y_2^2 + 2x_2 y_1 y_2 : x_1 y_0^2 - x_0 y_2^2 + 2x_0 y_0 y_1 - 2x_2 y_0 y_2 : \\ &\qquad x_2 y_2^2 : x_2 y_0^2 + 2x_0 y_0 y_2 : 2x_0 y_1 y_2 + 2x_1 y_0 y_2 + 2x_2 y_0 y_1]. \end{aligned}$$

In particular, $\gamma(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,2)) = 3$ and $\gamma(\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}(1,2)) \leq 8$.

PROOF. By Lemma 1.4.1, the morphism

$$\psi_{1,(1,2)} \colon \mathbb{P}(\operatorname{Sym}^1 \mathbb{C}^2) \times \mathbb{P}(\operatorname{Sym}^1 \mathbb{C}^2) \to \mathbb{P}(\operatorname{Sym}^3 \mathbb{C}^2)$$

is injective and its image is the tangential surface of the twisted cubic curve in $\mathbb{P}(\operatorname{Sym}^3 \mathbb{C}^2)$. Picking a basis $\{T_0, T_1\}$ of $\mathbb{C}^2$, we consider the $\{T_0^3, T_0^2 T_1, T_0 T_1^2, T_1^3\}$ as a basis of $\operatorname{Sym}^3 \mathbb{C}^2$. In these bases, the morphism $\psi_{1,(1,2)} \colon \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\text{inj}} \mathbb{P}^3$ maps $[x_0 : x_1] \times [y_0 : y_1]$ to the point whose coordinates are the coefficients in T_0 and T_1 of the expression

$(x_0T_0+x_1T_1)(y_0T_0+y_1T_1)^2$. Explicitly, this is the injection written out above. Note that it is given by global sections of $\mathcal{O}(1,2)$.

Similarly, the morphism $\psi_{2,(1,2)}$ injectively maps to the tangential variety of the Veronese surface and it is known that the secant variety of this tangential fourfold is of dimension 8 only, see [CGG02, Proposition 3.2]. By Lemma 1.1.13, a projection of the image of $\psi_{2,(1,2)}$ from a general linear space of codimension 9 gives an injection $\mathbb{P}^2 \times \mathbb{P}^2 \xrightarrow{\text{inj}} \mathbb{P}^8$ given by global sections of $\mathcal{O}(1,2)$.

Explicitly, with respect to a basis $\{T_0, T_1, T_2\} \subseteq \mathbb{C}^3$ and the corresponding basis $\{T_iT_jT_k \mid i,j,k\} \subseteq \operatorname{Sym}^3 \mathbb{C}^3$, the injection $\psi_{2,(1,2)}$ is the morphism $\mathbb{P}^2 \times \mathbb{P}^2 \xrightarrow{\text{inj}} \mathbb{P}^9$ mapping $[x_0 : x_1 : x_2] \times [y_0 : y_1 : y_2]$ to the 10 coefficients in T_0, T_1, T_2 of the expression

$$(x_0T_0 + x_1T_1 + x_2T_2)(y_0T_0 + y_1T_1 + y_2T_2)^2.$$

The secant variety of its image in $\mathbb{P}(\operatorname{Sym}^3 \mathbb{C}^3)$ does not fill the entire 9-dimensional projective space. Explicitly, one checks that the point $p := [T_0^2T_1+T_1^2T_2+T_2^2T_1] \in \mathbb{P}(\operatorname{Sym}^3 \mathbb{C}^3)$ does not lie on a secant line. In particular, the composition of $\psi_{2,(1,2)}$ with the projection from p gives an injective morphism $\mathbb{P}^2 \times \mathbb{P}^2 \xrightarrow{\text{inj}} \mathbb{P}^8$. This is the morphism written out above. $\square$

Another application of Lemma 1.4.1 is the following explicit construction:

Proposition 1.4.3. *Let $d \geq 3$. Then the following morphism is injective:*

$$\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\text{inj}} \mathbb{P}^4,$$

$$[x_0 : x_1] \times [y_0 : y_1] \mapsto \left[x_0y_0^d : dx_0y_0^dy_1 + x_1y_0^d : \tbinom{d}{2}x_0y_0^{d-1}y_1^2 + dx_1y_0^dy_1 : x_0y_1^d + dx_1y_0y_1^{d-1} : x_1y_1^d\right].$$

In particular, $\gamma(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,d)) \leq 4$.

Proof. We use Lemma 1.4.1 and construct an injection of the tangential variety of the rational normal curve of degree $d+1$: The morphism $\psi_{1,(1,d)} \colon \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}(\operatorname{Sym}^{d+1} \mathbb{C}^2)$ is injective. Fixing a basis $\{T_0, T_1\} \subseteq \mathbb{C}^2$ and the corresponding basis $\{T_{i_0} \cdots T_{i_d}\} \subseteq \operatorname{Sym}^{d+1} \mathbb{C}^2$, it is explicitly given by mapping $[x_0 : x_1] \times [y_0 : y_1]$ to the $d+1$ coefficients in T_0, T_1 of the expression $(x_0T_0 + x_1T_1)^d(y_0T_0 + y_1T_1)$. The morphism written out above only maps to the point in $\mathbb{P}^4$ whose coordinates are the coefficients of the monomials T_0^{d+1}, $T_0^dT_1$, $T_0^{d-1}T_1^2$, $T_0T_1^d$ and T_1^{d+1}. This is the composition of the injection $\psi_{1,(1,d)}$ with the projection $\mathbb{P}(\operatorname{Sym}^{d+1} \mathbb{C}^2) \to \mathbb{P}(\operatorname{Sym}^{d+1} \mathbb{C}^2/W)$, where $W := \langle T_0^{d-2}T_1^3, \ldots, T_0^2T_1^{d-1}\rangle$.

By Lemma 1.1.13, we need to see that $\mathbb{P}(W)$ does not meet the secant locus of the tangential variety of the rational normal curve of degree $d+1$. We check this explicitly: Assume that $f^dg - f'^dg' \in W \setminus \{0\}$ for some $f, f', g, g' \in \mathbb{C}^2 \setminus \{0\}$. Since every element of W is divisible by $T_0^2T_1^2$, we note that f is proportional to T_0 (resp. T_1) if and only if f' is. But this cannot be the case, since no nonzero element of W is

divisible by T_0^d (resp. T_1^d). Similarly, g is proportional to T_0 (resp. T_1) if and only if g' is. Since $W_0 := \langle T_0^{d-2}T_1^2, \dots, T_0^2T_1^{d-2}, T_0T_1^{d-1}\rangle$ defines a linear space $\mathbb{P}(W_0) \subseteq \mathbb{P}(\operatorname{Sym}^d \mathbb{C}^2)$ not intersecting the secant locus of the rational normal curve $\nu_d(\mathbb{P}^1) \subseteq \mathbb{P}(\operatorname{Sym}^d \mathbb{C}^2)$, no nonzero element of $W \subseteq T_0W_0 \cap T_1W_0$ can be written as $T_0(f^d - f'^d)$ or $T_1(f^d - f'^d)$.

We may therefore now assume that f, f', g, g' are not proportional to T_0, T_1, so we can write (after rescaling):

$$f = T_0 - \alpha T_1, \qquad g = T_0 - \beta T_1, \qquad f' = T_0 - \alpha' T_1, \qquad g' = T_0 - \beta' T_1$$

for non-zero $\alpha, \alpha', \beta, \beta' \in \mathbb{C}$. Since $f^d g - f'^d g' \in W$, it has zero coefficients for T_0^{d+1}, $T_0^d T_1$, $T_0^{d-1}T_1^2$, $T_0T_1^d$ and T_1^{d+1}, i.e.,

$$d\alpha + \beta = d\alpha' + \beta', \qquad \tbinom{d}{2}\alpha^2 + d\alpha\beta = \tbinom{d}{2}\alpha'^2 + d\alpha\beta',$$
$$\alpha^d + d\alpha^{d-1}\beta = \alpha'^d + d\alpha'^{d-1}\beta', \qquad \alpha^d\beta = \alpha'^d\beta'.$$

We wish to conclude $\alpha = \alpha'$ and $\beta = \beta'$. As

$$d\alpha^2 + \beta^2 = (d\alpha + \beta)^2 - 2\Big(\tbinom{d}{2}\alpha^2 + d\alpha\beta\Big) \quad \text{and} \quad \frac{d}{\alpha} + \frac{1}{\beta} = \frac{\alpha^d + d\alpha^{d-1}\beta}{\alpha^d\beta},$$

this follows from the lemma that follows. □

Lemma 1.4.4. *For every $d \geq 2$, the morphism $(\mathbb{C}^*)^2 \to \mathbb{C}^3$ mapping $(x, y) \mapsto (x^{-1} + dy^{-1}, x + dy, x^2 + dy^2)$ is injective.*

PROOF. The fiber over a point $(a, b, c) \in \mathbb{C}^3$ is given by the vanishing of

$$f := x^{-1} + dy^{-1} - a, \qquad g := x + dy - b \qquad \text{and} \qquad h := x^2 + dy^2 - c.$$

The ideal $(f, g, h) \subseteq \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ contains

$$(d+1)xyf + a(x+y)g - ah = (d(d+1) - ab)x + (d + 1 - ab)y + ac.$$

If the fiber consisted of more than one point, then this linear polynomial needs to be proportional to the linear polynomial g. Then in particular $d + 1 - ab = d(d(d+1) - ab)$, i.e., $ab = (d+1)^2$. In this case,

$$axyf + (a^2y - ad)g = a^2dy^2 + (a - ad^2 - a^2b)y - abd = d(ay - d - 1)^2,$$

hence $ay - d - 1 = 0$ for all points in $(\mathbb{C}^*)^2$ on which f, g, h vanish. Then the unique point mapping to (a, b, c) is $(\frac{d+1}{a}, \frac{d+1}{a})$. □

1.4.2. Inductive constructions

In the following, we provide explicit injections of some weighted projective spaces matching the lower bounds on injection dimensions in Corollary 1.3.9. This generalizes the case of (non-weighted) projective spaces worked out in [Duf08, Proposition 5.2.2].

Theorem 1.4.5. *Consider a weighted projective space* $\mathbb{P}(q_0, q_1, \ldots, q_n)$ *with* $q_0 = 1$ *and such that* $\operatorname{lcm}(q_i, q_j, q_k) = d$ *for all* i, j, k *distinct, where* $d := \operatorname{lcm}(q_1, \ldots, q_n)$. *Let* $a_i := d/q_i$ *and* $b_{ij} := \operatorname{lcm}(q_i, q_j)/q_i$ *for all* $i, j \in \{0, 1, \ldots, n\}$. *Then the following are injective morphisms:*

$$\begin{aligned}
&\varphi_1 \colon \mathbb{P}(q_0, q_1, \ldots, q_n) \to \mathbb{P}^{n+1}, \\
&\qquad [x_0 : \ldots : x_n] \mapsto \Big[x_0^d : x_0^{d-q_1} x_1 : x_1^{a_1} + x_0^{d-q_2} x_2 : \\
&\qquad\qquad x_2^{a_2} + x_0^{d-q_3} x_3 : \ldots : x_{n-1}^{a_{n-1}} + x_0^{d-q_n} x_n : x_n^{a_n}\Big], \\
&\varphi_k \colon \mathbb{P}(q_0, q_1, \ldots, q_n) \to \mathbb{P}^{2n}, \\
{}^{(k\geq 2)} &\qquad [x_0 : \ldots : x_n] \mapsto \Big[\sum_{\substack{i+j=\ell \\ i\leq j}} x_i^{ka_i - b_{ij}} x_j^{b_{ji}} \mid \ell = 0, 1, \ldots, 2n\Big].
\end{aligned}$$

In particular, the injection dimensions for these weighted projective spaces are as follows:

$$\gamma(\mathbb{P}(q_0, q_1, \ldots, q_n), \mathcal{O}(kd)) = \begin{cases} \infty & \text{if } k \leq 0, \\ n & \text{if } k = 1;\ q_1 = \ldots = q_n, \\ n+1 & \text{if } k = 1;\ q_1, \ldots, q_n \text{ not all equal}, \\ 2n & \text{if } k \geq 2. \end{cases}$$

PROOF. Recall that for any $r_0, \ldots, r_m \in \mathbb{Z}_{>0}$ and $c \in \mathbb{Z}_{>0}$ with $(r_0, c) = 1$, there is an isomorphism

$$\begin{aligned}
&\mathbb{P}(r_0, cr_1, \ldots, cr_n) \cong \mathbb{P}(r_0, r_1, \ldots, r_m), \\
&[x_0 : x_1 : \ldots : x_m] \mapsto [x_0^c : x_1 : \ldots : x_m].
\end{aligned} \tag{1.6}$$

In particular, we may assume that $\gcd(q_1, \ldots, q_n) = 1$, noting that the descriptions of φ_1 and φ_k do not change under composition with the isomorphism (1.6).

The restriction of φ_1 to $V(x_0) \cong \mathbb{P}(q_1, \ldots, q_n)$ is given by

$$\varphi_1([0 : x_1 : \ldots : x_n]) = [0 : 0 : x_1^{a_1} : x_2^{a_2} : \ldots : x_n^{a_n}].$$

By assumption, $\operatorname{lcm}(q_i, q_j) = \operatorname{lcm}(q_0, q_i, q_j) = d$ for all positive $i \neq j$. In particular, if p^r is a prime power in d not dividing q_i for some $i > 0$, then p^r must divide all q_j for $j > 0$, $j \neq i$. Composing the corresponding isomorphisms (1.6), we see that $\mathbb{P}(q_1, \ldots, q_n) \to \mathbb{P}^n$, $[x_1 : \ldots : x_n] \mapsto [x_1^{a_1} : x_2^{a_2} : \ldots : x_n^{a_n}]$ is an isomorphism. In particular, $\varphi_1|_{V(x_0)}$ is injective.

The restriction of φ_1 to the affine open $D(x_0)$ can be checked to be injective by setting $x_0 = 1$. We have

$$\begin{aligned}
&\varphi_1|_{x_0=1} \colon \mathbb{A}^n \to \mathbb{A}^{n+1} \\
&(x_1, \ldots, x_n) \mapsto (x_1, x_2 + x_1^{a_1}, x_3 + x_2^{a_2}, \ldots, x_n + x_{n-1}^{a_{n-1}}, x_n^{a_n}),
\end{aligned}$$

which is a closed embedding, since it is of triangular shape. Note that for points in $\varphi(V(x_0))$ the first coordinate is zero, while for points in

$\varphi(D(x_0))$ it does not. Hence, the images of $\varphi|_{V(x_0)}$ and $\varphi|_{D(x_0)}$ do not intersect. We conclude that φ is injective.

For φ_k with $k \geq 2$, we also consider the affine open $D(x_0)$ by setting $x_0 = 1$, giving

$$\begin{aligned}
&\varphi_k|_{x_0=1} \colon \mathbb{A}^n \to \mathbb{A}^{2n} \\
&(x_1, \ldots, x_n) \mapsto (x_1, x_2 + x_1^{ka_1}, x_3 + x_1^{ka_1 - b_{12}} x_2^{b_{21}}, \\
&\qquad\qquad\qquad\qquad x_4 + x_1^{ka_1 - b_{13}} x_3^{b_{31}} + x_2^{ka_2}, \ \ldots),
\end{aligned}$$

which is of triangular shape and therefore a closed embedding. The first coordinate for points in $\varphi_k(V(x_0))$ vanishes, while this is not the case for points in $\varphi_k(D(x_0))$. Hence, with the above, is enough to show that the restriction of φ_k to $V(x_0)$ is injective. As above, using (1.6), we have the isomorphism $\varphi_1|_{V(x_0)} \colon V(x_0) \xrightarrow{\cong} \mathbb{P}^n$. Composing with this isomorphism, it only remains to show that

$$\psi \colon \mathbb{P}^{n-1} \to \mathbb{P}^{2(n-1)}, \quad [x_1 : \ldots : x_n] \mapsto \Big[\sum_{\substack{i+j=\ell \\ i \leq j}} x_i^{k-1} x_j \mid \ell = 2, 3 \ldots, 2n \Big]$$

is injective. This was originally proved in [Duf08, Proposition 5.2.2]. It follows by induction on n as follows: The case $n = 1$ is trivial. Let $n \geq 2$. The restriction of ψ to $V(x_1) \cong \mathbb{P}^{n-2}$ is injective by the induction hypothesis. The restriction of ψ to the affine open $D(x_1)$ can be checked to be injective by setting $x_1 = 1$. We have

$$\begin{aligned}
&\varphi|_{x_1=1} \colon \mathbb{A}^{n-1} \to \mathbb{A}^{2(n-1)} \\
&(x_2, \ldots, x_n) \mapsto (x_2, x_3 + x_2^k, x_4 + x_2^{k-1} x_3, x_5 + \ldots, \ldots),
\end{aligned}$$

which is a closed embedding. Since $\psi(V(x_1)) \cap \psi(D(x_1)) = \emptyset$, we conclude that ψ is injective. $\square$

We employ a similar technique to construct an injective morphism $\mathbb{P}^1 \times \mathbb{P}^n \xrightarrow{\text{inj}} \mathbb{P}^{2(n+1)}$ given by global sections of $\mathcal{O}(d, 1)$:

Theorem 1.4.6. *Let $n, d \geq 1$. The following is an injective morphism:*

$$\begin{aligned}
&\mathbb{P}^1 \times \mathbb{P}^n \xrightarrow{\text{inj}} \mathbb{P}^{2n+2}, \\
[x_0 : x_1] \times [y_0 : \ldots : y_n] \mapsto [&x_0^d y_0 : x_1^d y_0 + d x_0 x_1^{d-1} y_1 : \\
&x_0^d y_1 : x_1^d y_1 + d x_0 x_1^{d-1} y_2 : \ldots : \\
&x_0^d y_{n-1} : x_1^d y_{n-1} + d x_0 x_1^{d-1} y_n : \\
&x_0^d y_n : x_1^d y_n : d x_0 x_1^{d-1} y_0].
\end{aligned}$$

In particular, $\gamma(\mathbb{P}^1 \times \mathbb{P}^n, \mathcal{O}(d, 1)) \leq 2(n+1)$.

PROOF. The morphism $\varphi \colon \mathbb{P}^1 \times \mathbb{P}^n \xrightarrow{\text{inj}} \mathbb{P}^{2n+2}$ written out above maps $[x_0 : x_1] \times [y_0 : \ldots : y_n]$ to the point in $\mathbb{P}^{2n+2}$ whose coordinates are the coefficients of T^{d+1} and TS^d in the expressions

$$\{f_i := (x_0 T + x_1 S)^d (y_i T + y_{i+1} S) \mid i = -1, 0, 1, \ldots, n\},$$

where $y_{-1} := y_{n+1} := 0$. Restricting φ to $\{[0:1]\} \times \mathbb{P}^n$, we obtain the closed embedding

$$\begin{aligned} \{[0:1]\} \times \mathbb{P}^n &\hookrightarrow \mathbb{P}^{2n+2}, \\ [0:1] \times [y_0 : \ldots : y_n] &\mapsto [0 : y_0 : 0 : y_1 : \ldots : 0 : y_{n-1} : 0 : y_n : 0]. \end{aligned}$$

Note that the zero-pattern of the points in the image implies that the image of $[0:1] \times \mathbb{P}^n$ does not intersect $\varphi(D(x_0) \times \mathbb{P}^n)$. In particular, it is sufficient to show injectivity of the restriction of φ to $D(x_0) \times \mathbb{P}^n$, for which we may simply set $x_0 = 1$. The first of the expressions $f_i = (x_0T + x_1S)^d(y_iT + y_{i+1}S)$ with a non-zero coefficient of T^{d+1} uniquely determines the minimal k such that $y_k \neq 0$. Similarly, the last expression in which TS^d appears with non-zero coefficient determines the maximal m with $y_m \neq 0$. Hence, for a point in the image of

$$\begin{aligned} \varphi|_{x_0=1} \colon \mathbb{A}^1 \times \mathbb{P}^n &\to \mathbb{P}^{2n+2}, \\ [1 : x_1] \times [y_0 : \ldots : y_n] &\mapsto [y_0 : x_1^d y_0 + d x_1^{d-1} y_1 : y_1 : x_1^d y_1 + d x_1^{d-1} y_2 : \ldots : \\ &\qquad y_{n-1} : x_1^d y_{n-1} + d x_1^{d-1} y_n : y_n : x_1^d y_n : d x_1^{d-1} y_0] \end{aligned}$$

the values k and m can be read off the zero-pattern of its coordinates. Note also that $y_0, \ldots, y_n$ are determined by the even-indexed coordinates. Finally, x_1 can be reconstructed from the coordinates as

$$x_1 = d\,\frac{y_k}{y_m}\,\frac{x_1^d y_m}{d x_1^{d-1} y_k} = d\,\frac{x_0^d y_k}{x_0^d y_m}\,\frac{x_1^d y_m + d x_0 x_1^{d-1} y_{m+1}}{x_1^d y_{k-1} + d x_0 x_1^{d-1} y_k}.$$

This shows that any point in the image of $\varphi|_{x_0=1}$ determines a unique preimage, proving injectivity. □

Note that in the case $n = 1$, Theorem 1.4.6 gives another injection $\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\text{inj}} \mathbb{P}^4$ via global sections of $\mathcal{O}(d, 1)$, structurally different from the one constructed before in Proposition 1.4.3.

1.4.3. Graph-theoretic constructions

In this section, we give a combinatorial construction of an injection $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^m \hookrightarrow \mathbb{P}^{2m+4}$ by multilinear forms, showing that

$$\gamma(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^m, \mathcal{O}(1,1,1)) \leq 2m + 4.$$

The complete linear system $|\mathcal{O}(1,1,1)|$ embeds $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^m$ by the Segre embedding

$$\begin{aligned} \mathbb{P}(\mathbb{C}^2) \times \mathbb{P}(\mathbb{C}^2) \times \mathbb{P}(\mathbb{C}^{m+1}) &\hookrightarrow \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^{m+1}), \\ [u] \times [v] \times [w] &\mapsto [u \otimes v \otimes w]. \end{aligned}$$

We denote its image by $Y \subseteq \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^{m+1})$. For any $k \geq 1$, we denote the standard basis vectors of $\mathbb{C}^{k+1}$ by $e_0, e_1, \ldots, e_k$, and we write $e_0^*, \ldots, e_k^* \in (\mathbb{C}^{k+1})^*$ for the dual basis.

Contrary to the approach in Section 1.4.1 and Section 1.4.2, here, we do not construct the injection by writing out explicit polynomials defining the morphism and proving injectivity by exploiting their structure. Instead, we explicitly describe a linear subspace $L \subseteq \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^{m+1})$ of codimension $2m+3$ not intersecting the secant locus $\sigma_2^\circ(Y)$. Then, by Proposition 1.1.14, the projection of Y from L gives an injection to $\mathbb{P}^{2m+4}$.

Necessarily, a linear space of codimension $2m+3$ must intersect the $(2m+3)$-dimensional secant variety $\sigma_2(Y)$, and we need to ensure that this intersection does not lie in the secant locus $\sigma_2^\circ(Y)$. In fact, we construct a linear subspace whose intersection with the secant variety is of much higher than expected dimension:

Theorem 1.4.7. *Consider the Segre variety $Y \subseteq \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^{m+1})$. There exists a linear subspace $L \subseteq \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^{m+1})$ of codimension $2m+3$ not meeting $\sigma_2^\circ(Y)$ such that*

$$L \cap \sigma_2(Y) = L_1 \sqcup L_2,$$

where $L_1, L_2 \subseteq \sigma_2(Y) \setminus \sigma_2^\circ(Y)$ are disjoint linear spaces spanning L.

We recall that Y consists of the points corresponding to rank 1 tensors in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^{m+1}$ and $\sigma_2^\circ(Y)$ is the set of rank ≤ 2 tensors. The closure of the latter is the secant variety $\sigma_2(Y)$ parameterizing tensors of *border rank* at most 2. Therefore, reformulated in the theory of tensors, Theorem 1.4.7 states the existence of two disjoint large-dimensional linear subspaces consisting of border rank 2 tensors only, whose common span does not contain any rank 2 tensor.

In order to constructively prove Theorem 1.4.7, we first introduce combinatorial objects encoding in a useful way the tensor subspaces which we will consider.

Throughout, we fix a positive integer m. Let Γ be the directed graph with vertex set $\{0, 1, \ldots, m\}$ and edges $E := E_1 \sqcup E_2$, where

$$\begin{aligned} E_1 &:= \{(0,1), (1,2), (2,3), \ldots, (m-1, m)\} \qquad \text{and} \\ E_2 &:= \{(i, m-i) \mid 0 \leq i < \lfloor m/2 \rfloor\} \\ &\quad \cup \{(m-i, i+1) \mid 0 \leq i < \lfloor (m-1)/2 \rfloor\} \\ &= \{(0,m), (m,1), (1, m-1), (m-1, 2), \ldots\}. \end{aligned}$$

See Figure 1.1 for an illustration, where the edges of in E_1 are marked in red, the edges of E_2 in blue. Note that E_1 and E_2 each form a directed path in Γ.

Consider the vector space $\mathbb{C}^E = \{w \colon E \to \mathbb{C}\}$ of complex weight functions on the edges of Γ. For each vertex $k \in \{0, 1, \ldots, m\}$, consider

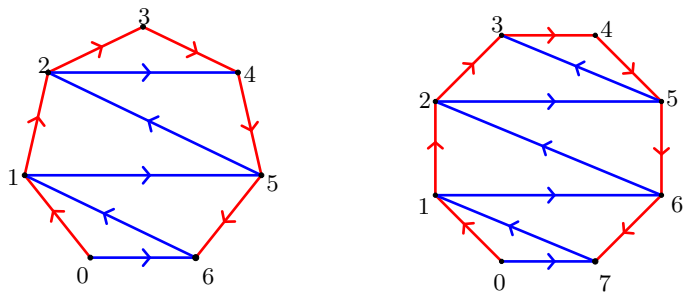


FIGURE 1.1. The graph Γ for $m = 6$ and $m = 7$.

the linear map

$$\Psi_k \colon \mathbb{C}^E \to \mathbb{C}^4,$$
$$w \mapsto \Big(\sum_{(i,k)\in E_1} w(i,k), \sum_{(k,i)\in E_1} w(k,i), \sum_{(i,k)\in E_2} w(i,k), \sum_{(k,i)\in E_2} w(k,i) \Big),$$

extracting from a weight function the total weights of incoming and outgoing edges at vertex k from E_1 and E_2, respectively. For every $w \in \mathbb{C}^E$, let $Z_w \subseteq \mathbb{C}^4$ denote the vector space spanned by $\Psi_0(w), \dots, \Psi_m(w)$.

We now define the linear space L which we will check to satisfy the properties of Theorem 1.4.7: Let $W_1 \subseteq \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^{m+1}$ be the m-dimensional vector space with the basis

$$u_{i,j} := e_0 \otimes e_0 \otimes e_j + e_0 \otimes e_1 \otimes e_i + e_1 \otimes e_0 \otimes e_i \quad \text{for all } (i,j) \in E_1$$

Similarly, let $W_2 \subseteq \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^{m+1}$ be the $(m-1)$-dimensional vector space with the basis

$$v_{i,j} := e_1 \otimes e_0 \otimes e_j + e_1 \otimes e_1 \otimes e_i + e_0 \otimes e_0 \otimes e_i \quad \text{for all } (i,j) \in E_2$$

Define $W := W_1 + W_2$, which is a vector space of dimension $2m-1$. Denote by $L_1, L_2, L \subseteq \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^{m+1})$ the corresponding linear spaces $L_i := \mathbb{P}(W_i)$ and $L := \mathbb{P}(W)$.

We may identify elements of W with elements of $\mathbb{C}^E$ under the linear isomorphism

$$\Phi \colon \mathbb{C}^E \xrightarrow{\cong} W, \qquad w \mapsto \sum_{(i,j)\in E_1} w(i,j)\, u_{i,j} + \sum_{(i,j)\in E_2} w(i,j)\, v_{i,j}.$$

This allows for a combinatorial reformulation of the condition that a tensor in W is of border rank ≤ 2:

Lemma 1.4.8. *A tensor $t \in W \subseteq \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^{m+1}$ is of border rank 2 if and only if $w := \Phi^{-1}(t) \in \mathbb{C}^E$ satisfies $\dim Z_w \leq 2$.*

PROOF. By [LM04, Theorem 5.1], a tensor $t \in W \subseteq \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^{m+1}$ is of border rank ≤ 2 if and only if the induced linear map

$$\varphi_t \colon (\mathbb{C}^{m+1})^* \to \mathbb{C}^2 \otimes \mathbb{C}^2, \qquad \ell \mapsto (\mathrm{id} \otimes \mathrm{id} \otimes \ell)(t)$$

has image $\mathrm{im}(\varphi_t) \subseteq \mathbb{C}^2 \otimes \mathbb{C}^2$ of dimension at most 2. Let $w := \Phi^{-1}(t)$. Composing φ_t with the isomorphism

$$(e_0^* \otimes e_0^* - e_1^* \otimes e_1^*,\ e_0^* \otimes e_1^*,\ e_1^* \otimes e_0^* - e_0^* \otimes e_1^*,\ e_1^* \otimes e_1^*) \colon \mathbb{C}^2 \otimes \mathbb{C}^2 \xrightarrow{\cong} \mathbb{C}^4$$

gives the linear map $(\mathbb{C}^{m+1})^* \to \mathbb{C}^4$, $e_k^* \mapsto \Psi_k(w)$, whose image is precisely Z_w. □

Based on Lemma 1.4.8, our proof of Theorem 1.4.7 will become very combinatorial. The main graph-theoretic observations are the content of the next two Lemmas:

Lemma 1.4.9. *Let $w \in \mathbb{C}^E$. If $Z_w \subseteq \mathbb{C}^4$ lies in one of the coordinate hyperplanes, then $\Phi(w) \in W_1$ or $\Phi(w) \in W_2$.*

PROOF. If $Z_w \subseteq V(e_0^*)$, then the first coordinate of $\Psi_k(w)$ is zero for all $k \in \{0, 1, \ldots, m\}$. Since every vertex of Γ has at most one incoming edge from E_1 and since every edge from E_1 is an incoming edge at some vertex, this shows that $w(i,j) = 0$ for all $(i,j) \in E_1$. Hence, $\Phi(w) \in W_2$. The other three cases $Z_w \subseteq V(e_i^*)$ for $i \in \{1,2,3\}$ follow from the same argument. □

Lemma 1.4.10. *Let $w \in \mathbb{C}^E$. If $\dim Z_w \leq 2$, then $Z_w \subseteq \mathbb{C}^4$ lies in one of the coordinate hyperplanes.*

PROOF. By induction on m. Assume that Z_w is not contained in any coordinate hyperplane. Consider the special vertex $k := \lceil m/2 \rceil$ in Γ, which has no adjacent edges in E_2.

First, we show that $\Psi_k(w) = 0$: Along the directed path of edges from E_2, let (i,j) is the first edge with non-zero weight. Then the two vertices i and k have both no weight on incoming edges from E_2, hence $\Psi_i(w)$ and $\Psi_k(w)$ lie in the coordinate hyperplane $V(e_2^*) \subseteq \mathbb{C}^4$. Since $\dim Z_w \leq 2$ and $Z_w \not\subseteq V(e_2^*)$, the vectors $\Psi_i(w)$ and $\Psi_k(w)$ must be proportional. But $\Psi_i(w) \notin V(e_3^*)$, while $\Psi_k(w) \in V(e_3^*)$, so we conclude that $\Psi_k(w) = 0$.

Secondly, we may assume that the edge in E_2 between the vertices $k-1$ and $k+1$ has non-zero weight: Otherwise, we may delete this edge as well as the the vertex k and its incident edges to obtain a weighted graph which can be viewed as a subgraph of the graph Γ for the case $m-1$. This case is covered by the induction hypothesis.

In particular, $\Psi_{k-1}(w) \neq 0$ and $\Psi_{k+1}(w) \neq 0$. One of the vertices $k-1$ and $k+1$ has no outgoing edge in E_1, while the other has an outgoing edge in E_2 with non-zero weight. Hence, $\Psi_{k-1}(w)$ and $\Psi_{k+1}(w)$ are not proportional. Because of $\dim Z_w \leq 2$, they must form a basis of Z_w.

We now need to distinguish between even and odd m.

Case 1: m is even. Then the last edge of the directed path formed by E_2-edges is $(k-1, k+1) \in E_2$. The vertex 0 has no incoming edge in E_1, so $\Psi_0(w) \in V(e_0^*)$. Since also $\Psi_{k+1}(w) \in V(e_0^*)$ and $Z_w \not\subseteq V(e_0^*)$

and $\dim Z_w = 2$, the vector $\Psi_0(w)$ must be a multiple of $\Psi_{k+1}(w)$. But the vertex 0 has no incoming edge in E_2, while the vertex $k+1$ has an incoming edge in E_2 with non-zero weight, so in fact, we must have $\Psi_0(w) = 0$.

If also $\Psi_m(w) = 0$, then we may delete the vertices 0 and m and their incident edges to obtain the graph Γ for the case of replacing m by $m-2$, so this case is already covered by the induction hypothesis. So, we may assume $\Psi_m(w) \neq 0$.

Since the vertex m has no outgoing edge in E_1, the vectors $\Psi_m(w)$ and $\Psi_{k-1}(w)$ are both non-zero vectors in $V(e_1^*)$, so they must be proportional by $\dim Z_w \leq 2$ and $Z_w \not\subseteq V(e_1^*)$. Since $\Psi_0(w) = 0$, we have $\Psi_m(w) \in V(e_2^*)$, so we must also have $\Psi_{k-1}(w) \in V(e_2^*)$. This means that the edge $(k+2, k-1) \in E_2$ has weight zero.

This in turn implies $\Psi_{k+2}(w) \in V(e_3^*)$. In particular, $\Psi_{k+2}(w)$ is proportional to $\Psi_{k+1}(w)$. Then $\Psi_{k+2}(w) \in V(e_0^*)$, hence the edge $(k+1, k+2) \in E_1$ has weight zero. But then $\Psi_{k-1}(w)$ and $\Psi_{k+1}(w)$ lie both in $V(e_1^*)$, contradicting $Z_w \not\subseteq V(e_1^*)$, since they form a basis. This concludes Case 1.

Case 2: m is odd. Here, the last edge along the directed path of edges from E_2 is $(k+1, k-1)$. We argue similar to Case 1: The vector $\Psi_m(w)$ is proportional to $\Psi_{k-1}(w)$, since the vertex m has no outgoing edge in E_1. The vertex 0 has no incoming edge in E_1, so $\Psi_0(w)$ is proportional to $\Psi_{k+1}(w)$.

If one of $\Psi_0(w)$ and $\Psi_n(w)$ is zero, then so is the other, because the edge in E_2 between them has weight zero, while the edge $(k+1, k-1)$ does not. But this case is covered by the induction hypothesis, as previously in Case 1.

Hence, $\Psi_0(w) \neq 0$ and $\Psi_m(w) \neq 0$. The edge $(m, 2) \in E_2$ has weight zero, since $\Psi_m(w)$ is proportional to $\Psi_{k-1}(w)$. But this implies that $\Psi_1(w)$ must be proportional to $\Psi_{k+1}(w)$. Then the edge $(0, 1) \in E_1$ must have weight zero. But since $\Psi_0(w)$ is a non-zero multiple of $\Psi_{k+1}(w)$, this implies that both $\Psi_{k-1}(w)$ and $\Psi_{k+1}(w)$ lie in $V(e_1^*)$, a contradiction. This concludes Case 2 and therefore the proof. $\square$

Proof of Theorem 1.4.7. Combining the previous Lemmas, we have established the following: A tensor $t \in W$ is of border rank ≤ 2 if and only if $t \in W_1$ or W_2. In other words, $\sigma_2(Y) \cap L = L_1 \sqcup L_2$.

It only remains to show that $L_k \cap \sigma_2^\circ(X) = \emptyset$ for $k = 1, 2$. For all $\lambda_{i,j} \in \mathbb{C}$, the vector $\sum_{(i,j)\in E_1} \lambda_{i,j} u_{i,j} \in W_1$ is the tangent vector to the Segre variety at the point $e_0 \otimes e_0 \otimes (\sum_{(i,j)\in E_1} \lambda_{i,j} e_i)$ in the direction $e_1 \otimes e_1 \otimes (\sum_{(i,j)\in E_1} \lambda_{i,j} e_j)$. Such a tangent vector is of rank 3 unless $\sum_{(i,j)\in E_1} \lambda_{i,j} e_i$ is proportional to $\sum_{(i,j)\in E_1} \lambda_{i,j} e_j$, which is the case if and only if $\lambda_{i,j} = 0$ for all $(i,j) \in E_1$. Hence $L_1 \cap \sigma_2^\circ(X) = \emptyset$. The same argument proves the claim for L_2. $\square$

We have given a geometric construction of an injective morphism $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^m \xrightarrow{\text{inj}} \mathbb{P}^{2m+4}$. Choosing appropriate bases, we arrive at the following explicit description:

Corollary 1.4.11. *The following morphism is injective:*

$$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^m \to \mathbb{P}^{2(m+2)},$$

$$\begin{aligned}
&[x_0 : x_1] \times [y_0 : y_1] \times [z_0 : \ldots : z_m] \\
&\mapsto \Big[x_0y_1z_m : x_1y_1z_{\lceil m/2\rceil} : x_1y_1z_{\lceil m/2\rceil+(-1)^m} : x_0y_0z_{i+1} - x_0y_1z_i : x_0y_0z_{i+1} - x_1y_0z_i : \\
&\qquad x_1y_0z_{m-j+\lfloor 2j/m\rfloor} - x_0y_0z_j : x_1y_0z_{m-j+\lfloor 2j/m\rfloor} - x_1y_1z_j \;\Big|\; i \in \{0,1,\ldots,m-1\}, \\
&\qquad\qquad j \in \{0,1,\ldots,m\} \setminus \{\lceil m/2\rceil, \lceil m/2\rceil + (-1)^m\}\Big].
\end{aligned}$$

In particular, $\gamma(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^m, \mathcal{O}(1,1,1)) \leq 2(m+2)$.

CHAPTER 2

Toric degenerations in families of unirational varieties

Toric degenerations are deformations of algebraic varieties to toric limits. They are an essential element in the toolkit of combinatorial algebraic geometry, as they allow to transition from the special class of toric varieties to more general settings and make purely combinatorial techniques applicable to a wider class of algebraic varieties.

Degeneration techniques are an important subject throughout algebraic geometry. In computational algebraic geometry, they are the geometric notion behind fundamental symbolic algorithms. In fact, the classical techniques for computing dimensions and degrees of varieties rely on degenerations to arrangements of linear spaces aligned with the coordinate axes. In general, classical Gröbner basis algorithms can be viewed as a tool to compute degenerations of arbitrary ideals in polynomial rings to monomial ideals [Eis95, §15.8].

Degenerations to *toric* varieties can be seen as a milder deformation technique with irreducible limits that carry a rich combinatorial structure described by lattice polytopes or fans of convex polyhedral cones. They often provide more structural insights, for example by describing Hilbert functions as counting lattice points in polytopes [SX10].

Computationally, embedded toric degenerations of an irreducible affine variety $X = V(I) \subseteq \mathbb{C}^n$ may be obtained via Gröbner basis techniques as the vanishing set of the initial ideal $\mathrm{in}_w(I)$, where $w \in \mathbb{Z}^n$ is a suitably chosen weight vector in the relative interior of a maximal cone of the tropical variety $\mathrm{Trop}(X)$. The family describing the deformation arises from rescaling coordinates of the ambient space corresponding to the weights w and passing to a limit. See [KM19] for a modern view on this construction in the more general context of the theory of Khovanskii bases. Central questions in this topic revolve around the (non-)existence of suitable weights w, possibly after re-embedding the variety [BLMM17; KM19; IW20].

In this chapter, we approach the topic of toric degenerations from an alternative direction inspired by Sturmfels and Xu's study of Cox rings of del Pezzo surfaces [SX10] and the subsequent work [BCDFM17]. We

This chapter grew out of the results in the article "Towards classifying toric degenerations of cubic surfaces" [DGW20] by Maria Donten-Bury, Milena Wrobel and the author of this thesis, accepted for publication in *Le Matematiche*, 2020.

consider a *family of unirational varieties* $\mathscr{X} \to S$ given as the closure of the image of a morphism

$$\Phi\colon \mathbb{C}^m \times S \to \mathbb{C}^n \times S,$$

where $S \subseteq (\mathbb{C}^*)^k$ is a very affine variety. The main geometric idea is to track the behavior of the family along curves in S under a simultaneous rescaling of the coordinates of the ambient space, and to use tropical geometry to classify toric limits arising from this.

Curves can in analytic neighborhoods of smooth points be described in terms of power series. Algebraically, this corresponds to $\mathbb{C}(\!(t)\!)$-valued point of S. More generally, if K is the fraction field of a discrete valuation ring R with residue field $\mathbb{C}$, then for any K-valued points of S, we consider the induced morphism

$$\Phi_K\colon \mathbb{C}^m \times \operatorname{Spec} K \to \mathbb{C}^n \times \operatorname{Spec} K.$$

After rescaling the coordinates of the ambient space $\mathbb{C}^n \times \operatorname{Spec} K = \mathbb{A}^n_K$ by suitable powers of the uniformizing parameter, this extends to a morphism $\Phi_R\colon \mathbb{C}^m \times \operatorname{Spec} R \to \mathbb{C}^n \times \operatorname{Spec} R$ such that over the closed point of $\operatorname{Spec} R$, the image of $\Phi_0\colon \mathbb{C}^m \to \mathbb{C}^n$ is not contained in the coordinate hyperplanes. The image of Φ_R describes a flat family over $\operatorname{Spec} R$. In fortunate circumstances, the special fiber may happen to agree with the closure of the image of Φ_0. If moreover, Φ_0 is defined by monomials, then the latter is a toric variety.

This geometric picture underlies the algebraic notion of *Khovanskii bases* over valued fields, which were in the present setup described in [SX10] under the name *SAGBI bases*. In that article, families of total coordinate spaces of del Pezzo surfaces were degenerated to provide interpretations of their multigraded Hilbert functions as counting lattice points of polytopes. The open classification problem [SX10, Problem 5.4] was the motivation behind our study.

This chapter is structured as follows: In Section 2.1, we introduce the general setup and give a detailed picture on the construction of a flat family outlined above. We raise a combinatorial problem for classifying when the special fiber is toric and relate it to the theory of Khovanskii bases. In Section 2.2, we describe an algorithm for resolving this classification problem based on a computational subdivision of the tropical variety of S. In Section 2.3, we specialize to the setting of [SX10, Problem 5.4] and comment on the algorithmic challenges and optimizations possible in this context. We carry out the classification algorithm with respect to a fixed embedding of S, partially answering [SX10, Problem 5.4]. In Section 2.4, we consider the issue of extending our classification to a complete answer to [SX10, Problem 5.4] and resolve it for one particular subclass.

2.1. Parameterized families and their degenerations

In this section, we describe a generalized geometric picture of a degeneration approach taken in [SX10] via the theory of SAGBI bases.

2.1.1. Families of unirational varieties and their degenerations

Throughout, we work with a family of unirational varieties $\mathscr{X} \to S$ arising as follows: Let S be an irreducible subvariety of $(\mathbb{C}^*)^k$ with coordinate ring $A = \mathbb{C}[z_1^{\pm}, \dots, z_k^{\pm}]/I_S$. Over the base S, we consider a morphism

$$\Phi\colon \mathbb{A}^m_S \to \mathbb{A}^n_S$$

explicitly given by n polynomials $f_1, \dots, f_n \in A[x_1, \dots, x_m]$, each of which is assumed to be homogeneous of positive degree. By $\mathscr{X} \subseteq \mathbb{A}^n_S$, we denote the closure of the image of Φ. We assume that

$$\mathscr{X} \times_S T = \overline{\operatorname{im}(\Phi \times_S T\colon \mathbb{A}^m_T \to \mathbb{A}^n_T)} \quad \text{for all } S\text{-schemes } T. \tag{2.1}$$

This is for example satisfied if the morphism Φ has a closed image.

Our aim is to study the family $\mathscr{X}$ along infinitesimal curves in S and to classify toric degenerations naturally arising from them. Algebraically, analytic curves can be studied in terms of K-valued points of S, where K is a suitable valued field such as the field of Puiseux series $\mathbb{C}\{\{t\}\}$. In general, we make the following assumptions on the field K:

Convention 2.1.1. We fix an algebraically closed field K with a non-trivial valuation $\nu\colon K^* \to \mathbb{R}$, i.e., ν is a non-zero group homomorphism with $\nu(f+g) \geq \min\{\nu(f), \nu(g)\}$ for all $f, g \in K^*$. Consider its valuation ring $R := \{f \in K^* \mid \nu(f) \geq 0\} \cup \{0\}$. For convenience, we assume that $K \supseteq \mathbb{C}$ and that the residue field of R is $\mathbb{C}$. Since K is algebraically closed, we may fix a group homomorphism $\mu\colon \nu(K^*) \to K^*$ such that $\nu \circ \mu = \mathrm{id}_{\nu(K^*)}$, see [MS15, Lemma 2.1.15]. We abbreviate $t^\lambda := \mu(\lambda)$ for $\lambda \in \nu(K^*)$. Our main example is $K = \mathbb{C}\{\{t\}\}$.

Our degeneration approach is to build suitable families over $\operatorname{Spec} R$ from K-valued points of S, based on the construction that follows. We write $S(K)$ for the set of K-valued points of S. For each $w \in \mathbb{Z}^n$, consider the automorphism

$$\psi_w\colon \mathbb{A}^n_K \xrightarrow{\cong} \mathbb{A}^n_K, \qquad (y_1, \dots, y_n) \mapsto (t^{-w_1} y_1, \dots, t^{-w_n} y_n).$$

Lemma 2.1.2. *Let $a \in S(K)$ and consider the induced morphism $\Phi(a)\colon \mathbb{A}^m_K \to \mathbb{A}^n_K$. There exists at most one weight vector $w(a) \in \mathbb{R}^n$ for which the composition $\psi_{w(a)} \circ \Phi(a)$ lifts to a morphism $\theta(a)\colon \mathbb{A}^m_R \to \mathbb{A}^n_R$ such that, over the closed point, the image of $\theta(a)_0\colon \mathbb{A}^m_{\mathbb{C}} \to \mathbb{A}^n_{\mathbb{C}}$ does not lie in the coordinate hyperplanes.*

Proof. For $a \in S(K)$, the polynomials $f_i \in A[x_1, \dots, x_m]$ defining Φ specialize to the polynomials $f_i|_a \in K[x_1, \dots, x_m]$ with coefficients in K that define $\Phi(a)$. If $f_i|_a = 0$ for some i, then the image of $\psi_w \circ \Phi(a)$

is contained in a coordinate hyperplane of $\mathbb{A}^n_R$, hence the desired weight vector $w(a)$ cannot exist. Otherwise, we claim that $w(a)$ exists and is uniquely determined. In fact, consider

$$\tilde{\nu}\colon K[x_1,\dots,x_m]\setminus\{0\}\to\mathbb{R},\ \sum_\alpha c_\alpha \mathbf{x}^\alpha \mapsto \min\{\nu(c_\alpha)\mid c_\alpha\neq 0\}, \tag{2.2}$$

which extends to a valuation $\tilde{\nu}\colon K(x_1,\dots,x_m)^*\to\mathbb{R}$. Note that the composition $\psi_w\circ\Phi(a)$ lifts to a morphism $\mathbb{A}^m_R\to\mathbb{A}^n_R$ if and only if $w_i\geq\tilde{\nu}(f_i|_a)$ for all i. On the other hand, if $w_i>\tilde{\nu}(f_i|_a)$ for some i, then the i-th component of $\psi_w\circ\Phi(a)$ has positive valuation with respect to $\tilde{\nu}$ and is therefore zero over the closed point of $\operatorname{Spec} R$. This shows that

$$w(a):=\big(\tilde{\nu}(f_1|_a),\dots,\tilde{\nu}(f_n|_a)\big)\in\mathbb{R}^n$$

is the unique weight vector with the desired property. □

We call a K-valued point a of S **feasible** if the weight vector $w(a)$ in Lemma 2.1.2 exists, i.e., if $f_1|_a,\dots,f_n|_a$ are non-zero elements of $K[x_1,\dots,x_m]$. In this case, we consider

$$\mathcal{Z}(a):=\overline{\operatorname{im}(\theta(a))}\subseteq\mathbb{A}^n_R,$$

which is a flat family over $\operatorname{Spec} R$ by construction, as it is integral and dominates $\operatorname{Spec} R$. Note that its generic fiber can be described as

$$\mathcal{Z}(a)_\eta=\overline{\operatorname{im}(\theta(a)_\eta\colon\mathbb{A}^m_K\to\mathbb{A}^n_K)},$$

while for the special fiber we only have the containment

$$\mathcal{Z}(a)_0\supseteq\overline{\operatorname{im}(\theta(a)_0\colon\mathbb{A}^m_{\mathbb{C}}\to\mathbb{A}^n_{\mathbb{C}})}. \tag{2.3}$$

For geometric intuition, $\mathcal{Z}(a)_\eta$ can be thought of (up to a rescaling of the ambient space) as a member of the family $\mathscr{X}$, more precisely as the fiber of $\mathscr{X}$ over a general point on a small analytic curve in S. Its properties relate through the family $\mathcal{Z}(a)$ to the special fiber which may be structurally simpler.

If equality holds in (2.3) and if the morphism $\theta(a)_0$ is given by monomials (possibly with non-trivial coefficients), then the special fiber is a (not necessarily normal) affine toric variety. In this case, $\mathcal{Z}(a)$ describes a *toric degeneration*. This leads to the following definition:

Definition 2.1.3. We call a feasible point $a\in S(K)$ **Khovanskii** if equality holds in (2.3). Moreover, a point $a\in S(K)$ is called **moneric** if it is feasible and if the morphism $\theta(a)_0\colon\mathbb{A}^m_{\mathbb{C}}\to\mathbb{A}^n_{\mathbb{C}}$ is of the form

$$(x_1,\dots,x_m)\mapsto(c_1\mathbf{x}^{\alpha_1},\dots,c_n\mathbf{x}^{\alpha_n})$$

for some $c_i\in\mathbb{C}^*$, $\alpha_i\in\mathbb{N}^m$. The **moneric class** of a is then the tuple of monomials $(\mathbf{x}^{\alpha_1},\dots,\mathbf{x}^{\alpha_n})$, disregarding the coefficients c_i.

The following observation translates the study of toric degenerations arising from the above construction into a finite classification problem.

Proposition 2.1.4. *There are only finitely many moneric classes. Let $a, b \in S(K)$ be two moneric points with the same moneric class. Then a is Khovanskii if and only if b is.*

Proof. The first claim follows from the simple observation that a moneric class $(\mathbf{x}^{\alpha_1}, \dots, \mathbf{x}^{\alpha_n})$ can only contain monomials present in the polynomials $f_1, \dots, f_n$. Indeed, it follows from the definition of $\theta(a)_0$ that the i-th monomial $\mathbf{x}^{\alpha_i}$ needs to occur among the terms of the polynomial $f_i \in A[x_1, \dots, x_m]$.

For the second claim, we argue by comparing the Hilbert polynomials of the coordinate rings involved. For this, we recall that the polynomials $f_1, \dots, f_n \in A[x_1, \dots, x_n]$ defining Φ are assumed to be homogeneous. Hence

$$\mathscr{X}_a,\ \mathcal{Z}(a)_\eta,\ \mathcal{Z}(a)_0 \text{ and } Y_a := \overline{\operatorname{im}(\theta(a)_0)}$$

have graded coordinate rings (over the field K or $\mathbb{C}$). Note that $\mathscr{X}_a := \mathscr{X} \times_S \operatorname{Spec} K$ (defined via $a\colon \operatorname{Spec} K \to S$) agrees with the closure of the image of $\Phi(a)$ because of (2.1). Since $\theta(a)_\eta = \psi_{w(a)} \circ \Phi(a)$, this shows that the coordinate ring of $\mathscr{X}_a$ has the same Hilbert polynomial as the coordinate ring of $\mathcal{Z}(a)_\eta = \operatorname{im}(\theta(a)_\eta)$. By flatness of $\mathcal{Z}(a) \to \operatorname{Spec} R$, this in turn coincides with the Hilbert polynomial for the coordinate ring of $\mathcal{Z}(a)_0$. We conclude that the coordinate rings of $\mathscr{X}_a$ and $\mathcal{Z}(a)_0$ have the same Hilbert polynomial, and similarly for b.

Moreover, note that $\mathscr{X} \to S$ is flat. Indeed, this follows from the valuative criterion for flatness: For a discrete valuation ring $\tilde{R}$ and an $\tilde{R}$-valued point of S, the assumption (2.1) implies that $\mathscr{X} \times_S \operatorname{Spec} \tilde{R}$ is an integral scheme surjecting onto $\operatorname{Spec} \tilde{R}$ and therefore flat over $\operatorname{Spec} \tilde{R}$.

In particular, the coordinate rings of $\mathscr{X}_a$ and $\mathscr{X}_b$ share their Hilbert polynomials. By the discussion above, the same is true for $\mathcal{Z}(a)_0$ and $\mathcal{Z}(b)_0$. Moreover, since a and b describe the same moneric class, the varieties $\overline{\operatorname{im}\theta(a)_0}$ and $\overline{\operatorname{im}\theta(b)_0}$ only differ by a rescaling of the ambient space, so the Hilbert polynomials of their coordinate rings also agree. From these observations, we conclude that the inclusions

$$\mathcal{Z}(a)_0 \supseteq \overline{\operatorname{im}(\theta(a)_0)} \qquad \text{and} \qquad \mathcal{Z}(b)_0 \supseteq \overline{\operatorname{im}(\theta(b)_0)}$$

are either both strict or both equalities. I.e., a is Khovanskii if and only if b is. □

Note that Proposition 2.1.4 allows us to talk about **Khovanskii classes** $(\mathbf{x}^{\alpha_1}, \dots, \mathbf{x}^{\alpha_n})$, by which we mean moneric classes such that any corresponding moneric K-valued point of S is Khovanskii.

If $a \in S(K)$ is moneric and Khovanskii, then the flat family $\mathcal{Z}(a)$ has a toric special fiber. The following classification problem arises:

Problem 2.1.5. *Classify all moneric classes. Which of them are Khovanskii?*

2.1.2. Khovanskii bases over valued base fields

The term *Khovanskii* in Definition 2.1.3 alludes to the theory of Khovanskii bases as in [KM19]. For an integral K-algebra Q equipped with a valuation $\tilde{\nu}\colon Q\setminus\{0\}\to\mathbb{R}$ extending $\nu\colon K^*\to\mathbb{R}$, a **Khovanskii basis** is a subset $F\subseteq Q\setminus\{0\}$ such that the image of $K^*\cup F$ in

$$\mathrm{gr}_{\tilde{\nu}}(Q) := \bigoplus_{\lambda\in\mathbb{R}} Q_{\tilde{\nu}\geq\lambda}/Q_{\tilde{\nu}>\lambda}$$

generates the $\mathbb{C}$-algebra $\mathrm{gr}_{\tilde{\nu}}(Q)$. Here,

$$Q_{\tilde{\nu}\geq\lambda} := \{f\in Q\setminus\{0\} \mid \tilde{\nu}(f)\geq\lambda\}.$$

and analogously for $Q_{\tilde{\nu}>\lambda}$. This is the natural extension of [KM19, Definition 2.5] to the setting of a valued base fields (for valuations of rank 1). In our setting, for a feasible point $a\in S(K)$, we consider the subalgebra of $K[x_1,\dots,x_m]$ generated by $f_1|_a,\dots,f_n|_a$ and equip it with the valuation $\tilde{\nu}$ from (2.2). Below, we observe that a feasible point $a\in S(K)$ is Khovanskii if and only if $f_1|_a,\dots,f_n|_a$ form a Khovanskii basis with respect to $\tilde{\nu}$ for the K-algebra they generate. Therefore, Problem 2.1.5 can be interpreted as a classification problem studying when the given polynomials $f_1,\dots,f_n\in A[x_1,\dots,x_m]$ specialize to a Khovanskii basis over a valued field K.

We note that the notion of a Khovanskii basis for a subalgebra $Q\subseteq K[x_1,\dots,x_m]$ equipped with the coefficient valuation (2.2) has a simple algebraic description: For a polynomial $g\in K[x_1,\dots,x_m]\setminus\{0\}$ over K, its **initial form** is defined as

$$\mathrm{in}(g) := (t^{-\tilde{\nu}(g)}g)\Big|_{t=0}\in\mathbb{C}[x_1,\dots,x_m].$$

Then, a set $F\subseteq Q$ is a Khovanskii basis with respect to $\tilde{\nu}$ if and only if its set of initial forms $\{\mathrm{in}(f)\mid f\in F\}$ generates the **initial algebra**

$$\mathrm{in}(Q) := \mathbb{C}[\mathrm{in}(g)\mid g\in Q].$$

In our setting, a K-valued point $a\in S(K)$ defines a set of polynomials $\{f_1|_a,\dots,f_n|_a\}\subseteq K[x_1,\dots,x_m]$ which are non-zero if and only if a is feasible. By the construction in the proof of Lemma 2.1.2, the morphism $\theta(a)_0\colon\mathbb{A}^m_{\mathbb{C}}\to\mathbb{A}^n_{\mathbb{C}}$ is the map given by the initial forms $\mathrm{in}(f_1|_a),\dots,\mathrm{in}(f_n|_a)$. In particular, a is moneric if and only if all the initials $\mathrm{in}(f_i|_a)$ are monomials (with possibly non-trivial coefficients). Moreover, note that by construction,

$$\mathcal{Z}(a) = \operatorname{Spec} R\Big[t^{-\tilde{\nu}(g)}g \mid g\in K[f_1|_a,\dots,f_n|_a]\Big],$$

and in particular, the coordinate ring of $\mathcal{Z}(a)_0$ is the initial algebra $\mathrm{in}(K[f_1|_a,\dots,f_n|_a])$. Therefore, $a\in S(K)$ is Khovanskii if and only if $f_1|_a,\dots,f_n|_a$ form a Khovanskii basis of the algebra they generate.

Remark 2.1.6. This algebraic setup behind the geometric picture described in Section 2.1.1 was introduced in [SX10] in the study of Cox

rings of del Pezzo surfaces and their Hilbert functions. A moneric set of polynomials over K forming a Khovanskii basis was called a *SAGBI basis* there, generalizing previous notions of SAGBI bases [KM89; RS90]. The article [BCDFM17] builds upon [SX10] and replaces the term SAGBI basis by *Khovanskii basis*, which were still required to be moneric by definition. It should be pointed out that, in contrast, we formally separate the property of being moneric from forming a Khovanskii basis, since these two notions are independent and it seems to be more in line with the general theory of Khovanskii bases [KM19]. For the purpose of toric degenerations, we will of course only be interested in Khovanskii bases which are also moneric.

2.2. Tropical geometry as a classification tool

Tropical geometry studies combinatorial shadows of algebraic varieties. For a subvariety Y of $(\mathbb{C}^*)^\ell$, its **tropical variety** is the closure in the euclidean topology of the set of coordinate-wise valuations of its K-valued points:

$$\mathrm{Trop}(Y) := \overline{\{(\nu(a_1), \dots, \nu(a_\ell)) \in \mathbb{R}^\ell \mid (a_1, \dots, a_\ell) \in Y(K)\}} \subseteq \mathbb{R}^\ell.$$

This definition of tropical variety will be sufficient for our purposes in this chapter, yet it should be noted that the theory of tropical geometry goes beyond this set-theoretic notion used here, see [MS15]. Later, in Chapter 3 and Chapter 4, we will extend the above definition of tropical varieties to subvarieties of $(K^*)^\ell$ and use refined structures such as multiplicities and Gröbner polyhedra.

The main structural property of interest in this chapter is the fact that $\mathrm{Trop}(Y)$ is the support of a fan of rational polyhedral cones [MS15, Theorem 3.3.5], i.e., it can be written as the union over finitely many rational polyhedral cones in $\mathbb{R}^\ell$ that only intersect each other along faces. By a *fan structure* on $\mathrm{Trop}(Y)$, we mean the collection of all the polyhedral cones in such a description, the maximal cones as well as all their faces. There is generally not a unique coarsest fan structure on a tropical variety, see [MS15, Example 3.5.4]. Of importance for us is that from the vanishing ideal of Y, it is algorithmically possible to compute *some* fan of rational polyhedral cones defining $\mathrm{Trop}(Y)$, see [BJSST07]. For irreducible Y, the maximal cones of any fan structure on $\mathrm{Trop}(Y)$ are all of the same dimension, namely $\dim(Y)$.

Let us return to the setup of the previous section. We will use the tropical variety of $S \subseteq (\mathbb{C}^*)^k$ to aid the classification of moneric classes in Problem 2.1.5. For any K-valued point $a = (a_1, \dots, a_k) \in (K^*)^k$ of S, we denote its coordinate-wise valuation by

$$\nu(a) := (\nu(a_1), \dots, \nu(a_k)) \in \mathrm{Trop}(S)$$

and we say that *a tropicalizes to $\nu(a)$*.

Definition 2.2.1. We define the **tropical dimension** of a moneric class $C = (\mathbf{x}^{\alpha_1}, \dots, \mathbf{x}^{\alpha_n})$ as the smallest non-negative integer d such that the set

$$\Big\{\nu(a) \in \operatorname{Trop}(S) \mid a \in S(K) \text{ moneric with moneric class } C\Big\}$$

is contained in a finite union of d-dimensional linear subspaces of $\mathbb{R}^k$. We call a moneric class **full-dimensional** if its tropical dimension is $\dim(Y)$.

Proposition 2.2.2. *After a suitable re-embedding of $S \subseteq (\mathbb{C}^*)^k$, all moneric classes are full-dimensional.*

Proof. We observe that the definition of moneric classes does not depend on the embedding of S into $(\mathbb{C}^*)^k$, while the tropical dimension of a moneric class does.

We write each of the polynomials $f_1, \dots, f_n \in A[x_1, \dots, x_m]$ defining Φ as

$$f_i = \sum_{j=1}^{r_i} c_{ij}\, \mathbf{x}^{\gamma_{ij}} \qquad \text{with } c_{ij} \in A \setminus \{0\} \text{ and } \gamma_{ij} \in \mathbb{N}^m.$$

The $r := r_1 + \cdots + r_n$ many coefficients $c_{ij} \in A$ are regular functions on $S = \operatorname{Spec} A$ and hence define a re-embedding

$$S \xhookrightarrow{\operatorname{id} \times (c_{ij} \mid i,j)} (\mathbb{C}^*)^k \times \mathbb{C}^r.$$

Denote by S' the intersection of the image with $(\mathbb{C}^*)^{k+r}$. Note that S' is then isomorphic to the non-empty open subset of S given by the non-vanishing of all $c_{ij} \in A$. We may now replace $\Phi\colon \mathbb{A}^m_S \to \mathbb{A}^n_S$ by $\Phi \times_S S'\colon \mathbb{A}^m_{S'} \to \mathbb{A}^n_{S'}$ (and therefore $\mathscr{X} \to S$ by $\mathscr{X} \times_S S' \to S'$) without changing the moneric classes. We claim that after this replacement every moneric class is full-dimensional.

Let $a \in S'(K)$ be a feasible point. Note that its tropicalization $\nu(a) \in \operatorname{Trop}(S')$ uniquely determines the valuation of each coefficient $c_{ij}|_a$ in $f_i|_a$. Consider a tuple $C = (\mathbf{x}^{\gamma_{1s_1}}, \dots, \mathbf{x}^{\gamma_{ns_n}})$ with $1 \le s_i \le r_i$. By definition, $a \in S'(K)$ is moneric with moneric class C if and only if

$$\nu(c_{is_i}|_a) < \nu(c_{ij}|_a) \quad \text{for all } i = 1, \dots, n,\ j \in \{1, \dots, r_i\} \setminus \{s_i\}.$$

These strict inequalities describe an open subset $U \subseteq \mathbb{R}^{k+r}$. Its intersection with $\operatorname{Trop}(S')$ is non-empty if and only if C is a moneric class. In this case, the moneric class is full-dimensional: Each maximal cone of $\operatorname{Trop}(S')$ is of dimension $\dim(S') = \dim(S)$, so the intersection $U \cap \operatorname{Trop}(S')$ cannot be contained in a finite union of linear spaces of smaller dimension. $\square$

In the following, for any $c \in A$, we denote by $\Gamma(c) \subseteq \mathbb{R}^{k+1}$ the **tropical graph** of c, i.e., $\Gamma(c)$ is the tropical variety of the re-embedding

of $S = \operatorname{Spec} A$ into $(\mathbb{C}^*)^k \times \mathbb{C}$ given by $\mathrm{id} \times c$. By the **lower envelope** of a subset $Y \subseteq \mathbb{R}^{k+1}$, we mean the set

$$\Big\{(p,q) \in \mathbb{R}^k \times \mathbb{R} \mid \Big(\{p\} \times (-\infty, q]\Big) \cap Y = \{(p,q)\}\Big\}.$$

Note that the lower envelope of a finite union of fans of convex cones is again a fan of convex cones.

The algorithm that follows classifies full-dimensional moneric classes based on computational tropical geometry and computational convex geometry. It is a generalization of the approach pursued in [SX10, §4] for total coordinate spaces of del Pezzo surfaces of degree 4. For the successful application of this algorithm in a computationally much more challenging setting, see Section 2.3.

Algorithm 2.2.3 (Computing full-dimensional moneric classes)**.**
Input: A prime Laurent polynomial ideal

$$I_S \subseteq \mathbb{C}[z_1^{\pm}, \ldots, z_k^{\pm}]$$

defining $S \subseteq (\mathbb{C}^*)^n$, and non-zero homogeneous polynomials

$$f_1, \ldots, f_n \in (\mathbb{C}[z_1^{\pm}, \ldots, z_k^{\pm}]/I_S)[x_1, \ldots, x_m],$$

defining a morphism $\Phi\colon \mathbb{A}^m_S \to \mathbb{A}^n_S$ satisfying (2.1).
Output: The set of all full-dimensional moneric classes.

1: Compute the tropical variety $\mathrm{Trop}(S) \subseteq \mathbb{R}^k$.
2: **for** $i = 1, \ldots, n$ **do**
3: Compute the tropical graphs $\Gamma(c_{i1}), \ldots, \Gamma(c_{ir_i}) \subseteq \mathbb{R}^{k+1}$, where $c_{i1}, \ldots, c_{ir_i}$ are the coefficients of $f_i = \sum_j c_{ij}\mathbf{x}^{\gamma_{ij}}$.
4: $\Lambda_i :=$ lower envelope of $\Gamma(c_{i1}) \cup \ldots \cup \Gamma(c_{ir_i})$.
5: $T_i := \{(\sigma, s) \mid \sigma$ max. cone of Λ_i, $\sigma \subseteq \Gamma(c_{is})$, $\sigma \nsubseteq \Gamma(c_{ij})\ \forall j \neq s\}$.
6: $\Sigma_i :=$ subdivision of $\mathrm{Trop}(S)$ induced from Λ_i under the projection $\pi\colon \mathbb{R}^{k+1} \twoheadrightarrow \mathbb{R}^k$.
7: $\Sigma :=$ common refinement of the fans $\Sigma_1, \ldots, \Sigma_n$ on $\mathrm{Trop}(S)$.
8: **return** $\left\{(\mathbf{x}^{\gamma_{1s_1}}, \ldots, \mathbf{x}^{\gamma_{ns_n}}) \;\middle|\; \begin{array}{c}\pi(\sigma_1) \cap \ldots \cap \pi(\sigma_n) \text{ max. cone} \\ \text{of } \Sigma \text{ and } (\sigma_i, s_i) \in T_i\ \forall i\end{array}\right\}$.

Correctness of Algorithm 2.2.3. If a coefficient c_{ij} can be expressed as $c_{ij} = \sum_{\beta \in \mathbb{Z}^k} b_\beta \mathbf{z}^\beta$, then the lower envelope of $\Gamma(c_{ij})$ is the image of $\mathrm{Trop}(S)$ under the map $\mathbb{R}^k \hookrightarrow \mathbb{R}^{k+1}$, $v \mapsto (v, \mathrm{trop}(c_{ij})(v))$, where $\mathrm{trop}(c_{ij})$ denotes the piece-wise linear function

$$\mathrm{trop}(c_{ij})\colon \mathrm{Trop}(S) \to \mathbb{R}, \quad (v_1, \ldots, v_k) \mapsto \min\Big\{\sum_{i=1}^k \beta_i v_i \mid b_\beta \neq 0\Big\}.$$

Note that because of this, $\mathrm{trop}(c_{ij})$ is independent of the choice of the expression $\sum_{\beta \in \mathbb{Z}^k} b_\beta \mathbf{z}^\beta$ representing c_{ij} modulo I_S. The projections to $\mathbb{R}^k$ of the maximal cones of the lower envelope of $\Gamma(c_{ij})$ are the regions

of lineality of $\operatorname{trop}(c_{ij})$. For $a \in S(K)$ such that $\nu(a)$ lies in the relative interior of such a projected cone, we have $\nu(c_{ij}|_a) = \operatorname{trop}(c_{ij})(\nu(a))$.

In particular, if σ is a maximal cone of Λ_i for some i, then for any $a \in S(K)$ with $\nu(a) \in \operatorname{Relint}(\pi(\sigma))$, we have

$$\begin{aligned}\tilde{\nu}(f_i|_a) &= \min\{\nu(c_{i1}|_a), \dots, \nu(c_{ir_i}|_a)\} \\ &= \min\{\operatorname{trop}(c_{i1})(\nu(a)), \dots, \operatorname{trop}(c_{ir_i})(\nu(a))\} \\ &= \operatorname{trop}(c_{is})(\nu(a)) \quad \text{if } \sigma \subseteq \Gamma(c_{is}).\end{aligned}$$

In particular, the minimum is attained exactly once if and only if σ shows up in a pair (σ, s) of the set T_i, and in this case, $\operatorname{in}(f_i|_a)$ is a scalar multiple of $\mathbf{x}^{\gamma_{is}}$. Since Σ is the common refinement of all $\pi(\Lambda_i)$, this shows that every monomial tuple $(\mathbf{x}^{\gamma_{1s_1}}, \dots, \mathbf{x}^{\gamma_{ns_n}})$ output by the algorithm is in fact a full-dimensional moneric class.

Conversely, we need to argue that every full-dimensional moneric class is output by the algorithm. Maximal cones of Σ are of the form $\pi(\sigma_1) \cap \dots \cap \pi(\sigma_n)$ with σ_i being maximal cones of Λ_i. We have just argued that any $a \in S(K)$ tropicalizing to the relative interior of such a cone is moneric with moneric class $(\mathbf{x}^{\gamma_{1s_1}}, \dots, \mathbf{x}^{\gamma_{ns_n}})$ if and only if $(\sigma_i, s_i) \in T_i$ for all i. Therefore, if $a \in S(K)$ is a moneric point whose moneric classes we do not output, then $\nu(a)$ cannot lie in the relative interior of any maximal cone of Σ. Since Σ covers $\operatorname{Trop}(S)$, this implies that the moneric class is not full-dimensional. □

Algorithm 2.2.3 can be used to classify moneric classes. For tackling Problem 2.1.5 of describing toric degenerations, it is also necessary to check which moneric classes are Khovanskii. This can be done for example by comparing the Hilbert polynomial of the toric variety arising from the moneric class with that of the fibers of $\mathscr{X} \to \operatorname{Spec} R$, as in the proof of Proposition 2.1.4. In situations where this is undesirable (e.g. if the main purpose of constructing toric degenerations is the computation of Hilbert polynomials), one can check that the defining equations of the toric variety lift to equations of the family, as in [CHV96, Proposition 1.3]. The verification of such a lifting criterion can be carried out with a classical subduction algorithm, see [KM19, Algorithm 2.11 and Theorem 2.17] for more details.

2.3. Total coordinate spaces of cubic surfaces

Smooth cubic surfaces are among the most classical projective varieties. They can be described as the blow-ups of six points in the projective plane in general position (meaning that no three of the points lie on a line and that there is no conic passing through all the points). In this section, we consider the family of total coordinate spaces of these blow-ups and study toric degenerations in this family. This study was initiated in [SX10] in the context of finding Ehrhart-type formulas for

multigraded Hilbert functions. A classification of the toric degenerations was left as an open research problem. Progress was made in [BCDFM17] and we build upon their work, applying the setup and techniques described in the previous sections.

If $p_1, \dots, p_6 \in \mathbb{P}^2_{\mathbb{C}}$ are points in general position, then their blow-up $X = \mathrm{Bl}_{p_1,\dots,p_6} \mathbb{P}^2_{\mathbb{C}}$ is a smooth cubic surface. There are 27 lines on X. For one, there are the exceptional divisors $E_1, \dots, E_6$. Then, for each of the 15 pairs of points $p_i \neq p_j$, there is the strict transform of the line through p_i and p_j, which we denote by F_{ij}. Finally, there are six more lines $G_1, \dots, G_6$, where G_i is the strict transform of the unique conic passing through all points except for p_i. We further denote by H the strict transform of a fixed general line in the projective plane not passing through any of the points $p_1, \dots, p_6$.

The divisors $H, E_1, \dots, E_6$ generate a subgroup $\mathcal{D}$ of $\mathrm{Div}(X)$ which maps isomorphically onto the free abelian group $\mathrm{Pic}(X)$. The **Cox ring** of the cubic surface X is defined as

$$\mathrm{Cox}(X) := \bigoplus_{D \in \mathcal{D}} H^0(X, \mathcal{O}_X(D)).$$

This definition is independent of the choice of $H, E_1, \dots, E_6$ as divisors generating the Picard group, see [ADHL15, Proposition 1.4.2.2]. The variety $\operatorname{Spec} \mathrm{Cox}(X)$ is called the **total coordinate space** of X. The $\mathbb{Z}^7$-grading on $\mathrm{Cox}(X)$ translates to a $(\mathbb{C}^*)^7$-action on the total coordinate space and by taking the GIT quotient of suitable subgroups, one can regain the homogeneous coordinate ring of X under any embedding by a complete linear system. In this sense, Cox rings carry information on all embeddings of X into projective spaces. See [ADHL15] for more background on the importance of Cox rings.

Of particular interest are toric degenerations of total coordinate spaces that are equivariant with respect to the $(\mathbb{C}^*)^7$-action, i.e., which reflect the $\mathbb{Z}^7$-graded structure on the Cox ring. Such toric degenerations can be used to give combinatorial interpretations for the multigraded Hilbert function in terms of counting lattice points in polytopes, see [SX10].

Cubic surfaces and therefore also their total coordinate spaces naturally arise in families, as one may allow the blown up points to vary in the projective plane.

As shown by Nagata in [Nag59], the Cox ring of a smooth cubic surface can be interpreted as a certain invariant ring: Let $M \in \mathbb{C}^{3\times 6}$ be a matrix whose columns define six points $p_1, \dots, p_6 \in \mathbb{P}^2_{\mathbb{C}}$ in general position, and let X be the blow-up of $\mathbb{P}^2_{\mathbb{C}}$ at these points. We consider the kernel of the matrix M as an additive subgroup G of $\mathbb{C}^6$ and consider the action of G on $\mathbb{A}^6_{\mathbb{C}} \times \mathbb{A}^6_{\mathbb{C}}$ given by

$$\begin{aligned} G \times \mathbb{A}^6_{\mathbb{C}} \times \mathbb{A}^6_{\mathbb{C}} &\to \mathbb{A}^6_{\mathbb{C}} \times \mathbb{A}^6_{\mathbb{C}}, \\ (\lambda, x, y) &\mapsto (x, y + \lambda \cdot x), \end{aligned}$$

where $\lambda \cdot x := (\lambda_1 x_1, \dots, \lambda_6 x_6)$. The corresponding invariant subring of $\mathbb{C}[\mathbf{x}, \mathbf{y}] := \mathbb{C}[x_1, \dots, x_6, y_1, \dots, y_6]$ under this action is called the **Cox–Nagata ring** and we have

$$\mathrm{Cox}(X) \cong \mathbb{C}[\mathbf{x}, \mathbf{y}]^G \subseteq \mathbb{C}[\mathbf{x}, \mathbf{y}]. \tag{2.4}$$

In [Muk04], Mukai explicitely describes the isomorphism in (2.4). Moreover, Batyrev and Popov [BP04] show that the Cox ring of a smooth cubic surface is generated by global sections corresponding to the 27 lines. Using this, we obtain distinguished generating set of the Cox–Nagata ring $\mathbb{C}[\mathbf{x}, \mathbf{y}]^G$, which, by abuse of notation, we denote simply as

$$\mathcal{F} = \{E_1, \dots, E_6, F_{12}, \dots, F_{56}, G_1, \dots, G_6\} \subseteq \mathbb{C}[\mathbf{x}, \mathbf{y}]^G.$$

Geometrically, this generating set and (2.4) together describe the total coordinate space $\operatorname{Spec} \mathrm{Cox}(X)$ as the closure of the image of the morphism

$$\varphi_M \colon \mathbb{A}^{12}_{\mathbb{C}} = \mathbb{A}^6_{\mathbb{C}} \times \mathbb{A}^6_{\mathbb{C}} \xrightarrow{\mathcal{F}} \mathbb{A}^{27}_{\mathbb{C}}.$$

We observe that multiplying the 3×6-matrix M from the left with an invertible 3×3-matrix leaves $G = \ker M$ invariant and in the geometric picture, interpreting the columns as points in $\mathbb{P}^2_{\mathbb{C}}$, it simply corresponds to applying an automorphism of $\mathbb{P}^2_{\mathbb{C}}$. In fact, the isomorphism (2.4) is equivariant with respect to this action of $\mathrm{GL}(3)$, hence the morphism φ_M only depends on the subspace $G = \ker M \subseteq \mathbb{C}^6$. Each of the polynomials in $\mathcal{F}$ can be expressed in terms of the Plücker coordinates p_{ijk} of this point in $\mathrm{Gr}(3, 6)$:

$$\begin{aligned}
E_1 &= x_1, \\
F_{12} &= p_{123} y_3 x_4 x_5 x_6 + p_{124} x_3 y_4 x_5 x_6 + p_{125} x_3 x_4 y_5 x_6 + p_{126} x_3 x_4 x_5 y_6, \\
G_1 &= p_{234} p_{235} p_{236} p_{456} \cdot y_2 y_3 x_4 x_5 x_6 x_1^2 + p_{234} p_{246} p_{245} p_{356} \cdot y_2 y_4 x_3 x_5 x_6 x_1^2 \\
&\quad + p_{235} p_{245} p_{256} p_{346} \cdot y_2 y_5 x_3 x_4 x_6 x_1^2 + p_{236} p_{246} p_{256} p_{345} \cdot y_2 y_6 x_3 x_4 x_5 x_1^2 \\
&\quad + p_{234} p_{345} p_{346} p_{256} \cdot y_3 y_4 x_2 x_5 x_6 x_1^2 + p_{235} p_{345} p_{356} p_{246} \cdot y_3 y_5 x_2 x_4 x_6 x_1^2 \\
&\quad + p_{236} p_{346} p_{356} p_{245} \cdot y_3 y_6 x_2 x_4 x_5 x_1^2 + p_{245} p_{345} p_{456} p_{236} \cdot y_4 y_5 x_2 x_3 x_6 x_1^2 \\
&\quad + p_{246} p_{346} p_{456} p_{235} \cdot y_4 y_6 x_2 x_3 x_5 x_1^2 + p_{256} p_{356} p_{456} p_{234} \cdot y_5 y_6 x_2 x_3 x_4 x_1^2 \\
&\quad + (p_{235} p_{346} p_{124} p_{256} - p_{234} p_{356} p_{125} p_{246}) \cdot y_2 y_1 x_3 x_4 x_5 x_6 x_1 \\
&\quad + (p_{235} p_{246} p_{134} p_{356} - p_{234} p_{256} p_{135} p_{346}) \cdot y_3 y_1 x_2 x_4 x_5 x_6 x_1 \\
&\quad + (p_{245} p_{236} p_{134} p_{456} + p_{234} p_{256} p_{145} p_{346}) \cdot y_4 y_1 x_2 x_3 x_5 x_6 x_1 \\
&\quad + (p_{235} p_{246} p_{145} p_{356} - p_{245} p_{236} p_{135} p_{456}) \cdot y_5 y_1 x_2 x_3 x_4 x_6 x_1 \\
&\quad + (p_{236} p_{245} p_{146} p_{356} - p_{246} p_{235} p_{136} p_{456}) \cdot y_6 y_1 x_2 x_3 x_4 x_5 x_1 \\
&\quad + (p_{235} p_{246} p_{134} p_{156} - p_{234} p_{256} p_{135} p_{146}) \cdot y_1^2 x_2 x_3 x_4 x_5 x_6,
\end{aligned}$$

and the other polynomials in $\mathcal{F}$ arise from letting the symmetric group $\mathfrak{S}_6$ permute the indices, see [SX10, §5].

Remark 2.3.1. On first sight, the above expressions may not seem to exhibit an invariance by the symmetric group $\mathfrak{S}_6$: For example, permuting the indices 2 and 6 changes the last binomial expression appearing as the coefficient of $y_1^2 x_2 x_3 x_4 x_5 x_6$ in G_1. However, modulo the Plücker relations the two differing expressions become equal:

$$p_{235}p_{246}p_{134}p_{156} - p_{234}p_{256}p_{135}p_{146} = p_{356}p_{246}p_{134}p_{125} - p_{346}p_{256}p_{135}p_{124}.$$

In fact, systematically exploiting this observation, we can for each of the binomial expressions above generate other binomial expressions equivalent modulo Plücker relations. Knowing several equivalent expressions for the coefficients turns out useful in the computations we describe below.

The binomial

$$q := p_{235}p_{246}p_{134}p_{156} - p_{234}p_{256}p_{135}p_{146}$$

plays a special role: It takes a non-zero value, as its vanishing would have the geometric meaning that the columns of the matrix M describe six points lying on a conic, [RSS16, (2.7)]. Moreover, each Plücker coordinate is non-zero, as no three of the six points were allowed to lie on a line.

This all gives rise to the following setup: Consider the affine cone in $\mathbb{A}^{20}_{\mathbb{C}}$ over the Grassmannian $\mathrm{Gr}(3,6)$ in its Plücker embedding. Re-embedding it into $\mathbb{A}^{21}_{\mathbb{C}}$ with the additional coordinate given by the binomial q and intersecting with the torus $(\mathbb{C}^*)^{21}$, we obtain a variety which we denote by S. We denote by $A = \mathbb{C}[p_{123}^{\pm}, \dots, p_{456}^{\pm}, q^{\pm}]/I_S$ its coordinate ring. Consider the morphism

$$\Phi\colon \mathbb{A}^{12}_S \xrightarrow{\mathcal{F}} \mathbb{A}^{27}_S$$

given by the polynomial expressions for $\mathcal{F}$ from above. This morphism describes the family of total coordinate spaces of smooth cubic surfaces as a family of unirational varieties as in the general setup in Section 2.1.

Tackling Problem 2.1.5 in this setting was suggested as an open research question in [SX10, Problem 5.4]. In [BCDFM17, Proposition 6.4], it was shown that not all moneric classes are full-dimensional, so directly applying Algorithm 2.2.3 will not necessarily lead to a complete classification of toric degenerations.

Nevertheless, carrying through a classification of all degenerations arising from *full-dimensional* moneric classes is already interesting in itself and can lead to further insights. In the following theorem, we work this out. For computational convenience, we restricted to those moneric classes that map full-dimensionally onto the tropical Grassmannian $\mathrm{TGr}(3,6)$, the tropical variety associated to the affine cone over $\mathrm{Gr}(3,6)$ in its Plücker embedding. This corresponds to replacing $\mathrm{Trop}(S)$ by $\mathrm{TGr}(3,6)$ in Algorithm 2.2.3, yet we do not expect this restriction to be essential for the feasibility of the computations.

Theorem 2.3.2. *The tropical Grassmannian* $\mathrm{TGr}(3,6)$ *can be given a fan structure with* f*-vector*

$$(0,\ 1,\ 987,\ 25605,\ 245280,\ 1195815,\ 3380380,\ 5827950,\ 6076590,\ 3524580,\ 870840)$$

such that every K*-valued point of* $\mathrm{Gr}(3,6)$ *tropicalizing to the relative interior of a maximal cone is moneric for the family of total coordinate spaces. There are* 32880 *distinct full-dimensional moneric classes, which fall into* 78 *orbits under the* $\mathfrak{S}_6$*-symmetry. Among these, there are* 38 *orbits of Khovanskii classes, describing toric degenerations of total coordinate spaces.*

Proof. The direct calculation of the fan structure Σ with Algorithm 2.2.3 is a computationally difficult task, as evidenced by its large f-vector. However, by exploiting symmetries at several stages and making use of parallel computations, we were able to carry out the computation with the computer algebra system Singular [DGPS19] using its interface to Gfan [Jen17] and its library for computing tropical varieties [JMMR19]. All of our computer code, extensive classification output and supplementary material is made available at

`https://software.mis.mpg.de`.

We proceed as follows: We consider $\mathrm{TGr}(3,6)$ with the fan structure described in [SS04]. To determine Σ, we compute for every single maximal cone in $\mathrm{TGr}(3,6)$ its subdivision in Σ. Up to $\mathfrak{S}_6$-symmetry, $\mathrm{TGr}(3,6)$ consists only of seven maximal cones [SS04]. These seven cones themselves are stabilized under a subgroup of $\mathfrak{S}_6$ of order 48, 24, 8, 8, 4, 3 and 2, respectively. In the following steps of the computation, we fix one of these seven maximal cones C and its symmetry group $\mathfrak{S}(C) \subseteq \mathfrak{S}_6$.

Most of the coefficients c_{ij} in $\mathcal{F} = \{E_1, \dots, G_6\}$ are monomial expressions in the Plücker coordinates. For these c_{ij}, the tropical graph $T(c_{ij})$ is simply the image of $\mathrm{TGr}(3,6)$ under the linear embedding $\mathbb{R}^{20} \hookrightarrow \mathbb{R}^{21}$, $v \mapsto (v,\ \mathrm{trop}(c_{ij})(v))$.

The remaining coefficients can be expressed as $c_{ij} = f \pm g$ with f, g monomials in $\mathbb{C}[p_{123}, \dots, p_{456}]$, possibly in several different ways, see Remark 2.3.1. If the linear functions $\mathrm{trop}(f)$, $\mathrm{trop}(g)\colon \mathbb{R}^{20} \to \mathbb{R}$ do not coincide on the entire cone C for at least one of the equivalent binomial expressions for c_{ij} obtained from Remark 2.3.1, then the lower envelope of $T(c_{ij}) \cap \pi^{-1}(C)$ is the graph of the piece-wise linear function $\min\{\mathrm{trop}(f), \mathrm{trop}(g)\}|_C \colon C \to \mathbb{R}$. Computations show this to happen in fact in all of the cases except for one: For one of the seven cones C, there exists one coefficient c_{ij} for which all known expressions as binomials $f \pm g$ satisfy $\mathrm{trop}(f)|_C = \mathrm{trop}(g)|_C$. For this one choice of C and c_{ij}, we computed the tropical variety $T(c_{ij})$ explicitly in Singular and read off that the lower envelope of $T(c_{ij}) \cap \pi^{-1}(C)$ is in fact the graph of a linear function $C \to \mathbb{R}$.

From these descriptions of the lower envelopes of $T(c_{ij}) \cap \pi^{-1}(C)$ for all c_{ij}, we compute $\Lambda_i \cap \pi^{-1}(C)$ for each $i = 1, \ldots, 27$ and the induced subdivisions $\Sigma_i|_C$ of C. Here, the computation reveals that the relative interior of each maximal cone of Λ_i intersects only one of the tropical graphs $T(c_{i1}), \ldots, T(c_{ir_i})$. Our next step is to compute the common refinement of $\Sigma_1|_C, \ldots, \Sigma_{27}|_C$, which is combinatorially large, so we find it imperative to exploit symmetries and use parallelization in the computation, as described in the following algorithm. Here, each subdivision $\Sigma_i|_C$ is understood as a set of maximal cones.

Input: The subdivisions $\Sigma_1|_C, \ldots, \Sigma_{27}|_C$ and the symmetry group $\mathfrak{S}(C)$
Output: $\Sigma|_C$, the common refinement of $\Sigma_1|_C, \ldots, \Sigma_{27}|_C$

```
Ω := {C}.
for i from 1 to 27 do
    for all σ ∈ Σ_i|_C do          // To be carried out in parallel
        Compute Ω_σ := {σ ∩ σ' | σ' ∈ Ω such that dim(σ ∩ σ') = dim(σ)}.
    Ω := {τ_1, ..., τ_k}, representatives for the S(C)-orbits of ⋃_{σ∈Σ_i|_C} Ω_σ.
return ⋃_{g∈S(C)} {g · σ | σ ∈ Ω}.
```

With this common refinement algorithm, we were able to compute the maximal cones in the subdivision $\Sigma_i|_C$ for each of the seven maximal cones C representing the $\mathfrak{S}_6$-symmetry classes in $\mathrm{TGr}(3,6)$. After letting the symmetric group act on these results and computing the fan structure induced by the set of maximal cones, we obtain Σ and read off the f-vector claimed above.

During the computation, we remember at every stage for each maximal cone the choice of initial monomials it corresponds to. At the end, the computation reveals that the 870840 maximal cones only describe 32880 distinct moneric classes, and they fall into only 78 orbits under the $\mathfrak{S}_6$-symmetry. More detailed statistics on the subdivisions for each of the seven symmetry classes of cones are presented in Figure 2.1. An explicit description of the subdivision for one of the seven maximal cones C in $\mathrm{TGr}(3,6)$ is contained among the discussion in Section 2.4.

It remains to test which of the moneric classes $(\mathbf{x}^{\alpha_1}\mathbf{y}^{\beta_1}, \ldots, \mathbf{x}^{\alpha_{27}}\mathbf{y}^{\beta_{27}})$ form a Khovanskii basis. For this, we proceed with the method presented in [SX10, Theorem 5.1] and consider the binomial ideal J of relations among these 27 monomials. As in the proof of Proposition 2.1.4, it suffices to compare the Hilbert function of $\mathbb{C}[\mathbf{x},\mathbf{y}]/J$ to that of the Cox ring of a smooth cubic surface. We are in a multigraded setting: The Cox–Nagata ring is graded by $\mathbb{Z}^7$ and this grading can be induced by a $\mathbb{Z}^7$-grading on $\mathbb{C}[\mathbf{x},\mathbf{y}]$ which is preserved under the degenerations we consider, see [SX10; BCDFM17]. In particular, we may consider the $\mathbb{Z}^7$-graded Hilbert function of $\mathbb{C}[\mathbf{x},\mathbf{y}]/J$ and compare it to the explicit formula for Cox rings given in [SX10, Corollary 5.2]. In fact, it is sufficient to compare the values of the multigraded Hilbert functions at the multidegrees of the minimal binomial generating set of J. Carrying

Type of cone C	FFFGG	EEEE	EEFF1	EEFF2	EFFG	EEEG	EEFG
size of $\mathfrak{S}_6$-orbit $\mathfrak{S}_6 \cdot C$	15	30	90	90	180	240	360
order of symmetry group $\mathfrak{S}(C)$	48	24	8	8	4	3	2
$\#\{\text{cones in } \Sigma\vert_C\}$	1240	864	1248	860	806	830	812
$\#\{\text{cones in } \Sigma\vert_C\}/\mathfrak{S}(C)$	38	36	205	142	259	278	460
$\#\{\text{moneric classes from } C\}/\mathfrak{S}(C)$	31	35	162	135	253	274	449
$\#\{\text{moneric classes from } \mathfrak{S}_6 \cdot C\}/\mathfrak{S}_6$	25	17	45	47	60	37	64
$\#\{\text{moneric classes from } \mathfrak{S}_6 \cdot C\}$	11040	7080	16080	17880	24600	17040	26400

FIGURE 2.1. Statistics on moneric classes arising from Σ

this out for representatives of the 78 moneric symmetry classes, we find that 38 of them are Khovanskii. In particular, we obtain 38 different combinatorial types of toric degenerations of total coordinate spaces of smooth cubic surfaces. □

2.4. Beyond full-dimensional classes

In Section 2.3, we have classified the toric degenerations of total coordinate spaces of smooth cubic surfaces arising from Khovanskii classes which are full-dimensional in $\mathrm{TGr}(3,6)$. To complete the classification asked for in [SX10, Problem 5.4], it would also be necessary to classify Khovanskii classes whose tropical dimension in $\mathrm{TGr}(3,6)$ is strictly smaller than $\dim(\mathrm{Gr}(3,6))$. Proposition 2.2.2 shows how this could in theory be achieved with Algorithm 2.2.3 by considering the re-embedding of the affine cone over the Grassmannian with all binomials in the Plücker coordinates that appear in the expressions $G_1, \ldots, G_6$. This would be a variety in a 51-dimensional ambient space whose tropicalization is computationally out of reach. In [BCDFM17, §6], the problem of finding a more tractable re-embedding of the base carrying full information about moneric classes was considered, yet no positive conclusion could be reached.

We believe that based on the fan structure described in Theorem 2.3.2, a refined approach examining in detail the behavior along non-maximal cones can lead to a complete answer to [SX10, Problem 5.4]. In this section, we present partial results in this direction.

The fan structure on $\mathrm{TGr}(3,6)$ described in [SS04] consists of 1005 maximal cones which fall into seven orbits under the $\mathfrak{S}_6$-action. In the following, we focus our attention to one of these seven symmetry classes, called EEEE in [SS04]. Going beyond Theorem 2.3.2, we study all moneric points tropicalizing to the relative interior of cones of this symmetry class, also classifying the behavior along the lower-dimensional cones in the subdivision. We suspect that a similar approach would be useful for treating all cones in $\mathrm{TGr}(3,6)$.

By [SS04], a cone C representing the symmetry class EEEE is the image of the cone $\mathbb{R}^6 \times \mathbb{R}^4_{\geq 0}$ under the linear map $\varphi\colon \mathbb{R}^6 \times \mathbb{R}^4 \hookrightarrow \mathbb{R}^{20}$,

$$\varphi(\lambda,\mu) := \sum_{i<j<k} (\lambda_i+\lambda_j+\lambda_k)e_{ijk}+\mu_1 e_{123}+\mu_2 e_{145}+\mu_3 e_{246}+\mu_4 e_{356} \in \mathbb{R}^{20},$$

where the standard basis vectors of $\mathbb{R}^{20}$ are indexed in the same way as the Plücker coordinates. The symmetry group $\mathfrak{S}(C) \subseteq \mathfrak{S}_6$ of this cone C is the subgroup of $\mathfrak{S}_6$ of order 24 generated by (12)(56), (16)(25), (13)(46) and (16)(34). It is the stabilizer of

$$(e_1 \wedge e_6) \cdot (e_2 \wedge e_5) \cdot (e_3 \wedge e_4) \in \operatorname{Sym}^3 \textstyle\bigwedge^2 \mathbb{R}^6.$$

In the following, let $a \in S(K)$ be a moneric point such that its corresponding tropical point in $\operatorname{TGr}(3,6)$ is $\nu(a) = \varphi(\lambda,\mu)$ for some $(\lambda,\mu) \in \mathbb{R}^6 \times \mathbb{R}^4_{>0}$. For notational simplicity, we denote the polynomials in

$$\mathcal{F}|_a = \{E_1|_a, \dots, E_6|_a, F_{12}|_a, \dots, F_{56}|_a, G_1|_a, \dots, G_6|_a\} \subseteq K[\mathbf{x},\mathbf{y}]$$

in the following just by E_i, F_{ij}, G_i. Since a is assumed to be moneric, their initial forms are monomials (with possibly non-trivial complex coefficients).

First, we observe that the coefficients in F_{16} are $p_{126}, p_{136}, p_{146}$ and p_{156}, so their valuations are $\{\lambda_1+\lambda_6+\lambda_i \mid i = 2,3,4,5\}$. Since $\operatorname{in}(F_{16})$ is a monomial, this implies that there is a unique smallest number among $\{\lambda_2,\lambda_3,\lambda_4,\lambda_5\}$. In the same way, considering the coefficients of F_{25} and F_{34} reveals that both $\{\lambda_1,\lambda_3,\lambda_4,\lambda_6\}$ and $\{\lambda_1,\lambda_2,\lambda_5,\lambda_6\}$ have a unique smallest element. Up to $\mathfrak{S}(C)$-symmetry, we may assume that $\lambda_1 < \lambda_i$ for $i = 2,\dots,5$ and that $\lambda_2 < \lambda_j$ for $j = 3,4,5$. The coefficients of F_{14} have valuations $\lambda_1+\lambda_4+\{\lambda_2,\lambda_3,\lambda_5+\mu_2,\lambda_6\}$, so $\lambda_2 \neq \lambda_6$ holds, as otherwise $\operatorname{in}(F_{14})$ would not be a monomial. This leads to two cases to consider: $\lambda_6 < \lambda_2$ and $\lambda_2 < \lambda_6$.

Case 1: $\lambda_1 < \lambda_6 < \lambda_2 < \lambda_i$ for all $i = 3,4,5$.

Considering the expression for the polynomial F_{ij}, we observe that the four coefficients of F_{ij} have valuations at least $\lambda_i+\lambda_j+\lambda_k$, where k ranges over $\{1,\dots,6\} \setminus \{i,j\}$, with equality if and only if

$$\{i,j,k\} \neq \{1,2,3\},\{1,4,5\},\{2,4,6\},\{3,5,6\}.$$

In particular, the above inequalities for $\lambda_1,\dots,\lambda_6$ uniquely determine

$$\operatorname{in}(F_{ij}) = \frac{x_2x_3x_4x_5x_6}{x_ix_j}y_1 \qquad \text{for } 1<i<j,\ (i,j) \neq (2,3),(4,5),$$

$$\operatorname{in}(F_{1i}) = \frac{x_2x_3x_4x_5}{x_i}y_6 \quad \text{for } i = 2,3,4,5 \quad \text{and} \quad \operatorname{in}(F_{16}) = x_3x_4x_5y_2.$$

For F_{23} and F_{45}, we observe that their coefficients have valuations $\lambda_2+\lambda_3+\{\lambda_1+\mu_1,\lambda_4,\lambda_5,\lambda_6\}$ and $\lambda_4+\lambda_5+\{\lambda_1+\mu_2,\lambda_2,\lambda_3,\lambda_6\}$, respectively, leading to the following four possibilities for the initial monomials $\operatorname{in}(F_{23})$, $\operatorname{in}(F_{45})$:

	$\mu_1 < \lambda_6 - \lambda_1$	$\mu_1 > \lambda_6 - \lambda_1$
$\mu_2 < \lambda_2 - \lambda_1$	$x_4x_5x_6y_1,\ x_2x_3x_6y_1$	$x_1x_4x_5y_6,\ x_2x_3x_6y_1$
$\mu_2 > \lambda_2 - \lambda_1$	$x_4x_5x_6y_1,\ x_1x_2x_3y_6$	$x_1x_4x_5y_6,\ x_1x_2x_3y_6.$

In the polynomial expression for G_i, the monomials are of the form $\frac{x_1\cdots x_6x_i}{x_jx_k}y_jy_k$, where (j,k) ranges over tuples with $j \neq k$ or $j = k = i$. Examining the coefficient of such a monomial, we observe that it has valuation at least $2\sum_{\ell\neq i}\lambda_\ell + \lambda_j + \lambda_k$. From this, we conclude that $\mathrm{in}(G_i) = x_2x_3x_4x_5x_iy_1y_6$ for $i = 2,3,4,5$, as the corresponding coefficients exactly attain this bound, while the inequalities among $\lambda_1,\ldots,\lambda_6$ force the remaining coefficients of G_i to have higher valuation.

For G_1, we observe that the coefficient of $y_1^2x_2x_3x_4x_5x_6$ is a binomial $u-v$ in the Plücker coordinates, where $\nu(v) = \nu(u)+\mu_3$. In particular, $\nu(u-v) = \nu(u) = 2\sum_i\lambda_i$, which is smaller than the valuations of all other coefficients of G_1. Hence, $\mathrm{in}(G_1) = x_2x_3x_4x_5x_6y_1^2$. Consider

$$G_6 = (u_1-u_2)x_2x_3x_4x_5x_6y_1y_6 + u_3x_3x_4x_5x_6^2y_1y_2 + u_4x_1x_2x_3x_4x_5y_6^2 + \ldots,$$

where $\{u_1,u_2,u_3,u_4\}$ are monomials in the Plücker coordinates of valuations

$$2\lambda_1 + \ldots + 2\lambda_5 + \{\lambda_1+\lambda_6+\mu_1,\ \lambda_1+\lambda_6+\mu_2,\ \lambda_1+\lambda_2+\mu_1,\ 2\lambda_6\}$$

and the remaining terms of G_6 are each of higher valuation than at least one of them. Refining the conditions imposed by the four possibilities for $\mathrm{in}(F_{23}), \mathrm{in}(F_{45})$ leads to the following possible leading monomials for G_6:

conditions on $a \in \mathbb{R}^6$, $b \in \mathbb{R}^4_{>0}$	valuations of $u_1,\ldots,u_4$	$\mathrm{in}(G_6)$
$\mu_1 > \lambda_6-\lambda_1,\ \mu_2 > \lambda_2-\lambda_1$	$\nu(u_4) < \nu(u_1),\nu(u_2),\nu(u_3)$	$x_1x_2x_3x_4x_5y_6^2$
$\mu_1 > \lambda_6-\lambda_1,\ \mu_2 < \lambda_2-\lambda_1$	$\nu(u_2) < \nu(u_4) < \nu(u_1) < \nu(u_3)$	$x_2x_3x_4x_5x_6y_1y_6$
$\mu_1 < \lambda_6-\lambda_1,\ \mu_2 > \lambda_2-\lambda_1$	$\nu(u_1) < \nu(u_2),\nu(u_3),\nu(u_4)$	$x_2x_3x_4x_5x_6y_1y_6$
$\mu_1 < \lambda_6-\lambda_1,\ \mu_2 < \lambda_2-\lambda_1,\ \mu_1 < \mu_2$	$\nu(u_1) < \nu(u_2),\nu(u_3),\nu(u_4)$	$x_2x_3x_4x_5x_6y_1y_6$
$\mu_2 < \mu_1 < \lambda_6-\lambda_1$	$\nu(u_2) < \nu(u_1) < \nu(u_3),\nu(u_4)$	$x_2x_3x_4x_5x_6y_1y_6$
$\mu_1 = \mu_2 < \lambda_6-\lambda_1$	$\nu(u_1) = \nu(u_2) < \nu(u_3),\nu(u_4)$	?

In the last case $\mu_1 = \mu_2 < \lambda_6-\lambda_1$, the binomial u_1-u_2 can attain any valuation $h \geq \mu_1 + 2\sum_{i=1}^6\lambda_i - (\lambda_6-\lambda_1)$. Indeed, the tropical graph $T(u_1-u_2) \subseteq \mathbb{R}^{21}$ contains all points of the form $(\varphi(a',b'),h')$ with $a' \in \mathbb{R}^6$, $b' \in \mathbb{R}^4_{>0}$, $b'_1 = b'_2$ and $h \geq b'_1 + 2\sum_{i=1}^6\lambda_i - (\lambda_6-\lambda_1)$. In particular, the following three cases occur under the specified condition on $h_0 := \nu(u_1-u_2) - 2\sum_{i=1}^6\lambda_i + (\lambda_6-\lambda_1) - \mu_1$:

conditions on $(a,b,h_0) \in \mathbb{R}^6 \times \mathbb{R}^4_{>0} \times \mathbb{R}_{>0}$	$\mathrm{in}(G_6)$
$\mu_1 = \mu_2 < \lambda_6-\lambda_1,\ h_0 < \lambda_2-\lambda_6,\ h_0 < (\lambda_6-\lambda_1)-\mu_1$	$x_2x_3x_4x_5x_6y_1y_6$
$\mu_1 = \mu_2 < (\lambda_6-\lambda_1)-(\lambda_2-\lambda_6),\ h_0 > \lambda_2-\lambda_6$	$x_3x_4x_5x_6^2y_1y_2$
$(\lambda_6-\lambda_1)-(\lambda_2-\lambda_6) < \mu_1 = \mu_2 < \lambda_6-\lambda_1,\ h_0 > (\lambda_6-\lambda_1)-\mu_1$	$x_1x_2x_3x_4x_5y_6^2.$

Case 2: $\lambda_1 < \lambda_2 < \lambda_i$ for all $i = 3, 4, 5, 6$.

We proceed as in the previous case, examining the restrictions imposed by the inequalities $\lambda_1 < \lambda_2 < \lambda_3, \lambda_4, \lambda_5, \lambda_6$ on the leading terms of $\mathcal{F}$. We always have

$$\mathrm{in}(F_{1j}) = \frac{x_3x_4x_5x_6}{x_j}y_2, \qquad \mathrm{in}(F_{ij}) = \frac{x_2x_3x_4x_5x_6}{x_ix_j}y_1$$

for all $j \geq 4$, $i \notin \{1, 6\}$ with $(i, j) \neq (4, 5)$ as well as

$$\mathrm{in}(G_1) = x_2x_3x_4x_5x_6y_1^2,\ \mathrm{in}(G_2) = x_2x_3x_4x_5x_6y_1y_2,\ \mathrm{in}(G_3) = x_3^2x_4x_5x_6y_1y_2,$$

while for the remaining generators, we obtain

$$\begin{aligned}
&\mathrm{in}(F_{45}) \in \{x_2x_3x_6y_1,\ x_1x_3x_6y_2\} =: X_1,\\
&\mathrm{in}(F_{12}) \in \{x_4x_5x_6y_3,\ x_3x_5x_6y_4,\ x_3x_4x_6y_5,\ x_3x_4x_5y_6\} =: X_2,\\
&\mathrm{in}(F_{13}) \in \{x_4x_5x_6y_2,\ x_2x_5x_6y_4,\ x_2x_4x_6y_5,\ x_2x_4x_5y_6\} =: X_3,\\
&\mathrm{in}(F_{23}) \in \{x_4x_5x_6y_1,\ x_1x_5x_6y_4,\ x_1x_4x_6y_5,\ x_1x_4x_5y_6\} =: X_4,\\
&\mathrm{in}(G_5) \in \{x_3x_4x_5^2x_6y_1y_2,\ x_2x_3x_5^2x_6y_1y_4,\ x_2x_3x_4x_5x_6y_1y_5,\ x_2x_3x_4x_5^2y_1y_6\} := X_5,\\
&\mathrm{in}(G_4) \in \{x_3x_4^2x_5x_6y_1y_2,\ x_2x_4^2x_5x_6y_1y_3,\ x_2x_3x_4x_5x_6y_1y_4,\ x_2x_3x_4^2x_6y_1y_5,\\
&\qquad x_2x_3x_4^2x_5y_1y_6\} =: X_6,\\
&\mathrm{in}(G_6) \in \{x_3x_4x_5x_6^2y_1y_2,\ x_2x_3x_5x_6^2y_1y_4,\ x_1x_3x_5x_6^2y_2y_4,\ x_2x_3x_4x_6^2y_1y_5,\\
&\qquad x_1x_3x_4x_6^2y_2y_5,\ x_2x_3x_4x_5x_6y_1y_6,\ x_1x_3x_4x_5x_6y_2y_6\} =: X_7.
\end{aligned}$$

Opposed to case 1, one also observes that here no cancellation of lowest order terms in binomials can affect the initial term of any generator. In particular, the initials of the generators $\mathcal{F}$ only depend on a and b, leading to a distinction of 31 cases of moneric choices, visualized in Figure 2.2. There, the choice of initial monomials from the sets $X_1, \ldots, X_7$ is specified by a 7-tuple $(i_1, \ldots, i_7)$ indicating that the i_j-th element of X_j forms the leading monomial.

In total, cases 1 and 2 combined, we obtain 39 sets of inequalities classifying all moneric subspaces inside C up to $\mathfrak{S}(C)$-symmetry. By considering their $\mathfrak{S}_6$-orbits, we obtain:

Theorem 2.4.1. *For the family of total coordinate spaces of smooth cubic surfaces, there are exactly* 7320 *moneric classes that arise from points $a \in S(K)$ whose corresponding tropicalization in* $\mathrm{TGr}(3, 6)$ *lies in the relative interior of a maximal cone of type EEEE. These classes fall into* 19 *orbits under the action of* $\mathfrak{S}_6$.

We envision that systematically expanding our approach for Theorem 2.4.1 to other cones of $\mathrm{TGr}(3, 6)$ leads to a complete classification of all moneric classes, entirely settling [SX10, Problem 5.4]. An efficient extension of Algorithm 2.2.3 to also cover lower-dimensional moneric classes would also be of interest for degenerations of other families of unirational varieties; this would be a subject worthy of further research.

	$\mu_2<\lambda_2-\lambda_1$		$\mu_2>\lambda_2-\lambda_1$	
$\lambda_4<\lambda_5,\lambda_6$, $\lambda_4<\mu_1+\lambda_1$	2222232		1222233	
$\lambda_5<\lambda_4,\lambda_6$, $\lambda_5<\mu_1+\lambda_1$	2333344		1333345	
$\lambda_6<\lambda_4,\lambda_5$, $\lambda_6<\mu_1+\lambda_1$	2444456		1444457	
$\lambda_4<\lambda_5,\lambda_6$, $\lambda_4\in\mu_1+(\lambda_1,\lambda_2)$	$\mu_2<(\lambda_2+\mu_1)-\lambda_4$ 1221232	$\mu_2>(\lambda_2+\mu_1)-\lambda_4$ 1221231	2221231	
$\lambda_5<\lambda_4,\lambda_6$, $\lambda_5\in\mu_1+(\lambda_1,\lambda_2)$	$\mu_2<(\lambda_2+\mu_1)-\lambda_5$ 1331344	$\mu_2>(\lambda_2+\mu_1)-\lambda_5$ 1331341	2331341	
$\lambda_6<\lambda_4,\lambda_5$, $\lambda_6\in\mu_1+(\lambda_1,\lambda_2)$	$\mu_2<(\lambda_2+\mu_1)-\lambda_6$ 1441456	$\mu_2>(\lambda_2+\mu_1)-\lambda_6$ 1441451	2441451	
$\lambda_4<\lambda_5,\lambda_6$, $\lambda_4\in\mu_1+(\lambda_2,\lambda_3)$	$\mu_4<\lambda_4-(\lambda_2+\mu_1)$ 1211111	$\mu_4>\lambda_4-(\lambda_2+\mu_1)$ 1211131	$\mu_4<\lambda_4-(\lambda_2+\mu_1)$ 2211111	$\mu_4>\lambda_4-(\lambda_2+\mu_1)$ 2211131
$\lambda_5<\lambda_4,\lambda_6$, $\lambda_5\in\mu_1+(\lambda_2,\lambda_3)$	$\mu_4<\lambda_5-(\lambda_2+\mu_1)$ 1311111	$\mu_4>\lambda_5-(\lambda_2+\mu_1)$ 1311141	$\mu_4<\lambda_5-(\lambda_2+\mu_1)$ 2311111	$\mu_4>\lambda_5-(\lambda_2+\mu_1)$ 2311141
$\lambda_6<\lambda_4,\lambda_5$, $\lambda_6\in\mu_1+(\lambda_2,\lambda_3)$	$\mu_4<\lambda_6-(\lambda_2+\mu_1)$ 1411111	$\mu_4>\lambda_6-(\lambda_2+\mu_1)$ 1411151	$\mu_4<\lambda_6-(\lambda_2+\mu_1)$ 2411111	$\mu_4>\lambda_6-(\lambda_2+\mu_1)$ 2411151
$\lambda_4,\lambda_5,\lambda_6>\mu_1+\lambda_3$	$\mu_4<\lambda_3-\lambda_2$ 1111111	$\mu_4>\lambda_3-\lambda_2$ 1111121	$\mu_4<\lambda_3-\lambda_2$ 2111111	$\mu_4>\lambda_3-\lambda_2$ 2111121

Figure 2.2. Classification of monerics for $\lambda_1 < \lambda_2 < \lambda_3, \lambda_4, \lambda_5, \lambda_6$.

CHAPTER 3

Computing zero-dimensional tropical varieties

Tropical varieties are piecewise linear structures which arise from polynomial equations. They appear naturally in many areas of mathematics and beyond, such as geometry [Mik05], combinatorics [AK06; FS05], and optimization [ABGJ18], as well as phylogenetics [SS04; LMY19], celestial mechanics [HM06; HJ11], and auction theory [BK19; TY19]. Wherever they emerge, tropical varieties often provide a fresh insight into existing computational problems, which is why efficient algorithms and optimized implementations are of great importance.

Computing tropical varieties from polynomial ideals is a fundamentally important yet algorithmically challenging task, requiring sophisticated techniques from computational algebra and convex geometry. Currently, Gfan [Jen17] and Singular [DGPS19] are the only two programs capable of computing general tropical varieties. Both programs rely on a traversal of the Gröbner complex as initially suggested by Bogart, Jensen, Speyer, Sturmfels, and Thomas [BJSST07], and for both programs the initial bottleneck had been the computation of so-called tropical links. Experiments suggest that this bottleneck was resolved with the recent development of new algorithms [Cha13; HR18]. However the new approaches still rely on computations that are known to be very hard, [Cha13] on elimination and [HR18] on root approximation to an unknown precision.

In this chapter, we study the computation of zero-dimensional tropical varieties, which is the key computational ingredient in [HR18], but using projections, which is the key conceptual idea in [Cha13]. We create a new algorithm for computing zero-dimensional tropical varieties that only requires a polynomial amount of field operations if we start with a Gröbner basis, and whose timings compare favorably with other implementations even if we do not. In particular, we argue that in the computation of general tropical varieties, the calculation of so-called tropical links becomes computationally insignificant compared to the Gröbner walk required to traverse the tropical variety.

Note that projections are a well-studied approach in polynomial systems solving, see [Stu02; DE05] for an overview on various techniques. Our approach can be regarded as a non-Archimedean analogue of that strategy, since tropical varieties can be regarded as zeroth-order

This chapter is based on the article [GRZ19] by Yue Ren, Leon Zhang and the author of this thesis.

approximation of the solutions in the topology induced by the valuation.

This chapter is organized as follows: In Section 3.2, we introduce a special class of unimodular transformations and study how they act on generic lexicographical Gröbner bases. In Section 3.3, we explain our main algorithm for reconstructing zero-dimensional tropical varieties from their projections, while Section 3.4 touches upon some technical details of the implementation. In Section 3.5, we compare the performance of our algorithm against the root approximation approach, while Section 3.6 analyses the complexity of our algorithm.

All algorithms have been implemented in the SINGULAR library `tropicalProjection.lib`. Together with the data for the timings, it is available at `https://software.mis.mpg.de`, and will in the future also be made available as part of the official SINGULAR distribution.

3.1. Background

For the sake of notation, we briefly recall some basic notions of tropical algebraic geometry and computational algebra that are of immediate relevance to us. In tropical geometry, our notation closely follows that of [MS15].

Convention 3.1.1. Throughout this chapter, let K be a field with non-trivial valuation $\nu\colon K^* \to \mathbb{R}$, i.e., ν is a non-zero group homomorphism with $\nu(f+g) \geq \min\{\nu(f), \nu(g)\}$ for all $f, g \in K^*$. Typical examples include:

(i) $K = \mathbb{C}(t)$, where ν maps $\alpha \in \mathbb{C}(t)^*$ to the unique integer k such that $\alpha = t^k\, f/g$ for some $f, g \in \mathbb{C}[t]$ not divisible by t,

(ii) $K = \mathbb{Q}$ with the p-adic valuation for some prime p: Here, ν maps $\alpha \in \mathbb{Q}^*$ to the unique integer k such that $\alpha = p^k\, f/g$ for some $f, g \in \mathbb{Z}$ not divisible by p,

(iii) valued field extensions of (i) and (ii).

We fix a multivariate polynomial ring $K[\mathbf{x}] := K[x_1, \ldots, x_n]$ as well as a multivariate Laurent polynomial ring $K[\mathbf{x}^{\pm}] := K[x_1^{\pm}, \ldots, x_n^{\pm}]$. Moreover, given a Laurent polynomial ideal $I \subseteq K[\mathbf{x}^{\pm}]$, we call a finite subset $G \subseteq I$ a **Gröbner basis** with respect to a monomial ordering $\prec$ on $K[\mathbf{x}]$ if G consists of polynomials and forms a Gröbner basis of the polynomial ideal $I \cap K[\mathbf{x}]$ with respect to $\prec$ in the conventional sense, see for example [GP02, §1.6]. All our Gröbner bases are reduced.

Finally, a **lexicographical Gröbner basis** will be a (reduced) Gröbner basis with respect to the lexicographical ordering $\prec_{\text{lex}}$ with $x_n \prec_{\text{lex}} \cdots \prec_{\text{lex}} x_1$.

For the purposes of this chapter, the following definition of tropical varieties in terms of coordinate-wise valuations of points in solution sets suffices.

Definition 3.1.2 (Tropical variety)**.** Let $I \subseteq K[\mathbf{x}^{\pm}]$ be a Laurent polynomial ideal. The **tropical variety** $\operatorname{Trop}(I) \subseteq \mathbb{R}^n$ is given by

$$\operatorname{Trop}(I) := \bigcup_{(L,\nu_L)} \Big\{ (\nu_L(p_1), \dots, \nu_L(p_n)) \in \mathbb{R}^n \;\Big|\; p \in (L^*)^n \text{ s.t. } f(p) = 0 \quad \forall f \in I \Big\},$$

where the union is over all valued field extensions of K, i.e., $K \subseteq L$ and $\nu_L \colon L^* \to \mathbb{R}$ is a valuation extending ν.

In a more combinatorial way, which we will exploit in Chapter 4, the tropical variety can also be described as the set of weight vectors $w \in \mathbb{R}^n$ such that the initial ideal $\operatorname{in}_w(I \cap K[\mathbf{x}])$ does not contain any monomial in $x_1, \dots, x_n$. See [MS15, Theorem 3.2.3] for the equivalence of these two descriptions and for further context. In particular, the definition above is independent of the valued extension (L, ν_L) of (K, ν), see also [MS15, Theorem 3.2.4]. In fact, up to euclidean closure, in the description of Definition 3.1.2, it suffices to consider $L = \widehat{K}^{\mathrm{al}}$, the algebraic closure of the completion of K, with the natural extension $\widehat{\nu}^{\mathrm{al}}$ of the valuation ν.

In this chapter, our focus lies on zero-dimensional ideals $I \subseteq K[\mathbf{x}^{\pm}]$, in which case $\operatorname{Trop}(I)$ is a finite set of $\deg(I)$ points if each point $w \in \operatorname{Trop}(I)$ is counted with the multiplicity corresponding to the number of solutions $p \in V_{\widehat{K}^{\mathrm{al}}}(I)$ with $\widehat{\nu}^{\mathrm{al}}(p) = w$.

In the univariate case, the tropical variety of an ideal $I = (f)$ in $K[x_1^{\pm}]$ simply consists of the negated slopes in the Newton polygon of f [Neu99, Proposition II.6.3]. Our approach for computing zero-dimensional tropical varieties of multivariate ideals is based on reducing computations to the univariate case.

Definition 3.1.3. We say that a zero-dimensional ideal $I \subseteq K[\mathbf{x}^{\pm}]$ is in **shape position** if the projection morphism onto the last coordinate $p_n \colon (K^*)^n \to K^*$, $(a_1, \dots, a_n) \mapsto a_n$ defines a closed embedding $p_n|_{V(I)} \colon V(I) \hookrightarrow K^*$.

In this chapter, we will concentrate on ideals that are in shape position. Lemma 3.1.4 shows an easy criterion to decide whether a given ideal is in shape position, while Lemma 3.1.5 shows how to coax degenerate ideals into shape position.

Lemma 3.1.4 ([CLO05, §4 Exercise 16])**.** *A zero-dimensional ideal $I \subseteq K[\mathbf{x}^{\pm}]$ is in shape position if and only if its (reduced) lexicographical Gröbner basis is of the form*

$$\text{(SP)} \qquad G = \{f_n,\, x_{n-1} - f_{n-1},\, \dots,\, x_2 - f_2,\, x_1 - f_1\}$$

for some univariate polynomials $f_1, \dots, f_n \in K[x_n]$. The polynomials f_i are unique.

Lemma 3.1.5. *Let $I \subseteq K[\mathbf{x}^{\pm}]$ be a zero-dimensional ideal. Then there exists a dense open subset $\mathcal{V} \subseteq \mathbb{R}^{n-1}$ such that for any $(u_1, \dots, u_{n-1})$ in $\mathcal{V} \cap \mathbb{Z}^{n-1}$ the unimodular transformation*

$$\Phi_u\colon \quad K[\mathbf{x}^{\pm}] \to K[\mathbf{x}^{\pm}], \quad x_i \mapsto \begin{cases} x_i & \text{if } i < n, \\ x_n \prod_{i=1}^{n-1} x_i^{u_i} & \text{if } i = n \end{cases}$$

maps I to an ideal in shape position.

PROOF. Without loss of generality, we may assume that the field K is algebraically closed. For any $u = (u_1, \dots, u_{n-1}) \in \mathbb{Z}^{n-1}$, let $f_u\colon (K^*)^n \to (K^*)^n$ be the torus automorphism induced by Φ_u, so that $V(\Phi_u(I)) = f_u^{-1}(V(I))$. Then the transformed ideal $\Phi_u(I)$ is in shape position if and only if the map $p_n \circ f_u^{-1}\colon (K^*)^n \to K^*$ given by $(a_1, \dots, a_n) \mapsto a_n \cdot \prod_{i=1}^{n-1} a_i^{-u_i}$ is injective on the finite set $V(I)$.

For $a \in (K^*)^n \setminus \{(1, \dots, 1)\}$, the set

$$N_a := \{w \in \mathbb{Z}^n \mid a_1^{w_1} \dots a_n^{w_n} = 1\}$$

is a $\mathbb{Z}^n$-sublattice of positive corank. Hence,

$$W_a := \{w \in \mathbb{R}^{n-1} \mid (w_1, \dots, w_{n-1}, -1) \in N_a \otimes_{\mathbb{Z}} \mathbb{R}\}$$

is a proper affine subspace of $\mathbb{R}^{n-1}$. By definition, for any two elements $b \neq c \in V(I)$, we have

$$p_n \circ f_u^{-1}(b) = p_n \circ f_u^{-1}(c) \quad \Leftrightarrow \quad (u_1, \dots, u_{n-1}) \in W_{b^{-1}c}.$$

Thus, $\Phi_u(I)$ is in shape position if and only if $(u_1, \dots, u_{n-1}) \in \mathcal{V} \cap \mathbb{Z}^{n-1}$, where $\mathcal{V} := \mathbb{R}^{n-1} \setminus \bigcup_{b \neq c \in V(I)} W_{b^{-1}c}$ is a dense open subset of $\mathbb{R}^{n-1}$. $\square$

3.2. Unimodular transformations on Gröbner bases

In this section, we introduce a special class of unimodular transformations and describe how they operate on lexicographical Gröbner bases in shape position.

Definition 3.2.1. We will consider unimodular transformations indexed by the set

$$\mathcal{U} := \{u \in \mathbb{Z}^n \mid \exists\, 1 \leq \ell \lneq n : u_\ell = -1 \text{ and } u_i \geq 0 \text{ for all } i \neq \ell\}.$$

For any $u \in \mathcal{U}$, we define a unimodular ring automorphism

$$\varphi_u\colon K[\mathbf{x}^{\pm}] \to K[\mathbf{x}^{\pm}], \quad x_i \mapsto \begin{cases} x_1^{u_1} \cdots x_{\ell-1}^{u_{\ell-1}} x_\ell^1 x_{\ell+1}^{u_{\ell+1}} \cdots x_n^{u_n} & \text{if } i = \ell, \\ x_i & \text{otherwise,} \end{cases}$$

and a linear projection

$$\pi_u\colon \mathbb{R}^n \twoheadrightarrow \mathbb{R}, \qquad (w_1, \dots, w_n) \mapsto -\sum_{i=1}^{n} u_i w_i.$$

We call such a φ_u a **slim (unimodular) transformation concentrated at ℓ.**

While our slim unimodular transformations might seem overly restrictive, the next lemma states that they are sufficient to compute arbitrary projections of tropical varieties, which is what we will need in Section 3.3.

Lemma 3.2.2. *Let φ_u be a slim transformation concentrated at ℓ. Then*

$$\pi_u(\mathrm{Trop}(I)) = \mathrm{Trop}(\varphi_u(I) \cap K[x_\ell^{\pm}]).$$

Proof. We may assume that K is algebraically closed. The ring automorphism φ_u describes a torus automorphism $f_u\colon (K^*)^n \to (K^*)^n$ with $f_u^{-1}(V(I)) = V(\varphi_u(I))$, which in turn gives rise a linear transformation $h_u\colon \mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^n$ mapping $\mathrm{Trop}(\varphi_u(I))$ to $\mathrm{Trop}(I)$:

$$\begin{array}{ccc} x_\ell \prod_{i\neq\ell} x_i^{u_i} & \longleftarrow\!\shortmid & x_\ell \\ K[\mathbf{x}^{\pm}] & \xleftarrow[\varphi_u]{} & K[\mathbf{x}^{\pm}] \end{array}$$

induces

$$\begin{array}{ccc} (z_1,\dots,z_n) & \longmapsto & (z_1,\dots,z_\ell \cdot \prod_{i\neq\ell} z_i^{u_i},\dots,z_n) \\ (K^*)^n & \xrightarrow[f_u]{} & (K^*)^n \\ \nu \big\downarrow & & \big\downarrow \nu \\ \mathbb{R}^n & \xrightarrow[h_u]{} & \mathbb{R}^n \\ (w_1,\dots,w_n) & \longmapsto & (w_1,\dots,w_\ell + \sum_{i\neq\ell} u_i w_i,\dots,w_n). \end{array}$$

Hence, with $p_\ell\colon \mathbb{R}^n \twoheadrightarrow \mathbb{R}$ denoting the projection onto the ℓ-th coordinate, we have:

$$\begin{aligned} \mathrm{Trop}(\varphi_u(I) \cap K[x_\ell^{\pm}]) &= p_\ell(\mathrm{Trop}(\varphi_u(I))) \\ &= (p_\ell \circ h_u^{-1})(\mathrm{Trop}(I)) = \pi_u(\mathrm{Trop}(I)). \qquad \square \end{aligned}$$

The following easy properties of slim unimodular transformations serve as a basic motivation for their inception. They map polynomials to polynomials, which is important when working with software which only supports polynomial data. Moreover, they preserve saturation and shape position for zero-dimensional ideals, which is valuable as saturating and restoring shape position as in Lemma 3.1.5 are two expensive operations.

Lemma 3.2.3. *For any slim transformation φ_u and any zero-dimensional ideal $I \subseteq K[\mathbf{x}^{\pm}]$, we have*

(1) $\varphi_u(K[\mathbf{x}]) \subseteq K[\mathbf{x}]$,
(2) $\varphi_u(I) \cap K[\mathbf{x}] = \varphi_u(I \cap K[\mathbf{x}])$,
(3) I *in shape position* $\Leftrightarrow$ $\varphi_u(I)$ *in shape position.*

PROOF. From the definition, it is clear that φ_u maps polynomials to polynomials, showing (1). In particular, the homomorphism $\varphi_u|_{K[\mathbf{x}]}\colon K[\mathbf{x}] \to K[\mathbf{x}]$ induces a morphism $\hat{f}_u\colon K^n \to K^n$ which on the torus $(K^*)^n$ restricts to an automorphism $f_u\colon (K^*)^n \to (K^*)^n$ satisfying $f_u^{-1}(V(I)) = V(\varphi_u(I))$. To show (2), we need to see that $\hat{f}_u^{-1}(V(I \cap K[\mathbf{x}])) \subseteq K^n$ does not have irreducible components supported outside the torus $(K^*)^n$. However, $V(I \cap K[\mathbf{x}])$ is the closure of $V(I) \subseteq (K^*)^n$ in K^n, so by zero-dimensionality of I, we have that $V(I \cap K[\mathbf{x}]) \subseteq (K^*)^n$. Since $\hat{f}_u^{-1}((K^*)^n) = (K^*)^n$, this proves (2). Finally, we note that $\varphi_u(x_n) = x_n$, so we have $p_n \circ f_u = p_n$, where $p_n\colon (K^*)^n \to K^*$ denotes the projection onto the last coordinate. Hence, $p_n|_{V(I)}$ is a closed embedding if and only if $p_n|_{V(\varphi_u(I))}$ is, proving (3). □

The following Algorithm 3.2.4 allows us to efficiently transform a lexicographical Gröbner basis of I into a lexicographical Gröbner basis of $\varphi_u(I)$. This is the main advantage of slim unimodular transformations, which we will leverage to compute $\pi_u(\mathrm{Trop}(I))$.

Algorithm 3.2.4 (Slim unimodular transformations of Gröbner bases)**.**

Input: (φ_u, G), where

- φ_u is a slim transformation concentrated at ℓ,
- $G = \{f_n,\, x_{n-1} - f_{n-1},\, \ldots,\, x_2 - f_2,\, x_1 - f_1\}$ is the lexicographical Gröbner basis of an ideal $I \subseteq K[\mathbf{x}^{\pm}]$ in shape position as in (SP).

Output: G', the lexicographical Gröbner basis of $\varphi_u(I)$.

1: In the univariate polynomial ring $K[x_n]$, compute the element f'_ℓ with $\deg(f'_\ell) < \deg(f_n)$ and

$$f'_\ell \equiv \left(x_n^{u_n} \cdot \prod_{\substack{i=1\\ i\neq \ell}}^{n-1} f_i^{u_i}\right)^{-1} \cdot f_\ell \pmod{f_n}.$$

2: **return** $G' := \{f_n,\, x_{n-1} - f_{n-1},\, \ldots,\, x_\ell - f'_\ell,\, \ldots,\, x_1 - f_1\}$.

CORRECTNESS OF ALGORITHM 3.2.4. Note that the polynomial ideal $I \cap K[\mathbf{x}]$ is saturated with respect to the product of variables $x_1 \cdots x_n$, and is by assumption generated by G. This implies that f_n is relatively prime to each f_i for $i < n$ and to x_n. In particular, the inverse in $K[x_n]/(f_n)$ showing up in the definition of f'_ℓ is well-defined. The ideal $\varphi_u(I) \subseteq K[\mathbf{x}^{\pm}]$ is generated by

$$\varphi_u(G) = \left\{f_n,\, x_{n-1} - f_{n-1},\, \ldots,\, \Big(\prod_{\substack{i=1\\ i\neq \ell}}^{n} x_i^{u_i}\Big)x_\ell - f_\ell,\, \ldots,\, x_1 - f_1\right\}.$$

Note that the expression $(\prod_{i\neq \ell} x_i^{u_i})x_\ell - f_\ell$ is equivalent to $x_\ell - f'_\ell$ modulo the ideal $(f_n,\, x_i - f_i \mid i \neq \ell, n)$. It follows that $\varphi_u(I)$ is generated by G', and it is clear that G' is a lexicographical Gröbner basis. □

3.3. Zero-dimensional tropical varieties via projections

In this section, we assemble our algorithm for computing $\mathrm{Trop}(I)$ from a zero-dimensional ideal $I \subseteq K[\mathbf{x}^{\pm}]$. This is done in two stages, see Figure 3.1: In the first stage, we project $\mathrm{Trop}(I)$ onto all coordinate axes of $\mathbb{R}^n$. In the second stage, we iteratively glue the coordinate projections together until $\mathrm{Trop}(I)$ is fully assembled.

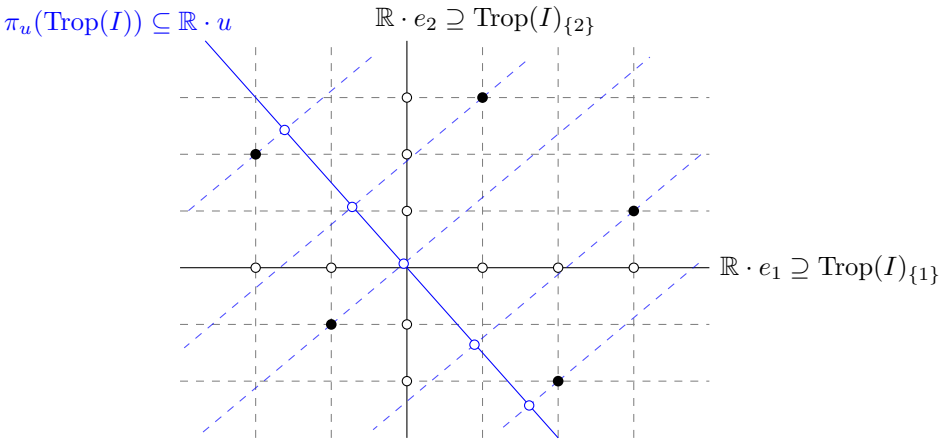


Figure 3.1. Computing zero-dimensional tropical varieties via projections.

For the sake of simplicity, all algorithms contain some elements of ambiguity to minimize the level of technical detail. To see how these ambiguities are resolved in the actual implementation, see Section 3.4. Moreover, we will only consider $\mathrm{Trop}(I)$ as points in $\mathbb{R}^n$ without multiplicities. It is straightforward to generalize the algorithms to work with $\mathrm{Trop}(I)$ as points in $\mathbb{R}^n$ with multiplicities, which is how we implement them in Singular.

The following algorithm merges several small projections into a single large projection. For clarity, given a finite subset $A \subseteq \{1, \ldots, n\}$, we use $\mathbb{R}^A$ to denote the linear subspace of $\mathbb{R}^n$ spanned by the unit vectors indexed by A and p_A to denote the projection $\mathbb{R}^n \twoheadrightarrow \mathbb{R}^A$. For $w \in \mathbb{R}^n$ and $I \subseteq K[\mathbf{x}^{\pm}]$, we denote $w_A := p_A(w) \in \mathbb{R}^A$ and $\mathrm{Trop}(I)_A := p_A(\mathrm{Trop}(I)) \subseteq \mathbb{R}^A$.

Algorithm 3.3.1 (gluing projections)**.**

Input: $(G, \mathrm{Trop}(I)_{A_1}, \ldots, \mathrm{Trop}(I)_{A_k})$, where

- G is the lexicographical Gröbner basis of a zero-dimensional ideal $I \subseteq K[\mathbf{x}^{\pm}]$ in shape position as in (SP),
- $A_1, \ldots, A_k \subseteq \{1, \ldots, n\}$ are non-empty sets.

Output: $\mathrm{Trop}(I)_A \subseteq \mathbb{R}^A$, where $A := A_1 \cup \ldots \cup A_k$.

1: Construct the candidate set

$$T := \left\{ w \in \mathbb{R}^A \;\middle|\; w_{A_i} \in \mathrm{Trop}(I)_{A_i} \text{ for } i = 1, \ldots, k \right\}.$$

2: Pick a slim transformation φ_u such that the following map is injective:
$$\pi_u|_T \colon T \to \mathbb{R}, \qquad (w_i)_{i\in A} \mapsto -\sum_{i\in A} u_i w_i.$$
3: Using Algorithm 3.2.4, transform G into a Gröbner basis G' of $\varphi_u(I)$:
$$G' := \{f_n, x_{n-1} - f_{n-1}, \dots, x_\ell - f'_\ell, \dots, x_1 - f_1\}.$$
4: Compute the minimal polynomial $\mu \in K[z]$ of $\overline{f'_\ell} \in K[x_n]/(f_n)$ over K and read off $\mathrm{Trop}(\mu) \subseteq \mathbb{R}$ from its Newton polygon.
5: **return** $\{w \in T \mid \pi_u(w) \in \mathrm{Trop}(\mu)\}$.

CORRECTNESS OF ALGORITHM 3.3.1. First, we argue that Line 2 can be realized, i.e., we show the existence of a slim unimodular transformation φ_u such that π_u is injective on the candidate set T. Pick $\ell \neq n$ and denote $B := \{1, \dots, n\} \setminus \{\ell\}$. It suffices to show that the set
$$Z := \{v \in \mathbb{R}^B_{\geq 0} \mid \pi_{v-e_\ell}|_T \text{ is injective}\} \subseteq \mathbb{R}^B$$
contains an integer point. By the definition of π_{v-e_ℓ}, we see that
$$Z = \mathbb{R}^B_{\geq 0} \setminus \bigcup_{w\neq w' \in T} H_{w-w'}, \text{ where } H_z := \Big\{ v \in \mathbb{R}^B \;\Big|\; \sum_{i\in B} z_i v_i = z_\ell \Big\}.$$
This describes Z as the complement of an affine hyperplane arrangement in $\mathbb{R}^B$ inside the positive orthant. Therefore, Z must contain an integer point.

Next, we note that the candidate set T contains $\mathrm{Trop}(I)_A$ by construction, so injectivity of $\pi_u|_T$ shows that
$$\mathrm{Trop}(I)_A = \{w \in T \mid \pi_u(w) \in \pi_u(\mathrm{Trop}(I))\}.$$
Therefore, the correctness of the output will follow from showing that $\pi_u(\mathrm{Trop}(I)) = \mathrm{Trop}(\mu)$. By Lemma 3.2.2, it suffices to prove that $\mu(x_\ell) \in K[x_\ell]$ generates the elimination ideal $\varphi_u(I) \cap K[x_\ell^{\pm}]$.

For this, we note that reducing a univariate polynomial $g \in K[x_\ell]$ with respect to the lexicographical Gröbner basis G' substitutes x_ℓ by f'_ℓ to obtain a univariate polynomial in $K[x_n]$ and then reduces the result modulo f_n. In particular, this shows that such a $g \in K[x_\ell]$ lies in the ideal $\varphi_u(I)$ if and only if $g(\overline{f'_\ell}) = 0$ in $K[x_n]/(f_n)$. Hence, the elimination ideal $\varphi_u(I) \cap K[x_\ell]$ is generated by $\mu(x_\ell)$. □

The next algorithm computes $\mathrm{Trop}(I)$ by projecting it onto all coordinate axes and gluing the projections together via Algorithm 3.3.1.

Algorithm 3.3.2 (tropical variety via projections)**.**

Input: $G = \{f_n, x_{n-1} - f_{n-1}, \dots, x_2 - f_2, x_1 - f_1\}$, the lexicographical Gröbner basis of a zero-dimensional ideal $I \subseteq K[\mathbf{x}^{\pm}]$ in shape position as in (SP).

Output: $\mathrm{Trop}(I) \subseteq \mathbb{R}^n$

1: Compute the projection onto the last coordinate:

$$\mathrm{Trop}(I)_{\{n\}} = \mathrm{Trop}(f_n).$$

2: **for** $k \in \{1, \dots, n-1\}$ **do**

3: Compute the minimal polynomial $\mu_k \in K[z]$ of $\overline{f_k} \in K[x_n]/(f_n)$ over K and read off the projection $\mathrm{Trop}(I)_{\{k\}} = \mathrm{Trop}(\mu_k)$.

4: Initialize a set of computed projections:

$$W := \{\mathrm{Trop}(I)_{\{1\}}, \dots, \mathrm{Trop}(I)_{\{n\}}\}.$$

5: **while** $W \not\ni \mathrm{Trop}(I)_{\{1,\dots,n\}}$ **do**

6: Pick projections $\mathrm{Trop}(I)_{A_1}, \dots, \mathrm{Trop}(I)_{A_k} \in W$ to be merged together such that $\mathrm{Trop}(I)_A \notin W$ for $A := A_1 \cup \dots \cup A_k$.

7: Using Algorithm 3.3.1, compute $\mathrm{Trop}(I)_A$.

8: $W := W \cup \{\mathrm{Trop}(I)_A\}$.

9: **return** $\mathrm{Trop}(I)_{\{1,\dots,n\}}$.

Correctness of Algorithm 3.3.2. Since G is the lexicographical Gröbner basis of I, the elimination ideal $I \cap K[x_n^{\pm}]$ is generated by f_n, so we indeed have the equality $\mathrm{Trop}(I)_{\{n\}} = \mathrm{Trop}(f_n)$ in Line 1. The equality $\mathrm{Trop}(I)_{\{k\}} = \mathrm{Trop}(\mu_k)$ in Line 3 holds because $\mu_k(x_\ell) \in K[x_\ell]$ generates the elimination ideal $I \cap K[x_\ell^{\pm}]$ by the same argument as in the proof of correctness of Algorithm 3.3.1.

In every iteration of the `while` loop, the set W grows in size. Since there are only finitely many coordinate sets $A \subseteq \{1, \dots, n\}$, we will after finitely many iterations compute $\mathrm{Trop}(I) = \mathrm{Trop}(I)_{\{1,\dots,n\}}$, hence the while loop terminates. □

Example 3.3.3. Consider $K = \mathbb{Q}$ equipped with the 2-adic valuation and the ideal

$$I = (\underbrace{2 + x_3 + x_3^2 + x_3^3 + 2x_3^4}_{=:f_3}, x_2 - \underbrace{2x_3}_{=:f_2}, x_1 - \underbrace{4x_3}_{=:f_1}) \subseteq K[x_1^{\pm}, x_2^{\pm}, x_3^{\pm}].$$

This ideal is in shape position by Lemma 3.1.4. From the Newton polygon of f_3, see Figure 3.2 (left), it is not hard to see that

$$\begin{aligned}
\mathrm{Trop}(I)_{\{3\}} &= \mathrm{Trop}(f_3) = \{-1, \mathbf{0}, 1\}, \\
\mathrm{Trop}(I)_{\{2\}} &= \{\lambda + 1 \mid \lambda \in \mathrm{Trop}(I)_{\{3\}}\} = \{0, \mathbf{1}, 2\}, \\
\mathrm{Trop}(I)_{\{1\}} &= \{\lambda + 2 \mid \lambda \in \mathrm{Trop}(I)_{\{3\}}\} = \{1, \mathbf{2}, 3\},
\end{aligned}$$

where points with multiplicity 2 are highlighted in bold. To merge $\mathrm{Trop}(I)_{\{1\}}$ and $\mathrm{Trop}(I)_{\{2\}}$, we consider the following projection that is injective on the candidate set $T := \mathrm{Trop}(I)_{\{1\}} \times \mathrm{Trop}(I)_{\{2\}}$:

$$\pi_{(-1,3,0)} : \quad T \longrightarrow \mathbb{R}, \quad (w_1, w_2) \longmapsto w_1 - 3w_2.$$

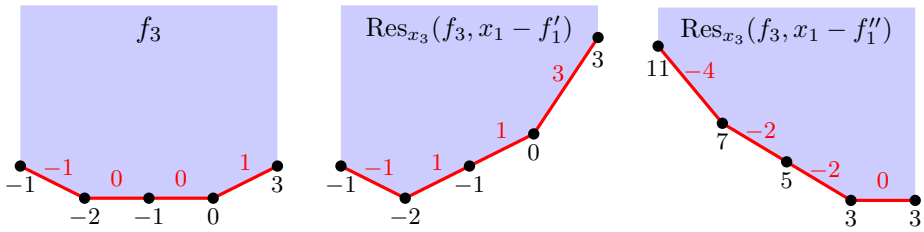


FIGURE 3.2. Newton polygons of f_3 and the two resultants in Example 3.3.3. Below each vertex is its height, above each edge is its slope.

The corresponding unimodular transformation $\varphi_{(-1,3,0)}$ sends x_1 to $x_1x_2^3$ and hence $\varphi_{(-1,3,0)}(I)$ is generated by $\{f_3, x_2 - f_2, x_1x_2^3 - 4x_3\}$, which Algorithm 3.2.4 transforms into the following lexicographical Gröbner basis:

$$\varphi_{(-1,3,0)}(I) = \Big(f_3,\ x_2 - f_2,\ x_1 - \underbrace{(\tfrac{1}{4}x_3^3 - \tfrac{3}{8}x_3^2 - \tfrac{1}{8}x_3 - \tfrac{1}{8})}_{=:f_1'}\Big).$$

The minimal polynomial of $\overline{f_1'}$ in $K[x_3]/(f_3)$ over K can be computed as the resultant

$$\mathrm{Res}_{x_3}(f_3, x_1 - f_1') = 8x_1^4 + 3x_1^3 + \tfrac{7}{2}x_1^2 + \tfrac{3}{4}x_1 + \tfrac{1}{2}.$$

Figure 3.2 (middle) shows the Newton polygon of the resultant, from which we see:

$$\mathrm{Trop}(\mathrm{Res}_{x_3}(f_3, x_1 - f_1')) = \{-3, -\mathbf{1}, 1\}.$$

Thus,

$$\mathrm{Trop}(I)_{\{1,2\}} = \{(3,2), (\mathbf{2},\mathbf{1}), (1,0)\}.$$

To merge $\mathrm{Trop}(I)_{\{1,2\}}$ and $\mathrm{Trop}(I)_{\{3\}}$, we consider the following projection that is injective on the candidate set $T := \mathrm{Trop}(I)_{\{1,2\}} \times \mathrm{Trop}(I)_{\{3\}}$:

$$\pi_{(-1,0,3)}: \quad T \longrightarrow \mathbb{R}, \quad (w_1, w_2, w_3) \longmapsto w_1 - 3w_3.$$

The corresponding unimodular transformation $\varphi_{(-1,0,3)}$ sends x_1 to $x_1x_3^3$ and hence $\varphi_{(1,0,3)}(I)$ is generated by $\{f_3, x_2 - f_2, x_1x_3^3 - 4x_3\}$, which Algorithm 3.2.4 transforms into the following lexicographical Gröbner basis:

$$\varphi_{(-1,0,3)}(I) = \Big(f_3,\ x_2 - f_2,\ x_1 - \underbrace{(2x_3^3 - 3x_3^2 - x_3 - 1)}_{=:f_1''}\Big).$$

Another resultant computation yields the minimal polynomial of $\overline{f_1''} \in K[x_3]/(f_3)$ over K:

$$\mathrm{Res}_{x_3}(f_3, x_1 - f_1'') = 8x_1^4 + 24x_1^3 + 224x_1^2 + 384x_1 + 2048.$$

Figure 3.2 (right) shows the Newton polygon of the resultant, from which we see:

$$\mathrm{Trop}(\mathrm{Res}_{x_3}(f_3, x_1 - f_1'')) = \{0, \mathbf{2}, 4\},$$

and thus

$$\mathrm{Trop}(I) = \mathrm{Trop}(I)_{\{1,2,3\}} = \{(3,2,1), (\mathbf{2},\mathbf{1},\mathbf{0}), (1,0,-1)\}.$$

3.4. Implementation

In this section, we reflect on some design decisions that were made in the implementation of the algorithms from the previous section in our Singular library `tropicalProjection.lib`. While the reader who is only interested in the algorithms, their performance, and their complexity may skip this section without impeding their understanding, we thought it important to include this section for the reader who is interested in the actual implementation.

3.4.1. Picking unimodular transforms in Algorithm 3.3.1, l. 2

As $\pi_u|_T$ is injective for generic $u \in \mathcal{U}$, it seems reasonable to sample random $u \in \mathcal{U}$ until the corresponding projection is injective on the candidate set. Our implementation however iterates over all $u \in \mathcal{U}$ in increasing ℓ_1-norm until the smallest one with injective $\pi_u|_T$ is found. This is made in an effort to keep the slim unimodular transformation $\varphi_u(I)$ as simple as possible, since Lines 3–4 are the main bottlenecks of our algorithm.

3.4.2. Transforming Gröbner bases in Algorithm 3.3.1, l. 3

As mentioned before, Lines 3–4 are the main bottlenecks of our algorithm. Two common reasons why polynomial computations may scale badly are an explosion in degree or in coefficient size. The degree of the polynomials is not problematic in our algorithm, as using Algorithm 3.2.4 in Line 3 only incurs basic arithmetic operations in $K[x_n]/(f_n)$ whose elements can be represented by polynomials of degree bounded by $\deg(f_n)$, while the degree of the minimal polynomial in Line 4 also is bounded by $\deg(f_n)$. Therefore, the only aspect that needs to be controlled in our computation is the size of the coefficients.

Coefficient explosion is a common problem for computing inverses in $K[x_n]/(f_n)$ via the Extended Euclidean Algorithm [GG13, §6.1]. To make matters worse, the polynomial $\overline{h} := \overline{x_n}^{u_n} \cdot \prod_{i \neq \ell, n} \overline{f_i}^{u_i} \in K[x_n]/(f_n)$ to be inverted in Algorithm 3.2.4 usually already has large coefficients. However, we can exploit the fact that the minimal polynomial $\sum_{i=0}^{k} a_i z^i$ of $\overline{f_\ell'} \in K[x_n]/(f_n)$ is the reflection of the minimal polynomial $\sum_{i=0}^{k} a_{k-i} z^i$ of $(\overline{f_\ell'})^{-1}$. Instead of computing $\overline{f_\ell'} = \overline{h}^{-1}\overline{f_\ell}$ in Algorithm 3.2.4, it thus suffices to compute $(\overline{f_\ell'})^{-1} = \overline{h} \cdot (\overline{f_\ell})^{-1} \in K[x_n]/(f_n)$,

which is easier as $\overline{f_\ell}$ has generally smaller coefficients than $\overline{h}$ and is independent of u, so its inversion modulo f_n is much faster.

3.4.3. Computing minimal polynomials in Algorithm 3.3.1, l. 4

The computation of minimal polynomials for elements in $K[x_n]/(f_n)$ can be carried out in many different ways, for example using:

Resultants: We can compute the resultant of the two polynomials f_n and $hx_\ell - f_\ell \in K[x_\ell, x_n]$ with respect to the variable x_n by standard resultant algorithms. The minimal polynomial $\mu(x_\ell) \in K[x_\ell]$ is the squarefree part of the resultant.

Linear algebra: Let k be the smallest positive integer such that in the finite-dimensional K-vector space $K[x_n]/(f_n)$ the set of polynomials $\{\overline{h}^{d-i}\overline{f_\ell}^{\,i} \mid i = 0, \ldots, k\}$ is linearly dependent, where $d := \deg(f_n)$. We can find a linear dependence $\sum_{i=0}^{k} a_i \overline{h}^{d-i}\overline{f_\ell}^{\,i} = 0$ and conclude that $\mu = \sum_{i=0}^{k} a_{k-i} z^i$.

Gröbner bases: Note that $\{f_n, x_\ell - f'_\ell\} \subseteq K[x_\ell, x_n]$ forms a Gröbner basis with respect to the lexicographical ordering with $x_n \prec x_\ell$. We can transform this to a Gröbner basis with respect to the lexicographical ordering with $x_\ell \prec x_n$ using `FGLM` [FGLM93] and read off the eliminant $\mu(x_\ell)$ as the generator of the elimination ideal $(x_\ell - f'_\ell, f_n) \cap K[x_\ell]$.

For polynomials with small coefficients, the implementation using SINGULAR's resultants seemed the fastest, but SINGULAR's `FGLM` seems to be best when dealing with very large coefficients.

For $K = \mathbb{Q}$ however, we can use a modular approach thanks to the SINGULAR library `modular.lib` [Ste19]: It computes the minimal polynomial over $\mathbb{F}_p$ for several primes p using any of the above methods, then lifts the results to $\mathbb{Q}$. This modular approach avoids problems caused by very large coefficients and works particularly well using the method based on linear algebra from above. We can check if the lifted μ is correct by testing whether $\mu(\overline{f'_\ell}) = 0$ in $K[x_n]/(f_n)$.

3.4.4. Picking gluing strategies in Algorithm 3.3.2, l. 6

Algorithm 3.3.2 is formulated in a flexible way: Different strategies of realizing the choice of coordinate sets $A_1, \ldots, A_k$ in Line 6 can adapt to the needs of a specific tropicalization problem. The four gluing strategies that follow seem very natural and are implemented in our SINGULAR library. See Figure 3.3 for an illustration in the case $n = 5$.

`oneProjection`: Only a single iteration of the `while` loop, in which we pick $k = n$ and $A_i = \{i\}$ for $i = 1, \ldots, n$.

`sequential`: $n - 1$ iterations of the `while` loop, during which we pick $k = 2$ and $A_1 = \{1, \ldots, i\}$ and $A_2 = \{i + 1\}$ in the i-th iteration.

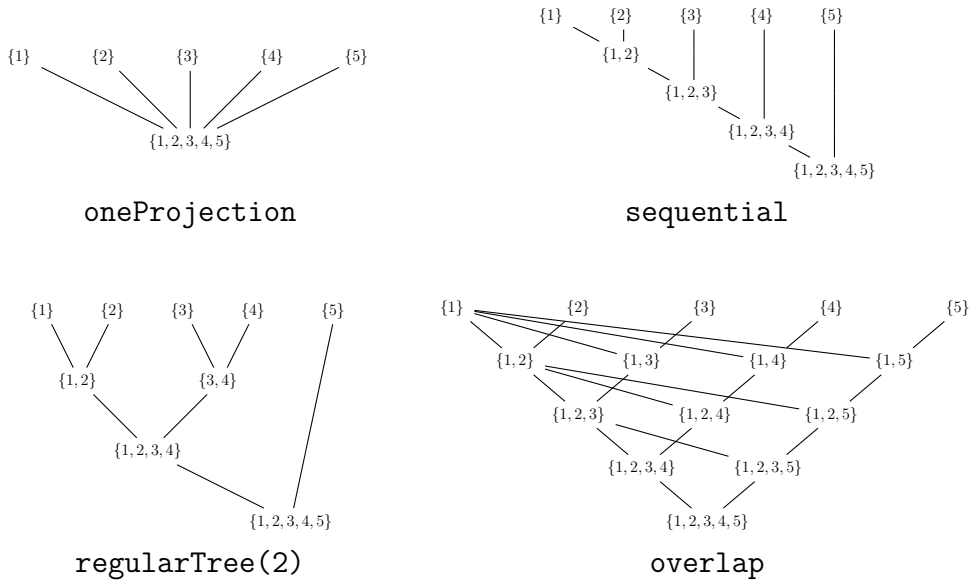


Figure 3.3. Visualization of different gluing strategies.

`regularTree(`k`)`: $n-1$ iterations of the `while` loop, which can be partially run in parallel in $\lceil \log_k n \rceil$ batches. In each batch we merge k of the previous projections.

`overlap`: $(n-1)n/2$ iterations of the `while` loop, which can be partially run in parallel in $n-1$ batches. During batch i, we pick $k=2$ and $A_1 = \{1, \ldots, i\}$, $A_2 = \{1, \ldots, i-1, j\}$ for $j > i$.

`oneProjection` is the simplest strategy, requiring only one unimodular transformation. For examples of very low degree, it is the best strategy due to its minimal overhead. For examples of higher degree d, the candidate set T in Algorithm 3.3.1 can become quite large, at worst $|T| = d^n$. This generally leads to larger $u \in \mathcal{U}$ in Line 2 and causes problems due to coefficient growth.

`sequential` avoids the problem of a large candidate set T by only gluing two projections at a time, guaranteeing $|T| \leq d^2$. This comes at the expense of computing $n-1$ unimodular transformations, but even for medium-sized instances we observe considerable improvements compared to `oneProjection`. In Section 3.6, we prove that `sequential` guarantees good complexity bounds on Algorithm 3.3.2.

`regularTree(`k`)` can achieve considerable speed-up by parallelization. Whereas every `while`-iteration in `sequential` depends on the output of the previous iteration, `regularTree(`k`)` allows us to compute all gluings in parallel in $\lceil \log_k n \rceil$ batches. The total number of gluings remains the same.

`overlap` further reduces the size of the candidate set T compared to `sequential`, while exploiting parallel computation like `regularTree(`k`)`.

It glues projections two at a time, but only those A_1 and A_2 which overlap significantly. This can lead to much smaller candidate sets T, at best $|T| = d$ which makes a unimodular transformation obsolete. The strategy `overlap` seems particularly successful in practice and is the one used for the timings in Section 3.5.

Our implementation in SINGULAR also allows for custom gluing strategies by means of specifying a graph as in Figure 3.3.

3.5. Timings

In this section we present timings of our SINGULAR implementation of Algorithm 3.3.2 for $K = \mathbb{Q}$ and the 2-adic valuation. We compare it to a MAGMA [BCP97] implementation by Avi Kulkarni which approximates the roots in the 2-adic norm using MAGMA's native `Roots` command, see the supplementary material provided in Section 3.7. While SINGULAR is also capable of the same task, we chose to compare to MAGMA instead as the latter is significantly faster due to its finite precision arithmetic over p-adic numbers. Our SINGULAR timings use the `overlap` strategy, a modular approach and parallelization with up to four threads. The SINGULAR timings we report on are total CPU times across all threads (for reference, the longest example in SINGULAR required 118 seconds total CPU time, but only 32 seconds real time). All computations were run on a server with 2 Intel Xeon Gold 6144 CPUs, 384GB RAM and Debian GNU/Linux 9.9 OS. All examples and scripts are available at `https://software.mis.mpg.de`.

3.5.1. Random Gröbner bases in shape position

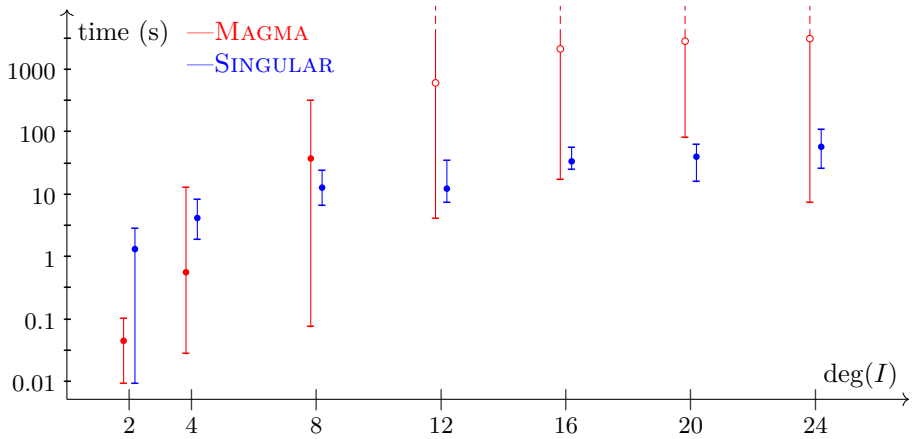
Given natural numbers d and n, a random lexicographical Gröbner basis G of an ideal $I \subseteq \mathbb{Q}[x_1, \ldots, x_n]$ of degree d in shape position will be a Gröbner basis of the form

$$G = \{f_n,\, x_{n-1} - f_{n-1},\, x_{n-2} - f_{n-2},\, \ldots,\, x_2 - f_2,\, x_1 - f_1\},$$

where $f_n, f_{n-1}, f_{n-2} \ldots, f_1$ are univariate polynomials in x_n of degree $d, d-1, d-1, \ldots, d-1$ respectively whose coefficients are of the form $2^\lambda \cdot (2k+1)$ for random $\lambda \in \{0, \ldots, 99\}$ and $k \in \{0, \ldots, 4999\}$.

Figure 3.4 shows timings for $n = 5$ and varying d. Each computation was aborted if it failed to terminate within one hour. We see that MAGMA is significantly faster for small examples, while SINGULAR scales better with increasing degree.

For many of the ideals I however, $\mathrm{Trop}(I)$ has fewer than d distinct points. This puts our algorithm at an advantage, as it allows for easier projections in Algorithm 3.3.2 Line 2. Mathematically, it is not an easy task to generate non-trivial examples with distinct tropical points. Picking f_n to have d roots with distinct valuation for example would make all roots live in $\mathbb{Q}_2$, in which case MAGMA terminates instantly.



deg(I)	2	4	8	12	16	20	24
#SINGULAR finished	100	100	100	100	100	100	100
#MAGMA finished	100	100	100	93	51	21	9
SINGULAR avg. (s)	1	5	14	19	37	44	63
MAGMA avg. (s)	0	1	41	>663	>2273	>3095	>3395

FIGURE 3.4. Timings for the randomly generated ideals in shape position.

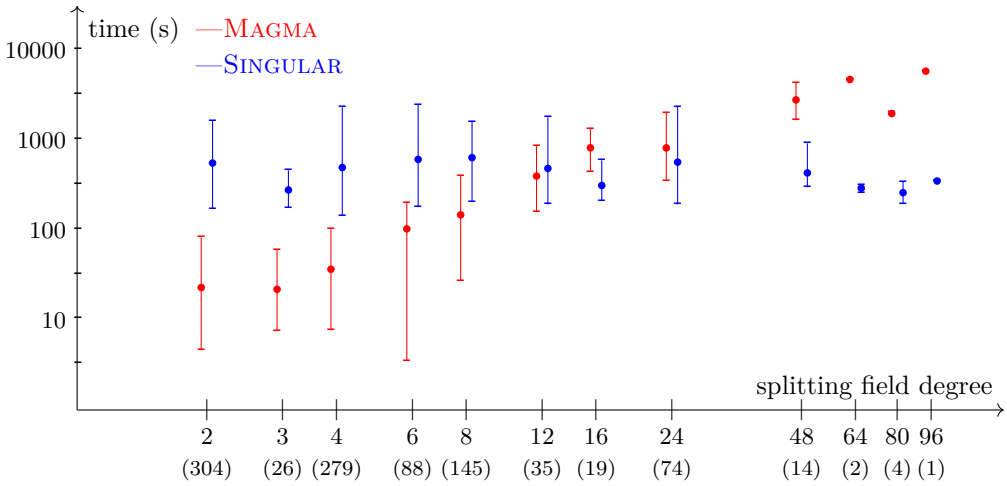
Our next special family of examples has criteria which guarantee distinct points.

3.5.2. Tropical lines on a random honeycomb cubic

Let $V(f) \subseteq \mathbb{P}^3$ be a smooth cubic surface. In [PV19], it is shown that $\mathrm{Trop}(f) \subseteq \mathbb{R}^3$ may contain infinitely many tropical lines. However, for general f whose coefficient valuations induce a honeycomb subdivision of its Newton polytope, $\mathrm{Trop}(f)$ will always contain exactly 27 distinct tropical lines [PV19, Theorem 27], which must therefore be the tropicalizations of the 27 lines on $V(f)$.

We used POLYMAKE [GJ00] to randomly generate 1000 cubic polynomials with honeycomb subdivisions whose coefficients are pure powers of 2. For each of these cubic polynomials f, we constructed the one-dimensional homogeneous ideal $\mathcal{L}_f \subseteq \mathbb{Q}[p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}]$ of degree 27 whose solutions are the lines on $V(f)$ in Plücker coordinates. Figure 3.5 shows the timings for computing $\mathrm{Trop}(L_f)$, where $L_f := \mathcal{L}_f + (p_{34} - 1)$ is a zero-dimensional ideal of degree 27. Out of our 1000 random cubics, 8 had to be discarded because L_f was of lower degree, i.e., $V(f)$ contained lines with $p_{34} = 0$.

Unsurprisingly, the SINGULAR timings are relatively stable, while the MAGMA timings heavily depend on the degree of the splitting field of L_f over $\mathbb{Q}_2$. Over $\mathbb{Q}$, the generic splitting field degree would be 51840 [EJ12]. Over $\mathbb{Q}_2$, the distinct tropical points of $\mathrm{Trop}(L_f)$ severely restrict the Galois group of the splitting field.



splitting deg.	2	3	4	6	8	12	16	24	48	64	80	96
Frequency	304	26	279	88	145	35	19	74	14	2	4	1
SINGULAR avg.	556	281	505	610	651	490	313	580	440	294	261	352
MAGMA avg.	23	22	37	104	149	403	831	830	2840	4791	1998	5935

FIGURE 3.5. Timings for the 27 tropical lines on a tropical honeycomb cubic.

3.6. Complexity

We now bound the complexity for computing a zero-dimensional tropical variety from a given Gröbner basis using Algorithm 3.3.2 with the `sequential` strategy. We show that the number of required arithmetic operations is polynomial in the degree of the ideal and the ambient dimension. Based on this, we argue that the complexity of computing a higher-dimensional tropical variety is dominated by the Gröbner walk required to traverse the Gröbner complex, as the computation of a tropical link is essentially polynomial time in the aforementioned sense.

Convention 3.6.1. For the remainder of the section, consider a zero-dimensional ideal $I \subseteq K[x_1^{\pm}, \dots, x_n^{\pm}]$ of degree d and assume that $\nu(K^*) \subseteq \mathbb{Q}$, which implies $\mathrm{Trop}(I) \subseteq \mathbb{Q}^n$.

For the sake of convenience, we recall some results on the complexity of arithmetic operations over algebraic extensions, a well-studied topic in the area of computational algebra.

Proposition 3.6.2 ([GG13, Corollary 4.6 + §4.3 + Exercise 12.10])**.** *Let $f, g \in K[z]$ be two univariate polynomials of degree $\leq d$. Then:*

(i) Addition, multiplication and inversion in $K[z]/(f)$ require $O(d^2)$ arithmetic operations in K.

(ii) Computing the k-th power of $\bar{g} \in K[z]/(f)$ requires $O(d^2 \log k)$ arithmetic operations in K.

(iii) Computing the minimal polynomial of an element $\bar{g} \in K[z]/(f)$ requires $O(d^2 \log d \log\log d)$ arithmetic operations in K.

Proposition 3.6.3. *Algorithm 3.2.4, which computes the lexicographical Gröbner basis of $\varphi_u(I)$ for some slim transformation φ_u, requires $O\Big(d^2 \sum_{u_i>0}(1+\log u_i)\Big)$ arithmetic operations in K.*

Proof. We need to count the number of field operations in which the following polynomial $f'_\ell \in K[x_n]$ can be computed:

$$f'_\ell \equiv \Big(x_n^{u_n} \cdot \prod_{\substack{i=1\\ i\neq\ell}}^{n-1} f_i^{u_i}\Big)^{-1} \cdot f_\ell \equiv \Big(x_n^{u_n} \cdot f_\ell^{-1} \cdot \prod_{\substack{i=1\\ i\neq\ell}}^{n-1} f_i^{u_i}\Big)^{-1} \pmod{f_n}.$$

Denoting $k := |\{i \in \{1,\dots,n\} \mid u_i \neq 0\}|$, this entails the following operations in $K[x_n]/(f_n)$:

- $k-1$ exponentiations $x_n^{u_n}$ and $f_i^{u_i}$ for $i \neq \ell, n$.
- 1 inversion for f_ℓ,
- $k-1$ multiplications for the product of f_ℓ^{-1}, $x_n^{u_n}$ and all other $f_i^{u_i}$,
- 1 final inversion.

An exponentiation to the power u_i requires $O(d^2 \log u_i)$ arithmetic operations in K, while every other operation requires $O(d^2)$ arithmetic operations in K by Proposition 3.6.2. In total, the number of required field operations in K is

$$O\Big(d^2 \sum_{u_i>0}(1+\log u_i) + d^2 + d^2(k-1) + d^2\Big) = O\Big(d^2 \sum_{u_i>0}(1+\log u_i)\Big). \quad \square$$

Lemma 3.6.4. *Let $X, Y \subseteq \mathbb{Q}$ be finite sets of cardinality $\leq d$. Then there exists a non-negative integer $m \leq \binom{d^2}{2}$ such that $X \times Y \to \mathbb{Q}$, $(a,b) \mapsto a - mb$ is injective. The smallest such m can be found in $O(d^4)$ arithmetic operations in $\mathbb{Q}$.*

Proof. The map $(a,b) \mapsto a - mb$ will fail to be injective if and only if there exists a pair of points in $X \times Y$ lying on an affine line with slope m. Since there are at most $\binom{d^2}{2}$ pairs of points, the statement follows by the pigeonhole principle.

We can determine all integral slopes attained by a line between any two points of $X \times Y$ with $O\Big(\binom{d^2}{2}\Big) = O(d^4)$ arithmetic operations in $\mathbb{Q}$. Picking the smallest natural number not occurring among these slopes gives the desired m. $\square$

Proposition 3.6.5. *Let $k \in \{2,\dots,n\}$ and assume that the following are known from a previous call of Algorithm 3.3.1 within Algorithm 3.3.2 running the* `sequential` *strategy:*

- $\mathrm{Trop}(I)_{\{1,\dots,k-1\}}$ *and* $\mathrm{Trop}(I)_{\{k\}}$,
- *a slim transformation φ_v concentrated at ℓ with $v_i = 0$ for $i \geq k$ such that π_v is injective on* $\mathrm{Trop}(I)_{\{1,\dots,k-1\}}$,
- *the lexicographical Gröbner basis of $\varphi_v(I)$.*

Then Algorithm 3.3.1 for gluing the two projections into $\mathrm{Trop}(I)_{\{1,\dots,k\}}$ *requires only* $O(d^2 \log d \log\log d)$ *and* $O(d^4)$ *arithmetic operations in* K *and* $\mathbb{Q}$ *respectively.*

PROOF. Applying Lemma 3.6.4 to $X := \pi_v(\mathrm{Trop}(I)_{\{1,\dots,k-1\}})$ and $Y := \mathrm{Trop}(I)_{\{k\}}$, we can compute a minimal $m \leq \binom{d^2}{2}$ such that $(a,b) \mapsto a - mb$ is injective on $X \times Y$ in $O(d^4)$ arithmetic $\mathbb{Q}$-operations. Setting $w := v + me_k$, this means that π_w is injective on the set $\mathrm{Trop}(I)_{\{1,\dots,k-1\}} \times \mathrm{Trop}(I)_{\{k\}}$.

Since $\varphi_w(I) = \varphi_u(\varphi_v(I))$ for $u := me_k - e_\ell$ and a lexicographical Gröbner basis of $\varphi_v(I)$ is already known, we may compute the lexicographical Gröbner basis of $\varphi_w(I)$ by applying Algorithm 3.2.4 to u and $\varphi_v(I)$. By Proposition 3.6.3, this requires $O(d^2 \log m) = O(d^2 \log d)$ arithmetic operations in K.

By Proposition 3.6.2, computing the minimal polynomial of the element $\overline{f_\ell} \in K[x_n]/(f_n)$ requires $O(d^2 \log d \log\log d)$ arithmetic operations in K, so the overall number of arithmetic K-operations in Algorithm 3.3.1 is also $O(d^2 \log d \log\log d)$. □

Theorem 3.6.6. *Algorithm 3.3.2, which computes the zero-dimensional tropical variety* $\mathrm{Trop}(I)$*, with the* `sequential` *strategy requires* $O(n\, d^2 \log d \log\log d)$ *and* $O(nd^4)$ *arithmetic operations in* K *and* $\mathbb{Q}$*, respectively.*

PROOF. Algorithm 3.3.2 using the `sequential` strategy consists of the following operations:

- computation of the minimal polynomials of $\overline{f_k} \in K[x_n]/(f_n)$ for $k = 1, \dots, n-1$,
- applying Algorithm 3.3.1 to $\mathrm{Trop}(I)_{\{1,\dots,k-1\}}$ and $\mathrm{Trop}(I)_{\{k\}}$ for $k = 2, \dots, n$.

We may store the information on the unimodular transformation computed in iteration $k-1$ during the computation of $\mathrm{Trop}(I)_{\{1,\dots,k-1\}}$ and this information may be used in the next iteration. Then Propositions 3.6.2 and 3.6.5 allow us to deduce the claimed bounds on arithmetic operations in Algorithm 3.3.2. □

Remark 3.6.7 (Computing positive-dimensional tropical varieties)**.** Currently, GFAN and SINGULAR are the only software systems capable of computing general tropical varieties, and both rely on a guided traversal of the Gröbner complex as introduced in [BJSST07]. Their frameworks roughly consist of two parts:

(i) the Gröbner walk to traverse the tropical variety,
(ii) the computation of tropical links to guide the Gröbner walk.

While the computation of tropical links had been a major bottleneck of the original algorithm and in early implementations, experiments suggest that it has since been resolved by new approaches [Cha13; HR18].

However, the algorithm in [Cha13, §4.2] relies heavily on projections, while [HR18, Algorithm 2.10] relies on root approximations to an unknown precision, so neither approach has good complexity bounds. In fact, [HR18, Timing 3.9] shows that the necessary precision can be exponential in the number of variables.

Algorithm 3.3.2 was designed with [HR18, Algorithm 2.10] in mind, and with Theorem 3.6.6 we argue that the complexity of calculating tropical links as in [HR18, Algorithm 4.6] is dominated by the complexity of the Gröbner basis computations required for the Gröbner walk. In the following, let $J \subseteq K[x_1^{\pm}, \dots, x_n^{\pm}]$ be a homogeneous ideal of codimension c and degree d.

(i) The Gröbner walk requires Gröbner bases of initial ideals $\mathrm{in}_w(J)$ with respect to weight vectors $w \in \mathrm{Trop}(J)$ with $\dim C_w(J) = \dim \mathrm{Trop}(J) - 1$, where $C_w(J)$ denotes the Gröbner polyhedron of J around w. Note that $\mathrm{in}_w(J)$ is neither monomial since $w \in \mathrm{Trop}(J)$ nor binomial as $\dim C_w(J) < \dim \mathrm{Trop}(J)$. Therefore, this is a general Gröbner basis computation which is commonly regarded as double exponential time.

(ii) Replacing [HR18, Algorithm 2.10] in [HR18, Algorithm 4.6] with our Algorithm 3.3.2 requires Gröbner bases of ideals of the form

$$\mathrm{in}_w(J)|_{x_1=\dots=x_{c-1}=1, x_c=\lambda} \subseteq K[x_{c+1}, \dots, x_n],$$

where w is chosen as before and $\lambda \in K$ is chosen to satisfy $\nu(\lambda) = \pm 1$. These ideals are zero-dimensional of degree at most d, and it is known that Gröbner bases of zero-dimensional ideals can be on average computed in polynomial time in the number of solutions [Lak91; LL91]. Thus, the entire computation of tropical links can on average be done in polynomial time.

3.7. Supplementary material: Magma code

The comparison of timings in Section 3.5 is based on the following Magma implementation by Avi Kulkarni. The function computes approximations of the solutions of a zero-dimensional affine scheme over a p-adic field. As input it requires a zero-dimensional ideal over the rational numbers in shape position and the completion of $\mathbb{Q}$ at a prime p. In addition to the solutions, it returns an extension of the input field over which the solutions are defined.

```
function pAdicSolutionsOverSplittingField(I, Qp)
  R := Generic(I);
  gs := GroebnerBasis(I);      //assumed to be in shape position

  u := UnivariatePolynomial(gs[#gs]);
  up := ChangeRing(u,Qp);
  K := SplittingField(up);     //main bottleneck of the algorithm

```

```
  vars_padic := Variables(ChangeRing(R,K));
  padic_rts := Roots(ChangeRing(up,K));

  function backSolve(rt)
    rt_coords := [rt];
    for i in [#gs-1 .. 1 by -1] do
      g := Evaluate(gs[i], vars_padic[1..i] cat rt_coords);
      rti := Roots(UnivariatePolynomial(g));
      assert #rti eq 1;
      Insert(~rt_coords, 1, rti[1][1]);
    end for;
    return rt_coords;
  end function;

  return [ backSolve(rt[1]) : rt in padic_rts], K;
end function;
```

CHAPTER 4

Detecting tropical defects of polynomial equations

The *tropical variety* $\mathrm{Trop}(I)$ of a polynomial ideal I is the image of its algebraic variety under component-wise valuation. Tropical varieties are commonly described as combinatorial shadows of their algebraic counterparts and arise naturally in many applications throughout mathematics and beyond. Inside mathematics for example, they enable new insights into important invariants in algebraic geometry [Mik05] or the complexity of central algorithms in linear optimization [ABGJ18]. Outside mathematics they arise as spaces of phylogenetic trees in biology [SS04; PS05], loci of indifference prizes in economics [BK19; TY19] or in the proof of the finiteness of central configurations in the $4, 5$-body problem in physics [HM06; HJ11].

As the image of an algebraic variety, a tropical variety equals the intersection of all tropical hypersurfaces of the polynomials inside the ideal. A natural question in this context is whether this equality already holds for a given finite generating set $F \subseteq I$, i.e.,

$$(\star) \qquad \mathrm{Trop}(I) = \bigcap_{f \in I} \mathrm{Trop}(f) \stackrel{?}{=} \bigcap_{f \in F} \mathrm{Trop}(f) =: \mathrm{Trop}(F).$$

We call $\mathrm{Trop}(F)$ a *tropical prevariety* and, if equality holds, F a *tropical basis*. This question is important for two main reasons. On the one hand, tropical prevarieties can provide upper dimension bounds where Gröbner bases are infeasible to compute, see [HM06; HJ11], and a tropical basis implies that this bound is actually sharp. On the other hand, the difference between a tropical variety and prevariety can be interesting in and of itself, e.g., tropical matrices of Kapranov rank r versus tropical matrices of tropical rank r [DSS05], tropical Grassmannians versus their Dressians [HJS14], or other realizability loci of combinatorial objects such as Δ-matroids [Rin12] or gaussoids [BDKS19].

Nevertheless, checking the equality in $(\star)$ is a computationally highly challenging task. Current algorithms for computing tropical varieties require a Gröbner basis for each maximal Gröbner polyehdron, of which there can be many even for tropicalization of linear spaces [JS18]. Additionally, it is known that deciding the equality in $(\star)$ is co-NP-hard, as is merely deciding whether $\mathrm{Trop}(F)$ is connected [The06].

In practice, testing the equality in $(\star)$ can fail for multiple reasons:

This chapter is based on the article [GRS19] by Yue Ren, Jeff Sommars and the author of this thesis, published in *Journal of Algebraic Combinatorics*, 2019.

(P1) Computing $\mathrm{Trop}(F)$ might not be possible due to its size or due to the number of intersections necessary to compute it.
(P2) Computing $\mathrm{Trop}(I)$ might not be feasible due to its size or due to problematic Gröbner cones in $\mathrm{Trop}(I)$ whose Gröbner bases are too hard to compute.

In this chapter, we introduce the notion of *tropical defects*, certificates for generating sets which are not tropical bases, and propose two randomized algorithms for computing tropical defects around affine subspaces of complementary dimension. An independent verification of these certificates will require a single Gröbner basis computation.

The basic idea is simple, relying on some recent results on (stable) intersections of tropical varieties [OP13; JY16]: to reduce the complexity of the computations, we (stably) intersect both sides of Equation ($\star$) with a random affine space of complementary dimension, and look for differences between the tropical variety and prevariety around it. Under certain genericity assumptions, this yields a zero-dimensional tropical variety on the left, which is not only simpler to compute than its positive-dimensional counterparts, but also implies that the tropical prevariety computation on the right can be aborted if a positive-dimensional polyhedron is found. Therefore, our algorithm operates within the realm where (P1) and (P2) are infeasible, but the following key computational ingredients are not:

(K1) computation of zero-dimensional tropical varieties in Singular [DGPS19; HR18],
(K2) computation of zero-dimensional tropical prevarieties in DynamicPrevariety [JSV17].

To a degree, our approach for finding tropical defects is related to the approach for studying tropical bases in [HT09; HT12]. In [HT09; HT12], the authors consider preimages of projections to $\mathbb{R}^{d+1}$, where $d := \dim \mathrm{Trop}(I)$. Our hyperplanes are generally given as preimages of points under a projection to $\mathbb{R}^d$, but can also be regarded as preimages of lines under a projection to $\mathbb{R}^{d+1}$. Hence our approach can be seen as a relaxation where instead of considering the preimage of the entire projection to $\mathbb{R}^{d+1}$ we only consider the parts of the projection which meet a fixed line.

In Sections 4.2 and 4.3, we present two tropical defects found using out algorithm, disproving Conjecture 5.3 in [RSS16] and Conjecture 8.4 in [BDKS19]. Note that the tropical defects were postprocessed for the ease of reproduction, see Remark 4.1.8.

Code and auxiliary materials for this chapter are available at

`https://software.mis.mpg.de`.

More information on gaussoids can be found at

`https://www.gaussoids.de`.

4.1. Tropical defects

In this section, we introduce the notion of tropical defects for generating sets of polynomial ideals, and two algorithms to find them around generic affine spaces $L = \mathrm{Trop}(H)$ of complementary dimension. To be precise, Algorithm 4.1.9 requires a generic tropicalization L, whereas Algorithm 4.1.13 merely requires a generic realization H.

Opposed to the case of zero-dimensional tropical varieties treated in Chapter 3, for the treatment in this chapter, polyhedral structures on tropical varieties arising from Gröbner basis theory are of central importance. In the following, we introduce and fix the necessary notation. Our conventions with those of [MS15], to which we again refer for a more in-depth treatment of the subject.

Convention 4.1.1. Throughout this chapter, fix an algebraically closed field K with a (possibly trivial) valuation $\nu\colon K^* \to \mathbb{R}$. We interpret the residue field

$$\kappa := \{f \in K \mid f = 0 \text{ or } \nu(f) \geq 0\} \big/ \{f \in K \mid f = 0 \text{ or } \nu(f) > 0\}$$

as a valued field equipped with a trivial valuation. Since K is algebraically closed, we may fix a group homomorphism $\mu\colon \nu(K^*) \to K^*$ such that $\nu \circ \mu = \mathrm{id}_{\nu(K^*)}$, see [MS15, Lemma 2.1.15]. We abbreviate $t^\lambda := \mu(\lambda)$ for $\lambda \in \nu(K^*)$. As before, we denote the multivariate Laurent polynomial ring $K[x_1^\pm, \ldots, x_n^\pm]$ by $K[\mathbf{x}^\pm]$.

Definition 4.1.2 (Initial forms, initial ideals)**.** Given a Laurent polynomial $f \in K[\mathbf{x}^\pm]$, say $f = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha \cdot x^\alpha$, its **initial form** with respect to a weight vector $w \in \mathbb{R}^n$ is

$$\mathrm{in}_w(f) := \textstyle\sum_{w\cdot\alpha+\nu(c_\alpha)\ \min.} \overline{t^{-\nu(c_\alpha)} c_\alpha} \cdot x^\alpha \qquad \in \kappa[\mathbf{x}^\pm].$$

For a finite set $F \subseteq K[\mathbf{x}^\pm]$ and an ideal $I \subseteq K[\mathbf{x}^\pm]$, we denote

$$\mathrm{in}_w(F) := \{\mathrm{in}_w(g) \mid g \in F\} \qquad \subseteq \kappa[\mathbf{x}^\pm],$$
$$\mathrm{in}_w(I) := (\mathrm{in}_w(g) \mid g \in I) \qquad \subseteq \kappa[\mathbf{x}^\pm].$$

Moreover, the **Gröbner polyhedron** of f, of I, or of a finite set $F \subseteq K[\mathbf{x}^\pm]$ around w is defined as

$$C_w(f) := \overline{\{v \in \mathbb{R}^n \mid \mathrm{in}_w(f) = \mathrm{in}_v(f)\}} \qquad \subseteq \mathbb{R}^n,$$
$$C_w(I) := \overline{\{v \in \mathbb{R}^n \mid \mathrm{in}_w(f) = \mathrm{in}_v(f) \text{ for all } f \in I\}} \qquad \subseteq \mathbb{R}^n,$$
$$C_w(F) := \overline{\{v \in \mathbb{R}^n \mid \mathrm{in}_w(f) = \mathrm{in}_v(f) \text{ for all } f \in F\}} \qquad \subseteq \mathbb{R}^n.$$

Both $C_w(f)$ and $C_w(F)$ are in fact convex polyhedra, while $C_w(I)$ is only guaranteed to be a convex polyhedron if I is homogeneous.

The combinatorial definition of tropical variety we will embrace in this chapter follows. The equivalence to the description in terms of component-wise valuations of algebraic solutions is a fundamental fact in tropical geometry, see [MS15, Theorem 3.2.3].

Definition 4.1.3 (Tropical variety, tropical prevariety)**.** Given a polynomial $f \in K[\mathbf{x}^{\pm}]$, an ideal $I \subseteq K[\mathbf{x}^{\pm}]$ and a finite set $F \subseteq K[\mathbf{x}^{\pm}]$, the **tropical varieties** of f and of I and the **tropical prevariety** of F are defined to be

$$\begin{aligned}\operatorname{Trop}(f) &:= \{w \in \mathbb{R}^n \mid \operatorname{in}_w(f) \text{ is not a monomial}\},\\ \operatorname{Trop}(I) &:= \{w \in \mathbb{R}^n \mid \operatorname{in}_w(f) \text{ is not a monomial for all } f \in I\},\\ \operatorname{Trop}(F) &:= \{w \in \mathbb{R}^n \mid \operatorname{in}_w(f) \text{ is not a monomial for all } f \in F\}.\end{aligned}$$

A finite generating set $F \subseteq I$ is called a **tropical basis** if

$$\operatorname{Trop}(F) = \operatorname{Trop}(I).$$

The sets $\operatorname{Trop}(f)$, $\operatorname{Trop}(I)$ and $\operatorname{Trop}(F)$ are supports of polyhedral complexes. For both $\operatorname{Trop}(f)$ and $\operatorname{Trop}(F)$, these polyhedral complexes can be chosen to be a collection of Gröbner polyhedra, and, if I is homogeneous, so can $\operatorname{Trop}(I)$.

Let $T \subseteq \mathbb{R}^n$ be the support of a polyhedral complex Σ. The **star** of T around a point $w \in \mathbb{R}^n$ is given by

$$\operatorname{Star}_w T := \{v \in \mathbb{R}^n \mid w + \varepsilon \cdot v \in T \text{ for } \varepsilon > 0 \text{ sufficiently small}\}$$

and that the **stable intersection** of T with respect to an affine subspace $H \subseteq \mathbb{R}^n$ is defined to be

$$T \cap_{\mathrm{st}} H := \bigcup_{\substack{\sigma \in \Sigma \\ \dim(\sigma + H) = n}} \sigma \cap H.$$

Example 4.1.4. Let $K = \mathbb{C}\{\{t\}\}$ be the field of complex Puiseux series and consider the ideal $I \subseteq K[x^{\pm}, y^{\pm}]$ which can be generated by either one of the following two generating sets:

$$I := (\underbrace{x + y + 1, x + t^{-1}y + 2}_{=:F_1}) = (\underbrace{x + y + 1, (t^{-1} - 1)y + 1}_{=:F_2})$$

Figure 4.1 compares the tropical prevarieties of both F_1 and F_2 with the tropical variety of I, showing that F_2 is a tropical basis while F_1 is not.

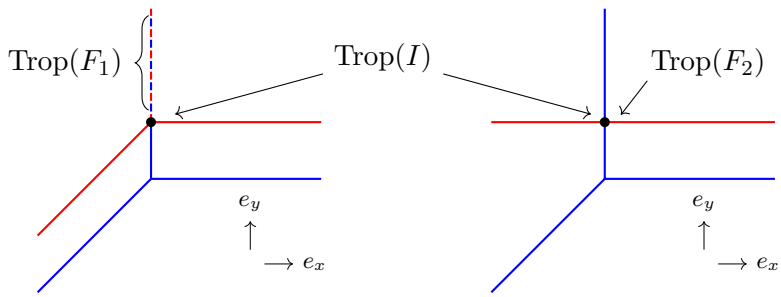


FIGURE 4.1. A tropical non-basis and a tropical basis.

For the following result, we refer to [MS15], where it is only shown for polynomial rings. However, the result extends directly to Laurent polynomial rings, since $\mathrm{in}_w(I \cap K[\mathbf{x}]) \cdot K[\mathbf{x}^{\pm}] = \mathrm{in}_w(I)$ for every ideal $I \subseteq K[\mathbf{x}^{\pm}]$.

Lemma 4.1.5 ([MS15, Lemma 2.4.6 and Corollary 2.4.10])**.** *Given an element $f \in K[\mathbf{x}^{\pm}]$ and a homogeneous ideal $I \subseteq K[\mathbf{x}^{\pm}]$, we have for any weight vectors $w, v \in \mathbb{R}^n$ and $\varepsilon > 0$ sufficiently small:*

$$\mathrm{in}_v \, \mathrm{in}_w(f) = \mathrm{in}_{w+\varepsilon \cdot v}(f) \text{ and } \mathrm{in}_v \, \mathrm{in}_w(I) = \mathrm{in}_{w+\varepsilon \cdot v}(I).$$

In particular, for a finite set $F \subseteq K[\mathbf{x}^{\pm}]$ or a homogeneous ideal $I \subseteq K[\mathbf{x}^{\pm}]$, this implies

$$\mathrm{Trop}(\mathrm{in}_w F) = \mathrm{Star}_w \, \mathrm{Trop}(F) \text{ and } \mathrm{Trop}(\mathrm{in}_w I) = \mathrm{Star}_w \, \mathrm{Trop}(I).$$

We will now introduce the notion of a tropical defect and two algorithms for finding them around affine spaces of complementary dimension to the tropical variety. For the sake of simplicity, we will restrict ourselves to affine spaces in the direction of the last few coordinates. General affine spaces can be realized via suitable unimodular transformations similar to Chapter 3, see Example 4.1.10.

Definition 4.1.6 (Tropical defects)**.** Let $I \subseteq K[\mathbf{x}^{\pm}]$ be a Laurent polynomial ideal generated by a finite set $F \subseteq I$. We call a finite tuple $\mathbf{w} := (w_0, \ldots, w_k) \in (\mathbb{R}^n)^{k+1}$ a **tropical defect** if for all $\varepsilon > 0$ sufficiently small we have

$$w_0 + \varepsilon w_1 + \cdots + \varepsilon^k w_k \in \mathrm{Trop}(F) \setminus \mathrm{Trop}(I).$$

Example 4.1.7. For $I = (F_1)$ from Example 4.1.4, the tuple (w, v) with $w := (0, 1)$ and $v := (0, 1)$ is a tropical defect, while the singleton (w) is not. On the other hand, the singleton (u) with $u := (0, 2)$ is a tropical defect, see Figure 4.2.

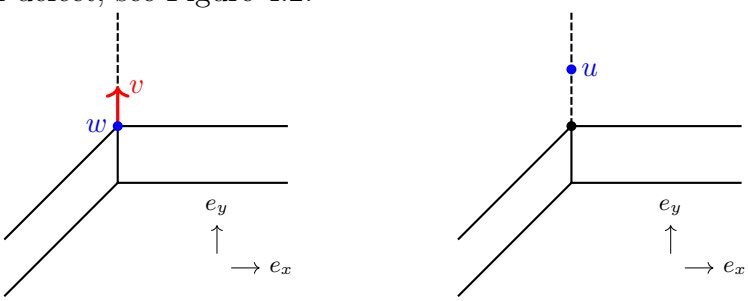


Figure 4.2. Two tropical defects.

Remark 4.1.8 (Singleton tropical defects)**.** Note that any tropical defect $(w_0, \ldots, w_k)$ of a homogeneous ideal can be transformed into a singleton tropical defect u through a single (tropical) Gröbner basis [CM19] or standard basis computation [MR19]:

One can simulate the weight vector $w_\varepsilon := w_0 + \varepsilon w_1 + \cdots + \varepsilon^k w_k$ for $\varepsilon > 0$ sufficiently small through a sequence of weights as in Lemma 4.1.5.

In particular, we can compute a Gröbner basis with respect to the sequence of weights, which gives us the inequalities and equations of the Gröbner cone $C_{w_\varepsilon}(I)$ by [MS15, proof of Proposition 2.5.2]. Any $u \in \operatorname{Relint} C_{w_\varepsilon}(I)$ is a singleton tropical defect.

For the ease of verification, the tropical defects we detect in Sections 4.2 and 4.3 have been transformed into singletons.

Algorithm 4.1.9 checks for tropical defects around affine subspaces which satisfy a strong genericity assumption.

Algorithm 4.1.9 (Testing for defects, strong genericity)**.**

Input: (F, v), where

(1) $F \subseteq K[\mathbf{x}^{\pm}]$, a finite generating set of a d-dimensional prime ideal $I \subseteq K[\mathbf{x}^{\pm}]$, and assume w.l.o.g. that

$$(*) \qquad \pi(\operatorname{Trop}(I)) = \mathbb{R}^d,$$

where $\pi\colon \mathbb{R}^n \to \mathbb{R}^d$ denotes the projection onto the first d coordinates.

(2) $v \in \mathbb{R}^d$, describing an affine subspace $H := \pi^{-1}(v) \subseteq \mathbb{R}^n$ of complementary dimension $n - d$ such that the following *strong genericity* assumption holds:

$$\text{(SG)} \qquad \operatorname{Trop}(I) \cap H = \operatorname{Trop}(I) \cap_{\mathrm{st}} H.$$

Output: $(b, \mathbf{w})$, such that

(1) if b=`true`, then $\mathbf{w}$ is a tropical defect,

(2) if b=`false`, then $\operatorname{Trop}(F) \cap H = \operatorname{Trop}(I) \cap H$ holds. (In this case, $\mathbf{w} := 0$.)

Set $F' := F \cup \{x_i - t^{v_i} \mid i = 1, \dots, d\}$ and $I' := I + (x_i - t^{v_i} \mid i = 1, \dots, d)$.
Compute the tropical prevariety $\operatorname{Trop}(F')$.
if $\exists w \in \operatorname{Trop}(F')$ with $\dim C_w(F') > 0$ **then**
  Pick $0 \neq u \in \operatorname{Lin}(C_w(F') - w)$. // $C_w(F') - w := \{v - w \mid v \in C_w(F')\}$
  return (`true`, (w, u)).
Compute the tropical variety $\operatorname{Trop}(I')$.
if $\exists w \in \operatorname{Trop}(F') \setminus \operatorname{Trop}(I')$ **then**
  return (`true`, w)
else
  return (`false`, 0)

CORRECTNESS OF ALGORITHM 4.1.9. Note that (SG) implies that $\operatorname{Trop}(I) \cap H$ is at most zero-dimensional, since H is of complementary dimension to $\operatorname{Trop}(I)$ and because of [MS15, Theorem 3.6.10], while $(*)$ ensures that it is not empty. By [OP13, Theorem 1.1], we therefore have

$$\begin{aligned}\operatorname{Trop}(I') &= \operatorname{Trop}(I + (x_i - t^{v_i} \mid i = 1, \dots, d)) \\ &= \operatorname{Trop}(I) \cap \operatorname{Trop}((x_i - t^{v_i} \mid i = 1, \dots, d)) = \operatorname{Trop}(I) \cap H.\end{aligned}$$

If the algorithm terminates at Line 5, then $C_w(F')$ is a positive-dimensional polyhedron contained in $\operatorname{Trop}(F') = \operatorname{Trop}(F) \cap H$, whereas $\operatorname{Trop}(I) \cap H$ consists of finitely many points. In particular, we have that $w + \varepsilon u \notin \operatorname{Trop}(I)$ for $\varepsilon > 0$ sufficiently small.

If the algorithm terminates at Line 8, then w is a tropical defect since

$$\begin{aligned} w \in \operatorname{Trop}(F') \setminus \operatorname{Trop}(I') &= (\operatorname{Trop}(F) \cap H) \setminus (\operatorname{Trop}(I) \cap H) \\ &\subseteq \operatorname{Trop}(F) \setminus \operatorname{Trop}(I). \end{aligned}$$

Finally, should the algorithm terminate at Line 10, then

$$\operatorname{Trop}(F) \cap H = \operatorname{Trop}(F') = \operatorname{Trop}(I') = \operatorname{Trop}(I) \cap H. \qquad \square$$

Example 4.1.10. Consider the generating set F of the following one-dimensional ideal:

$$I := (\underbrace{(x+1)(y+1), (x-1)(y+1)}_{=:F}) \subseteq \mathbb{C}[x^{\pm}, y^{\pm}],$$

and let $\pi \colon \mathbb{R}^{\{x,y\}} \to \mathbb{R}^{\{x\}}$ denote the projection onto the x-coordinate. Figure 4.3 shows the tropical variety $\operatorname{Trop}(I)$ and the tropical prevariety $\operatorname{Trop}(F)$.

Then for any $v \in \mathbb{R}$, the affine line $H_v := \pi^{-1}(v)$ satisfies (SG). Algorithm 4.1.9 yields a tropical defect if and only if $v = 0$, in which case it terminates at Line 5.

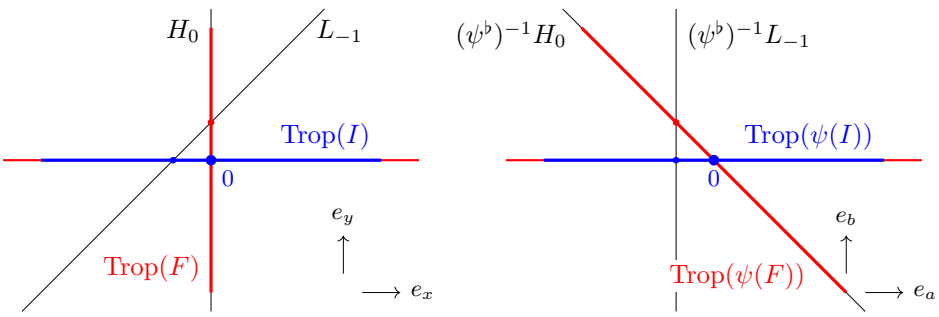


FIGURE 4.3. $\operatorname{Trop}(I) \subseteq \operatorname{Trop}(F)$ in Example 4.1.10.

We can also use arbitrary rational affine subspaces like

$$L_v := v \cdot e_x + \operatorname{Lin}(e_x + e_y)$$

by applying a unimodular transformation ψ on the ring of Laurent polynomials whose induced map $\psi^\flat$ on the weight space aligns L_v with the coordinate axes:

$$\begin{aligned} \psi &: \quad K[x^{\pm}, y^{\pm}] \xrightarrow{\sim} K[a^{\pm}, b^{\pm}], \qquad x \mapsto ab, \qquad y \mapsto b, \\ \psi^\flat &: \quad \mathbb{R}^{\{x,y\}} \xleftarrow{\sim} \mathbb{R}^{\{a,b\}}, \qquad e_x \leftarrow\!\shortmid e_a, \qquad e_x + e_y \leftarrow\!\shortmid e_b. \end{aligned}$$

This transformation yields

$$\psi(F) = \{(ab+1)(b+1), (ab-1)(b+1)\} \qquad \text{and}$$
$$(\psi^\flat)^{-1}(L_v) = v \cdot e_a + \mathrm{Lin}(e_b) \subseteq \mathbb{R}^{\{a,b\}},$$

which always satisfies (SG) and for which Algorithm 4.1.9 terminates at Line 8 if and only if $v \neq 0$, as $\mathrm{Trop}(\psi(F)) \cap (\psi^\flat)^{-1}(L_v)$ consists of two points of which only one belongs to the tropical variety $\mathrm{Trop}(\psi(I))$, see Figure 4.3.

Example 4.1.11. Consider the generating set F of the following one-dimensional ideal:

$$I := (\underbrace{x+z+2, y+z+1}_{=:F}) \subseteq \mathbb{C}[x^\pm, y^\pm, z^\pm],$$

and let $\pi\colon \mathbb{R}^{\{x,y,z\}} \to \mathbb{R}^{\{x\}}$ denote the projection onto the x-coordinate. Figure 4.4 shows $\mathrm{Trop}(I)$ as well as $\mathrm{Trop}(F)$. Consider the plane $H_v := \pi^{-1}(v)$ for some $v \in \mathbb{R}$. Note that while any H_v with $v \neq 0$ satisfies (SG), only H_v with $v > 0$ yields a tropical defect in Algorithm 4.1.9, Line 5.

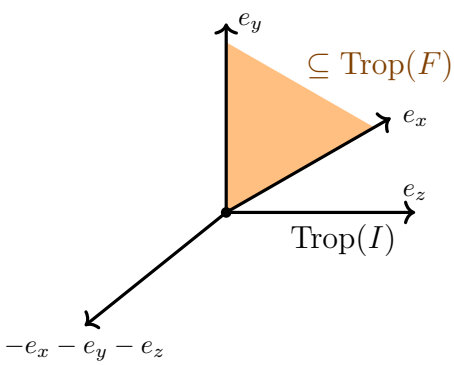


FIGURE 4.4. $\mathrm{Trop}(I) \subseteq \mathrm{Trop}(F)$ from Example 4.1.11.

Remark 4.1.12 (Strong genericity)**.** In Algorithm 4.1.9, the strong genericity assumption (SG) is only required for the correctness of the output at Line 5. If the algorithm does not terminate at Line 5, then (SG) must hold because $\mathrm{Trop}(F) \cap H = \mathrm{Trop}(F')$ is zero-dimensional, and hence so is $\mathrm{Trop}(I) \cap H \subseteq \mathrm{Trop}(F) \cap H$. This implies that for $\lambda_i \in K$ generic with $\nu(\lambda_i) = v_i$, we have

$$\mathrm{Trop}(I) \cap H = \mathrm{Trop}(I + (x_i - \lambda_i)) = \mathrm{Trop}(I) \cap_{\mathrm{st}} H,$$

where the first equality holds by [OP13, Theorem 1.1], and the second equality holds by [MS15, Theorem 3.6.1].

One possibility to ascertain whether (SG) holds upon termination at Line 5 is to compute the Gröbner polyhedron $C_w(I)$, if I is homogeneous. However, that requires a tropical Gröbner basis or standard basis, and hence might not be viable for large examples.

In practice, affine subspaces satisfying the strong genericity assumption induce several problems, see Remark 4.1.16. This is why we introduce Algorithm 4.1.13, which relies on a weakened genericity assumption. Note that, compared to Algorithm 4.1.9, Algorithm 4.1.13 requires the computation of $\mathrm{Trop}(\mathrm{in}_w(F))$ for some $w \in \mathrm{Trop}(F) \cap H$ at Line 5. This is unproblematic however, since $\mathrm{in}_w(f)$ has fewer terms than f for all $f \in F$, so that $\mathrm{Trop}(\mathrm{in}_w(f))$ will be simpler than $\mathrm{Trop}(f)$. In fact, generically $\mathrm{in}_w(f)$ will be a binomial and $\mathrm{Trop}(\mathrm{in}_w(f))$ a linear space.

Algorithm 4.1.13 (Testing for defects, weak genericity)**.**

Input: (F, λ), where

(1) $F \subseteq K[\mathbf{x}^{\pm}]$, a finite generating set of a d-dimensional prime ideal $I \subseteq K[\mathbf{x}^{\pm}]$, and assume w.l.o.g. that

$$\pi(\mathrm{Trop}(I)) = \mathbb{R}^d,$$

where $\pi\colon \mathbb{R}^n \to \mathbb{R}^d$ denotes the projection onto the first d coordinates.

(2) $\lambda \in (K^*)^d$, describing an affine subspace

$$H := \mathrm{Trop}(\{x_i - \lambda_i \mid i = 1, \dots, d\}) \subseteq \mathbb{R}^n$$

of complementary dimension $n - d$ such that the following *weak genericity* assumption holds:

$$\text{(WG)} \qquad \mathrm{Trop}(I + (x_i - \lambda_i \mid i = 1, \dots, d)) = \mathrm{Trop}(I) \cap_{\mathrm{st}} H.$$

Output: $(b, \mathbf{w})$, such that

(1) if b=`true`, then $\mathbf{w}$ is a tropical defect,

(2) if b=`false`, then $\mathrm{Trop}(F) \cap_{\mathrm{st}} H = \mathrm{Trop}(I) \cap_{\mathrm{st}} H$ holds. (In this case, $\mathbf{w} := 0$.)

1: Set $H := \mathrm{Trop}(\{x_i - \lambda_i \mid i = 1, \dots, d\})$ and $F' := F \cup \{x_i - \lambda_i \mid i = 1, \dots, d\}$.
2: Compute the tropical prevariety $\mathrm{Trop}(F')$. // $\mathrm{Trop}(F') = \mathrm{Trop}(F) \cap H$
3: Initialize $\Delta := \emptyset$. // *Δ will consist of tuples of weight vectors*
// *first entry: weight vector in* $\mathrm{Trop}(F) \cap_{\mathrm{st}} H$
// *further entries: bookkeeping of original cone in* $\mathrm{Trop}(F)$
4: **for** $w \in \mathrm{Trop}(F')$ with $\dim C_w(F') = 0$ **do**
5: Compute $\mathrm{Trop}(\mathrm{in}_w F)$.
6: **if** $\exists u \in \mathrm{Trop}(\mathrm{in}_w F) :\ \dim C_u(\mathrm{in}_w F) > d$ **then**
7: Let $v_1, \dots, v_k$ be a basis of $\mathrm{Lin}(C_u(\mathrm{in}_w F))$.
8: **return** (`true`, $(w, u, v_1, \dots, v_k)$).
9: **if** $\exists u \in \mathrm{Trop}(\mathrm{in}_w F)$ with $\dim(C_u(\mathrm{in}_w F) + H) = n$ **then**
10: Let $v_1, \dots, v_d$ be a basis of $\mathrm{Lin}(C_u(\mathrm{in}_w F))$.
11: $\Delta := \Delta \cup \{(w, u, v_1, \dots, v_d)\}$.
12: Compute $\mathrm{Trop}(I')$, where $I' := I + (x_i - \lambda_i \mid i = 1, \dots, d)$.
13: **if** $\exists (w, u, v_1, \dots, v_d) \in \Delta$ such that $w \notin \mathrm{Trop}(I')$ **then**

14: **return** (`true`, $(w, u, v_1, \ldots, v_d)$).
15: **else**
16: **return** (`false`, 0).

Correctness of Algorithm 4.1.13. Suppose that the algorithm terminates at Line 8. By Lemma 4.1.5, there exists $\delta > 0$ with

$$D := \{w + \varepsilon u + \varepsilon^2 v_1 + \cdots + \varepsilon^{k+1} v_k \mid 0 < \varepsilon < \delta\} \subseteq \operatorname{Trop}(F).$$

Because any infinite subset of D has affine span $w + \operatorname{Lin}(C_u(\operatorname{in}_w F))$ of dimension $k > d = \dim \operatorname{Trop}(I)$, any polyhedron on $\operatorname{Trop}(I)$ will have a finite intersection with D. In particular, this implies that $w + \varepsilon u + \varepsilon^2 v_1 + \cdots + \varepsilon^{k+1} v_k \notin \operatorname{Trop}(I)$ for $\varepsilon > 0$ sufficiently small.

Suppose the algorithm terminates at Line 14. Again, Lemma 4.1.5 shows that there exists $\delta > 0$ such that

$$D := \{w + \varepsilon u + \varepsilon^2 v_1 + \cdots + \varepsilon^{d+1} v_d \mid 0 < \varepsilon < \delta\} \subseteq \operatorname{Trop}(F).$$

Any infinite subset of D has affine span $w + \operatorname{Lin}(C_u(\operatorname{in}_w F))$, which intersects H stably. We have $w \notin \operatorname{Trop}(I') = \operatorname{Trop}(I) \cap_{\mathrm{st}} H$ by Assumption (WG), so any polyhedron on $\operatorname{Trop}(I)$ around w can only have a finite intersection with D. In particular, this implies that $w + \varepsilon u + \varepsilon^2 v_1 + \cdots + \varepsilon^{k+1} v_k \notin \operatorname{Trop}(I)$ for $\varepsilon > 0$ sufficiently small.

Finally, suppose the algorithm terminates at Line 16. Since $\operatorname{Trop}(F)$ contains $\operatorname{Trop}(I)$, we always have $\operatorname{Trop}(F) \cap_{\mathrm{st}} H \supseteq \operatorname{Trop}(I) \cap_{\mathrm{st}} H$. For the converse, assume there exists a weight $w \in \operatorname{Trop}(F) \cap_{\mathrm{st}} H$ not contained in $\operatorname{Trop}(I) \cap_{\mathrm{st}} H$. Let $C_u(F) \subseteq \operatorname{Trop}(F)$ be a Gröbner polyhedron of the prevariety with $w \in C_u(F) \cap H$ and $\dim(C_u(F) + H) = n$, which necessarily implies $\dim C_u(F) \geq d$. If $\dim C_u(F) > d$, then $\dim C_u(\operatorname{in}_w(F)) > d$ and we would have terminated at Line 8. If $\dim C_u(F) = d$, then w appears as the first entry of some tuple in Δ by Lemma 4.1.5 and Lines 9 to 11, hence we would have terminated at Line 14, as $\operatorname{Trop}(I') = \operatorname{Trop}(I) \cap_{\mathrm{st}} H$ by Assumption (WG). □

Remark 4.1.14 (Weak genericity)**.** If Algorithm 4.1.13 terminates at Line 8, then the output is correct even if the input did not satisfy the weak genericity assumption (WG), since a polyhedron in $\operatorname{Trop}(F)$ of too large dimension was found. On the other hand, the correctness of a tropical defect output at Step 14 does depend on the assumption (WG) on the input. In order to certify the correctness of the output regardless of the validity of (WG), one needs to check that there is no sufficiently small $\varepsilon > 0$ such that $w + \varepsilon u + \varepsilon^2 v_1 + \ldots + \varepsilon^{d+1} v_d \in \operatorname{Trop} I$. If I is homogeneous, this can by Lemma 4.1.5 be achieved by certifying that the iterated initial ideal $\operatorname{in}_{v_d} \cdots \operatorname{in}_{v_1} \operatorname{in}_u \operatorname{in}_w(I)$ is the entire Laurent polynomial ring $\kappa[\mathbf{x}^{\pm}]$.

Example 4.1.15. Consider the generating set from Example 4.1.10 (see also Figure 4.3):

$$I := (\underbrace{(x+1)(y+1), (x-1)(y+1)}_{=:F}) \subseteq \mathbb{C}[x^{\pm}, y^{\pm}].$$

Unlike before, Algorithm 4.1.13 will be unable to find a tropical defect around H_v even for $v = 0$, always terminating at Line 16. This is because without condition (SG), the line H_0 need not have a zero-dimensional intersection with $\mathrm{Trop}(I)$, so that its positive-dimensional intersection with $\mathrm{Trop}(F)$ need not arise from a tropical defect.

However Algorithm 4.1.13 will still find a tropical defect for L_v for $v \neq 0$, in which case it terminates at Line 14.

Remark 4.1.16 (Strong genericity vs. weak genericity from a practical point of view)**.** Theoretically, it is always possible to find tropical defects for generating sets which are not tropical bases using Algorithm 4.1.9 with the right choice of an affine subspace. In practice, however, it is much more reasonable to use Algorithm 4.1.13 instead. This is because generic $v \in \mathbb{R}^d$ for Algorithm 4.1.9 usually entail high exponents in the polynomial computations, whereas generic $\lambda \in (K^*)^d$ for Algorithm 4.1.13 only entail big coefficients, and most computer-algebra software systems such as Macaulay2 or Singular are better equipped to deal with the latter. For instance, our Singular experiments using Algorithm 4.1.9 regularly failed due to exponent overflows, since exponents in Singular are stored in the C++ type `signed short` (bounded by 2^{15} for most CPU architectures), while coefficients are stored with arbitrary precision.

Remark 4.1.17 (Comparison with existing techniques)**.** As hinted in the introduction, tropical basis verification is a problem that has been studied by many people. However, the only software currently capable of this task is gfan [Jen17], which for example has been used to prove that the 4×4-minors of a $5 \times n$ matrix form a tropical basis [CJR11]. Its command `gfan_tropicalbasis` computes a tropical basis of a tropical curve, and its command `gfan_tropicalintersection` for computing tropical prevarieties $\mathrm{Trop}(F)$ has an optional argument `--tropicalbasistest` to test whether $\mathrm{Trop}(F)$ equals the tropical variety $\mathrm{Trop}(I)$. Compared to the algorithms in gfan, our techniques have the following disadvantages and advantages.

Since our algorithms revolve around finding tropical defects, they are incapable to verify that a generating set is a tropical basis. As we only search around random hyperplanes of complementary dimension, we are also blind to lower-dimensional defects. This means, if $\dim(\mathrm{Trop}(I) \setminus \mathrm{Trop}(F)) < \dim(\mathrm{Trop}(I)) =: d$ then the probability for a random affine hyperplane of codimension d to intersect $\mathrm{Trop}(I) \setminus \mathrm{Trop}(F)$ is zero. One example where our algorithms failed to return a definite answer is [Rin12, Conjecture 4.8].

In return, our algorithms avoid the computation of both $\mathrm{Trop}(F)$ and $\mathrm{Trop}(I)$. Instead of $\mathrm{Trop}(F) = \bigcap_{f\in F} \mathrm{Trop}(f)$, we rather compute $\mathrm{Trop}(F') = \bigcap_{f\in F}(\mathrm{Trop}(f) \cap H)$. This is faster, since $\mathrm{Trop}(f) \cap H$ is covered by fewer polyhedra compared to $\mathrm{Trop}(f)$. Moreover, instead of $\mathrm{Trop}(I)$ we compute $\mathrm{Trop}(I')$, where $I' := I + (x_i - \lambda_i \mid i = 1, \ldots, d)$. This is easier since I' is zero-dimensional whereas I is not. Additionally, $\mathrm{Trop}(I')$ consists of up to $\deg(I)$ many points while $\mathrm{Trop}(I)$ is generally covered by many more polyhedra.

4.2. Application: Cox rings of cubic surfaces

Cox rings are global invariants of important classes of algebraic varieties. For example, they carry essential information about all morphisms to projective spaces. See [ADHL15] for further details and Chapter 2, where we studied their toric degenerations. In this section, we address [RSS16, Conjecture 5.3] which predicts a tropical basis for Cox rings of smooth cubic surfaces; we disprove this conjecture it with a tropical defect.

Definition 4.2.1. Consider six points $p_1, \ldots, p_6 \in \mathbb{P}^2$ in general position in the complex projective plane. Up to change of coordinates, we may assume that

$$p_i = (1 : d_i : d_i^3) \quad \text{for some } d_i \in \mathbb{C},$$

where d_i satisfy certain genericity conditions, see [RSS14, §6]. Blowing up $\mathbb{P}^2$ in these points results in a smooth cubic surface $X := \mathrm{Bl}_{p_1,\ldots,p_6} \mathbb{P}^2$. The geometry of this surface is captured by its **Cox ring**

$$\mathrm{Cox}(X) = \bigoplus_{(a_0,\ldots,a_6)\in\mathbb{Z}^7} H^0(X, \mathcal{O}_X(a_0E_0 + a_1E_1 + \ldots + a_6E_6)),$$

where

- $E_1, \ldots, E_6 \subseteq X$ are the exceptional divisors over the points $p_1, \ldots, p_6 \in \mathbb{P}^2$,
- $E_0 \subseteq X$ is the preimage of a line in $\mathbb{P}^2$ not containing $p_1, \ldots, p_6$,
- $H^0(X, \mathcal{O}_X(a_0E_0 + a_1E_1 + \ldots + a_6E_6)) \subseteq K(X)$ are the rational functions on X vanishing along E_i with multiplicity at least $-a_i$.[1]

For a smooth cubic surface X, the Cox ring $\mathrm{Cox}(X)$ is a finitely generated integral domain with a natural set of 27 generators. These generators are the rational functions on X establishing the linear equivalence of each of the 27 lines on the cubic surface X to a divisor of form $\sum_i a_iE_i \in \mathrm{Div}(X)$, see [BP04, Theorem 3.2]. Their ideal of relations was described by Ren, Shaw and Sturmfels as follows:

[1] Here, vanishing with a negative multiplicity $-k$ means should be understood as having poles of positive order k.

Proposition 4.2.2 ([RSS16, Proposition 2.2])**.** *For general complex numbers $d_1, \dots, d_6 \in \mathbb{C}$, let X be the cubic surface arising as the blowup of the points $(1 : d_i : d_i^3) \in \mathbb{P}^2$. Then*

$$\mathrm{Cox}(X) \cong \mathbb{C}[E_1, \dots, E_6, F_{12}, F_{13}, \dots, F_{56}, G_1, \dots, G_6]/I_X,$$

where, up to saturation at the product of all variables, I_X is generated by the following 10 trinomials and their 260 translates under the action of the Weyl group of type $\mathbf{E}_6$:

$$(d_3-d_4)(d_1+d_3+d_4)E_2F_{12}-(d_2-d_4)(d_1+d_2+d_4)E_3F_{13}+(d_2-d_3)(d_1+d_2+d_3)E_4F_{14},$$
$$(d_3-d_5)(d_1+d_3+d_5)E_2F_{12}-(d_2-d_5)(d_1+d_2+d_5)E_3F_{13}+(d_2-d_3)(d_1+d_2+d_3)E_5F_{15},$$
$$(d_3-d_6)(d_1+d_3+d_6)E_2F_{12}-(d_2-d_6)(d_1+d_2+d_6)E_3F_{13}+(d_2-d_3)(d_1+d_2+d_3)E_6F_{16},$$
$$(d_4-d_5)(d_1+d_4+d_5)E_2F_{12}-(d_2-d_5)(d_1+d_2+d_5)E_4F_{14}+(d_2-d_4)(d_1+d_2+d_4)E_5F_{15},$$
$$(d_4-d_6)(d_1+d_4+d_6)E_2F_{12}-(d_2-d_6)(d_1+d_2+d_6)E_4F_{14}+(d_2-d_4)(d_1+d_2+d_4)E_6F_{16},$$
$$(d_5-d_6)(d_1+d_5+d_6)E_2F_{12}-(d_2-d_6)(d_1+d_2+d_6)E_5F_{15}+(d_2-d_5)(d_1+d_2+d_5)E_6F_{16},$$
$$(d_4-d_5)(d_1+d_4+d_5)E_3F_{13}-(d_3-d_5)(d_1+d_3+d_5)E_4F_{14}+(d_3-d_4)(d_1+d_3+d_4)E_5F_{15},$$
$$(d_4-d_6)(d_1+d_4+d_6)E_3F_{13}-(d_3-d_6)(d_1+d_3+d_6)E_4F_{14}+(d_3-d_4)(d_1+d_3+d_4)E_6F_{16},$$
$$(d_5-d_6)(d_1+d_5+d_6)E_3F_{13}-(d_3-d_6)(d_1+d_3+d_6)E_5F_{15}+(d_3-d_5)(d_1+d_3+d_5)E_6F_{16},$$
$$(d_5-d_6)(d_1+d_5+d_6)E_4F_{14}-(d_4-d_6)(d_1+d_4+d_6)E_5F_{15}+(d_4-d_5)(d_1+d_4+d_5)E_6F_{16}.$$

Here, the generators corresponding to the 27 lines are denoted as follows:

- *E_i represents the exceptional divisor over the point p_i,*
- *F_{ij} represents the strict transform of the line through p_i and p_j,*
- *G_i represents the strict transform of the unique conic through $\{p_1, \dots, p_6\} \setminus \{p_i\}$.*

The following theorem answers [RSS16, Conjecture 5.3] negatively:

Theorem 4.2.3. *For general $d_1, \dots, d_6 \in \mathbb{C}$, the 270 trinomial generators of I_X described in Proposition 4.2.2 are not a tropical basis.*

Proof. Fix the following ordered set of variables:

$$S := \{E_1, E_2, E_3, E_4, E_5, E_6, F_{12}, F_{13}, F_{14}, F_{15}, F_{16}, F_{23}, F_{24}, F_{25}, F_{26}, F_{34}, F_{35}, F_{36}, F_{45}, F_{46}, F_{56}, G_1, G_2, G_3, G_4, G_5, G_6\}.$$

Let I_X be the ideal in the polynomial ring $\mathbb{C}(d_1, \dots, d_6)[S]$ generated by the 270 trinomials described in Proposition 4.2.2, and consider the weight vector

$$w := (2,1,0,1,1,1,0,2,0,0,0,1,0,0,0,1,1,1,0,0,0,0,0,0,0,0,0) \in \mathbb{R}^S.$$

One can computationally verify that w is a tropical defect, i.e., w lies in the tropical prevariety, since $\mathrm{in}_w(f)$ is at least binomial for each trinomial generator f, and outside the tropical variety, since $\mathrm{in}_w(I_X)$ contains the monomial $E_6F_{56}G_6$. □

Remark 4.2.4. The statements in the proof of Theorem 4.2.3 can be easily verified using a computer algebra system such as Singular. The following script is available on `https://software.mis.mpg.de`,

and the following shortened transcript was produced using Singular's online interface (version 4.1.2) available at

https://www.singular.uni-kl.de/tryonline.

```
> LIB "tropicalBasis.lib"; // initializes necessary libraries
                           // and helper functions
> intvec wMin = 2,1,0,1,1,1,0,2,0,0,0,1,0,0,0,1,1,1,0,0,0,0,0,
                0,0,0,0;          // wMin is in min-convention
> intvec wMax = -wMin;            // Singular uses max-convention
> intvec allOnes = onesVector(size(wMax));
> ring r = (0,d1,d2,d3,d4,d5,d6),(E1,E2,E3,E4,E5,E6,
.   F12,F13,F14,F15,F16,F23,F24,F25,F26,F34,F35,F36,F45,F46,F56,
.   G1,G2,G3,G4,G5,G6),(a(allOnes),a(wMax),lp);
// Prepending allOnes makes no difference mathematically as the ideal is
// homogeneous, but it helps computationally.
> ideal F =                 // Singular ideals are lists of polynomials
.  (d3-d4)*(d1+d3+d4)*E2*F12+(d2-d4)*(d1+d2+d4)*E3*F13
.    -(d2-d3)*(d1+d2+d3)*E4*F14,
⋮  [...]
.  -(d5-d6)*(d1+d3+d4)*F24*G4+(d4-d6)*(d1+d3+d5)*F25*G5
.    -(d4-d5)*(d1+d3+d6)*F26*G6;
> ideal inF = initial(F,wMax); // initial forms of the elements in F
                               // all are at least binomial, hence
                               // wMin ∈ Trop(F)
> ideal IX = groebner(F);
> ideal inIX = initial(IX,wMax) // initials of Gröbner basis elements
                                // this is a Gröbner basis of in_wMin(I_X)
> NF(E6*F56*G6,inIX);
0                        // normal form is 0, hence E6*F56*G6 ∈ in_wMin(I_X)
```

4.3. Application: Realizability of valuated gaussoids

Gaussoids are combinatorial structures introduced by Lněnička and Matúš [LM07] that encode conditional independence relations among Gaussian random variables. Reminiscent of the study of matroids, Boege, D'Alì, Kahle and Sturmfels [BDKS19] introduced the notions of oriented and valuated gaussoids. In this section, we address the question whether all valuated gaussoids on four elements are realizable, disproving it with a tropical defect. This was initially conjectured in the first version of [BDKS19], as found on arXiv. The published version has since been updated with our Theorem 4.3.3.

Definition 4.3.1 ([BDKS19, §1])**.** Fix $n \in \mathbb{N}$. Consider the Laurent polynomial ring

$$R_n := \mathbb{C}\Big[p_I^{\pm} \mid I \subseteq [n]\Big]\Big[a^{\pm}_{\{i,j\}|K} \mid i,j \in [n] \text{ distinct}, K \subseteq [n] \setminus \{i,j\}\Big],$$

in which we abbreviate $a_{\{i,j\}|K}$ to $a_{ij|K}$, and consider the ideal T_n generated by the following $2^{n-2}\binom{n}{2}$ so-called *square trinomials*

$$a_{ij|K}^2 - p_{K\cup\{i\}}\, p_{K\cup\{j\}} + p_{K\cup\{i,j\}}\, p_K \qquad \text{for } i,j \in [n] \text{ distinct, } K \subseteq [n] \setminus \{i,j\},$$

and the following $12 \cdot 2^{n-3}\binom{n}{3}$ so-called *edge trinomials*:

$$p_{L\cup\{k\}}\, a_{ij|L\setminus\{i,j\}} - p_L\, a_{ij|L\cup\{k\}\setminus\{i,j\}} - a_{ki|L\setminus\{i\}}\, a_{kj|L\setminus\{j\}} \qquad \text{for } i,j,k \in [n] \text{ distinct, } L \subseteq [n] \setminus \{k\}.$$

A **valuated gaussoid** is a point in the tropical prevariety defined by the square and edge trinomials. It is called **realizable** if it lies in the tropical variety $\mathrm{Trop}(T_n)$.

Remark 4.3.2. The variables of the ring R_n correspond to the principal and almost-principal minors of a symmetric $n \times n$-matrix (i.e., determinants of square submatrices whose row- and column index sets differ by at most one index). The ideal T_n corresponds to the polynomial relations among these minors for symmetric matrices with nonzero principal minors by [BDKS19, Proposition 6.2].

The following theorem negatively answers Conjecture 8.4 in the first arXiv-version of [BDKS19], and is now Theorem 8.4 in the final published version of [BDKS19]:

Theorem 4.3.3. *Not all valuated gaussoids on four elements are realizable, i.e., the square and edge trinomials in Definition 4.3.1 are not a tropical basis of T_4.*

Proof. Consider the following ordered set S of the variables of R_4 and weight vector $w \in \mathbb{R}^S$:

$$\begin{aligned} S := \{&p_\emptyset, p_1, p_{12}, p_{123}, p_{1234}, p_{124}, p_{13}, p_{134}, p_{14}, p_2, p_{23}, p_{234}, p_{24}, p_3, p_{34}, p_4, \\ &a_{12}, a_{12|3}, a_{12|34}, a_{12|4}, a_{13}, a_{13|2}, a_{13|24}, a_{13|4}, a_{14}, a_{14|2}, a_{14|23}, a_{14|3}, \\ &a_{23}, a_{23|1}, a_{23|14}, a_{23|4}, a_{24}, a_{24|1}, a_{24|13}, a_{24|3}, a_{34}, a_{34|1}, a_{34|12}, a_{34|2}\} \\ w := (&14, 10, 6, 0, 6, 8, 8, 2, 8, 6, 6, 2, 8, 8, 8, 8, 8, 4, 2, 10, 9, 3, 5, 5, 9, 11, \\ &1, 5, 7, 5, 5, 5, 7, 7, 1, 5, 8, 6, 4, 4) \in \mathbb{R}^S. \end{aligned}$$

One can check that w is a tropical defect, i.e., w lies in the tropical prevariety, since $\mathrm{in}_w(f)$ is at least binomial for all square and edge trinomials f, and lies outside the tropical variety, since $\mathrm{in}_w(T_4)$ contains the monomial $a_{23}a_{23|1}$. □

Remark 4.3.4. The statements in the proof of Theorem 4.3.3 can be easily verified using a computer algebra system such as Singular. The following script is available on `https://software.mis.mpg.de`, and the following shortened transcript was produced using Singular's online interface (version 4.1.2) available at

https://www.singular.uni-kl.de/tryonline.

```
> LIB "tropicalBasis.lib"; // initializes necessary libraries
                           // and helper functions
> intvec wMin = 14,10,6,0,6,8,8,2,8,6,6,2,8,8,8,8,8,4,2,10,9,3,
.               5,5,9,11,1,5,7,5,5,5,7,7,1,5,8,6,4,4;
                           // wMin is in min-convention
> intvec wMax = -wMin;     // Singular uses max-convention
> intvec allOnes = onesVector(size(wMax));
> ring r = 0,(p,p1,p12,p123,p1234,p124,p13,p134,p14,p2,p23,
.   p234,p24,p3,p34,p4,a12,a12_3,a12_34,a12_4,a13,a13_2,a13_24,
.   a13_4,a14,a14_2,a14_23,a14_3,a23,a23_1,a23_14,a23_4,a24,
.   a24_1,a24_13,a24_3,a34,a34_1,a34_12,a34_2),
.   (a(allOnes),a(wMax),lp);
// Prepending allOnes makes no difference mathematically as the ideal is
// homogeneous, but it helps computationally.
> ideal F =                // Singular ideals are lists of polynomials
.   a34_12*a13_24+p124*a14_23-a14_2*p1234,
⋮   [...]
.   -p1*p2+a12^2+p*p12;
> ideal inF = initial(F,wMax); // initial forms of the elements in F
                               // all are at least binomial, hence
                               // wMin ∈ Trop(F)
> ideal I = groebner(F);
> ideal inI = initial(I,wMax); // initials of Gröbner basis elements
                               // this is a Gröbner basis of in_wMin(I_X)
> NF(a23*a23_1,inI);
0                          // normal form is 0, hence a_23 a_23|1 ∈ in_wMin(I_X)
```

Remark 4.3.5 (sampling affine subspaces for tropical defects)**.** The tropical defects in Theorems 4.2.3 and 4.3.3 were found by repeatedly running Algorithm 4.1.13 on random affine subspaces $H \subseteq \mathbb{R}^n$. In the sampling of the affine subspaces, a situation which we tried to avoid are two subspaces intersecting the tropical variety in exactly the same Gröbner polyhedra. In the following, we describe our sampling approach which we based on this thought.

Even though we were unable to compute the tropical variety $\mathrm{Trop}(I)$ or the tropical prevariety $\mathrm{Trop}(F)$ in both problems, we were able to compute

(1) a Gröbner basis of I with respect to a graded reverse lexicographical ordering,
(2) for selected finite fields $\mathbb{F}_p$ and for $d+1 := \dim(I)+1$ variables $x_{i_0}, \dots, x_{i_d}$, the generator $\overline{g} \in \mathbb{F}_p[x_{i_0}, \dots, x_{i_d}]$ of the principal elimination ideal $(I \otimes_{\mathbb{Z}} \mathbb{F}_p) \cap \mathbb{F}_p[x_{i_0}, \dots, x_{i_d}]$.[2]

[2]Here, we abuse notation by regarding I as an ideal in $\mathbb{Z}[\mathbf{x}^{\pm}]$. Note that in the example of Cox rings, the generators of I have integral coefficients if we choose rational numbers $d_i \in \mathbb{Q}$ and clear denominators.

In other words, (2) allowed for educated guesses for generators g of principal elimination ideals $I \cap K[x_{i_0}, \ldots, x_{i_d}]$, while (1) allowed for tests whether the guesses were correct. Thus, we were able to compute tropical hypersurfaces $\mathrm{Trop}(g) \subseteq \mathbb{R}^{d+1}$ which are the images of $\mathrm{Trop}(I)$ under selected orthogonal projections $\pi : \mathbb{R}^n \to \mathbb{R}^{d+1}$.

For each projection, we then constructed affine lines $L_1, \ldots, L_k$ in $\mathbb{R}^{d+1}$ such that each maximal polyhedron of $\mathrm{Trop}(g)$ intersects at least one line. Their preimages $\pi^{-1}(L_1), \ldots, \pi^{-1}(L_k)$ are then affine subspaces of codimension d, which were our samples for H.

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(Paul Görlach)

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