# On the Cone Conjecture for Calabi–Yau threefolds with Picard number three

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Born February 10, 1992 in Berlin, Germany

June 3, 2016

Master's Thesis Mathematics

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CHAPTER 1

# Introduction

## 1.1. Motivation

A smooth projective variety over $\mathbb{C}$ is of *Calabi–Yau type* if its canonical bundle is numerically trivial. These varieties form an important class of objects in Higher-Dimensional Algebraic Geometry.

Indeed, it has been a common theme in the study of algebraic varieties in higher dimensions to investigate the geometry based on properties of the canonical bundle. An early instance of this is the Enriques–Kodaira classification of surfaces, where the basic invariant is the Kodaira dimension of the surface, which roughly measures how many global sections the canonical divisor and its multiples possess.

In Birational Geometry, a central result is the finite generation of the canonical ring and the Minimal Model Program tries to classify varieties up to birational equivalence and study their geometry by constructing algebraic fiber space structures and birational contractions based on numerical properties of the canonical bundle.

A class of relatively well-behaved varieties in any dimension is given by Fano varieties, which are characterized by the ampleness of the anti-canonical bundle. In general, it seems that special numerical properties of the canonical divisor may facilitate the study of the geometry of a given variety. This leads to the immediate question: What can we say about varieties whose canonical bundle is numerically trivial? This is the study of varieties of Calabi–Yau type.

Philosophically, the birational study of a variety of Calabi–Yau type $X$ seems less tractable at a first glance, since its Néron–Severi space $N^1(X)_{\mathbb{R}} := \mathrm{Pic}(X) \otimes \mathbb{R}/ \equiv_{\mathrm{num}}$ lacks an intrisically defined non-zero point corresponding to $K_X$, which could serve as a starting point for investigations inside $N^1(X)_{\mathbb{R}}$. On the other hand, there is a nice structural prediction for varieties of Calabi–Yau types, the Morrison–Kawamata Cone Conjecture.

In concordance with the general philosophy of Birational Geometry that for understanding the geometry of $X$ it is essential to characterize the convex geometry of certain cones inside $N^1(X)_{\mathbb{R}}$, the Morrison–Kawamata Cone Conjecture concerns the structure of the most important cone, the cone $\mathrm{Amp}(X) \subset N^1(X)_{\mathbb{R}}$ spanned by ample divisors, resp. its closure $\mathrm{Nef}(X)$. In general, one cannot expect that $\mathrm{Nef}(X)$ is a rational polyhedral cone, as is the case for Fano varieties. Indeed, there are examples that $\mathrm{Nef}(X)$ may be a round cone or an almost rational polyhedral cone generated by infinitely many extremal rays, see [**Tot12**, Section 4.1].

In fact, as the automorphism group of $X$ permutes the extremal rays of the cone $\mathrm{Nef}(X)$, it can be easily seen that $\mathrm{Nef}(X)$ cannot be a rational polyhedral cone when the action of $\mathrm{Aut}(X)$ on $N^1(X)_{\mathbb{R}}$ is given by an infinite group. The Cone Conjecture then predicts the next best: that the nef cone is rational polyhedral up to the action of the automorphism group. More precisely, [**Mor93**], [**Kaw97**]:

**Conjecture** (Morrison–Kawamata Cone Conjecture)**.** *Let $X$ be a variety of Calabi–Yau type. Then there exists a rational polyhedral cone which is a fundamental domain for the action of* $\mathrm{Aut}(X)$ *on* $\mathrm{Nef}(X) \cap \mathrm{Eff}(X)$*.*

This formulation is due to Kawamata and is a refined form of a conjecture of Morrison, which states the above result for $\mathrm{Nef}^+(X) := \mathrm{conv}(\mathrm{Nef}(X) \cap N^1(X)_{\mathbb{Q}})$ instead of $\mathrm{Nef}(X) \cap \mathrm{Eff}(X)$. We refer to Section 2.3 for a detailed discussion how these conjectures relate to each other and also for the convention for fundamental domains used in this Conjecture. There we will also state a weak version of the Cone Conjecture allowing for a fundamental domain which is rational polyhedral up to a finite number of exceptional faces.

Progress has been made on the Cone Conjecture as follows: Sterk [**Ste85**] proved the Cone Conjecture for $K3$ surfaces and it was observed by Kawamata [**Kaw97**] that this argument extends to arbitrary surfaces of Calabi–Yau type. This settles the case $\dim X = 2$. Kawamata also proved a relative version for threefolds fibered over a positive-dimensional base, [**Kaw97**]. For abelian varieties, the Cone Conjecture was proven by Prendergast-Smith [**PS12b**]. Moreover, the Conjecture has been verified several special varieties, e.g. [**OS01**], [**PS12a**]. In general, the case $\dim X = 3$ remains widely open.

This thesis is concerned with investigating the structure of Calabi–Yau threefolds with Picard number $\rho(X) = 3$ with a view towards the Kawamata–Morrison Cone Conjecture. Here, by a *Calabi–Yau variety* we mean – in contrast to a variety of Calabi–Yau type – a smooth projective variety $X$ with $K_X = 0$ and $H^1(X, \mathcal{O}_X) = 0$.

Calabi–Yau varieties are of large interest because of their importance in Theoretical Physics as well as the central role they play among varieties of Calabi–Yau type: According to the following Theorem, Calabi–Yau varieties are one of three fundamental building blocks of varieties with numerically trivial canonical bundle:

**Theorem 1.1** (Beauville–Bogomolov Decomposition)**.** *Let $X$ be a smooth projective variety of Calabi–Yau type. Then there exists a finite étale covering*

$$T \times \prod_{i=1}^{r} A_i \times \prod_{i=1}^{s} Y_i \to X,$$

*where $T$ is a complex torus, $A_i$ are hyperkähler manifolds and $Y_i$ are simply-connected Calabi–Yau varieties.*

## 1.2. Outline of the thesis

This thesis will be structured as follows: In Chapter 2, we give the basic definitions, specify notations and recall basic results on which this thesis is based. We give a precise statement of Morrison's and Kawamata's version of the Cone Conjecture and compare these two.

Chapter 3 is headed towards a classification result (Theorem 3.12) for the intersection form on threefolds with Picard number 3, on which our analysis of these varieties will be built. This classification is known for Calabi–Yau varieties by [**LOP13**] when the automorphism group $\mathrm{Aut}(X)$ is infinite. We give a proof of their result from a different perspective which allows an extension to the case of a finite automorphism group acting non-trivially on the Néron–Severi space $N^1(X)_{\mathbb{R}}$.

For this, we start out Chapter 3 by noting that the properties of the intersection product as a multisymmetric form on $N^1(X)_{\mathbb{R}}$ reflect in its associated vanishing set, the null cone. Then, we give general criteria when this vanishing set is reducible and splits off a hyperplane. The main result in this direction will be Corollary 3.9, showing that for odd-dimensional varieties of Picard number 3 such a splitting always occurs unless the action of $\mathrm{Aut}(X)$ on $N^1(X)_{\mathbb{R}}$ is very simple. Finally, we deduce from this the desired classification result for threefolds with Picard number three.

In Chapter 4, with the classification result established, we investigate the structure of each of the different types of Calabi–Yau threefolds with Picard number 3 in more detail. Based on this, we obtain results on the Cone Conjecture and the existence of rational curves. Our main focus will lie on the investigation of the case that the null cone splits as the union of three hyperplanes.

We conclude in Chapter 5 by indicating how questions about Diophantine Approximation problems may provide further restrictions on the structure of the nef cone of Calabi–Yau threefolds by exploiting a finiteness result due to Szendrői [**Sze99**].

## 1.3. Summary of the main results

The main results we establish in this thesis are the following:

**Theorem 1.2** (Theorem 4.1)**.** *Let $X$ be a threefold with Picard number 3 whose Chern classes satisfy $c_2(X) \not\equiv 0$ or $c_1(X)^2 \not\equiv 0$. Then*

$$\mathcal{A}_+(X) := \mathrm{im}(\mathrm{Aut}(X) \to \mathrm{GL}(N^1(X)) \cap \mathrm{SL}(N^1(X))$$

*is either trivial or a cyclic group.*

**Theorem 1.3** (Corollary 3.9)**.** *Let $X$ be a smooth projective variety of odd dimension $n$ with Picard number 3 and such that the group $\mathcal{A}_+(X)$ is neither trivial nor of order 3. Then the intersection form*

$$N^1(X)_\mathbb{R} \to \mathbb{R},\ x \to x^n$$

*factors off a linear factor over $\mathbb{R}$.*

**Theorem 1.4** (Theorems 4.18, 4.20, 4.22)**.** *Let $X$ be a Calabi–Yau threefold with Picard number 3 such that the group $\mathcal{A}_+(X)$ is non-trivial and assume that the intersection form*

$$N^1(X)_\mathbb{R} \to \mathbb{R},\ x \to x^3$$

*splits into linear factors over $\mathbb{R}$. Then the following holds:*

*(i) If $\mathrm{Aut}(X)$ is infinite, then Morrison's Cone Conjecture holds.*
*(ii) If $\mathrm{Aut}(X)$ is finite, then the Weak Cone Conjecture holds.*
*(iii) If $\mathrm{Aut}(X)$ is infinite and $X$ is simply-connected, then it contains a rational curve.*

Along our way, we will also encounter further results that may be of independent interest.

## CHAPTER 2

# Preliminaries

### 2.1. Basics, definitions and conventions

In this section we start by fixing conventions and basic definitions to be used in this thesis.

We denote $\mathbb{R}_{\geq 0} := \{x \in \mathbb{R} \mid x \geq 0\}$ and $\mathbb{R}_{>0} := \{x \in \mathbb{R} \mid x > 0\}$. If $\varphi \in \mathrm{GL}(V)$ for a vector space $V$ over $\Bbbk$ and $\lambda \in \Bbbk$ is an eigenvalue of $\varphi$, we define the generalized eigenspace of $\varphi$ with respect to $\lambda$ is $\ker(\varphi - \mathrm{id})^{\dim V} \subset V$. We denote the dual vector space by $V^\vee$ and if $\ell \in V^\vee$, then we denote

$$\{\ell = 0\} := \{v \in V \mid \ell(v) = 0\}.$$

**2.1.1. Preliminaries from Algebraic Geometry.** By a *variety* we mean an integral separated scheme of finite type over $\mathbb{C}$. For simplicity, we will restrict to smooth projective varieties in this Thesis, even though many results will hold with possibly some additional effort in a more general setting.

We denote the canonical divisor of a smooth variety $X$ by $K_X$. All subschemes and points we refer to are closed unless mentioned otherwise. By a *surface*, a *threefold* etc. we mean a smooth projective variety of the corresponding dimension.

For a smooth variety $X$ its Picard group $\mathrm{Pic}(X)$ plays a prominent role in the study of the geometry of $X$. Its importance is two-fold: On the one hand, global sections of a line bundle on $X$ determine a rational map to projective space. This simple fact is basic for the construction of morphisms between projective varieties. On the other hand, there is a well-behaved intersection product between divisors and 1-cycles and many properties of a morphism induced by a line bundle reflect in numerical properties of the intersection product.

For a smooth variety $X$ we denote the intersection product of a divisor $D$ and a 1-cycle $C$ by $D \cdot C$. We recall that a divisor (resp. a 1-cycle) is called numerically trivial (denoted $\equiv 0$) if its intersection with any 1-cycle (resp. divisor) is zero. Denoting the subgroup of numerically trivial divisors by $\mathrm{Div}^0(X) \subset \mathrm{Div}(X)$, the *Néron–Severi group* $N^1(X) := \mathrm{Div}(X)/\mathrm{Div}^0(X)$ is finitely generated. Similarly, we denote by $N_1(X)$ the quotient of the group of 1-cycles by the subgroup of numerically trivial 1-cycles. This gives rise to a perfect pairing of abelian groups

$$N^1(X) \times N_1(X) \to \mathbb{Z}, \; ([D],[C]) \mapsto D \cdot C.$$

In particular, the abelian groups $N^1(X)$ and $N_1(X)$ are free of the same rank and we obtain an natural identification with the dual $\mathbb{Z}$-module: $N^1(X)^\vee \cong N_1(X)$ and $N_1(X)^\vee \cong N^1(X)$. The finite rank of $N^1(X)$ (resp. $N_1(X)$) is called the *Picard number* of $X$, denoted by $\rho(X)$.

Even though the intersection product on divisors and 1-cycles gives a satisfying description for many phenomena on varieties and will be sufficient for most parts of the treatment in this thesis, it is worthwile to keep in mind the general construction from Intersection Theory involving cycles of arbitrary dimension: For an $n$-dimensional variety $X$ we let $\mathrm{CH}_i(X)$ be the Chow group of $i$-cycles modulo rational equivalence and denote $\mathrm{CH}^i(X) := \mathrm{CH}_{n-i}(X)$. Note that $\mathrm{CH}^i(X) = 0$ if $i < 0$ or $i > n$. The Chow ring of $X$ is the graded ring $\mathrm{CH}(X) := \bigoplus_i \mathrm{CH}^i(X)$, where the product is given by the intersection product

$$\mathrm{CH}^i(X) \times \mathrm{CH}^j(X) \to \mathrm{CH}^{i+j}(X), \; (\alpha, \beta) \mapsto \alpha \cdot \beta.$$

We then denote $N^i(X) := \mathrm{CH}^i(X)/\equiv$, where we take the quotient by numerical equivalence $\equiv$, and let $N_i(X) := N^{n-i}(X)$. In particular (identifying $N^n(X)$ with $\mathbb{Z}$), there is a well-behaved intersection product $\alpha_1 \cdot \ldots \cdot \alpha_k \in \mathbb{Z}$ for $\alpha_j \in N^{i_j}(X)$ with $\sum_{j=1}^k i_j = n$.

For any vector bundle $\mathcal{E}$ on a smooth variety $X$ of dimension $n$, Intersection Theory associates Chern classes $c_i(\mathcal{E}) \in \mathrm{CH}^i(X)$ for each $i \in \{1, \ldots, n\}$. In particular, a smooth variety $X$ carries intrinsically defined Chern classes $c_i(X) := c_i(\mathcal{T}_X) \in \mathrm{CH}^i(X)$ associated to its tangent bundle $\mathcal{T}_X$.

When working with the free abelian groups of finite rank $N^1(X)$ and $N_1(X)$, it is convenient to extend the scalars to a field, defining

$$N^1(X)_\Bbbk := N^1(X) \otimes_\mathbb{Z} \Bbbk$$

for a field $\Bbbk$ to obtain a finite-dimensional $\Bbbk$-vector space of dimension $\rho(X)$. Similarly we define $N_1(X)_\Bbbk$. This scalar extension is most natural for $\Bbbk = \mathbb{Q}$. However, many geometric properties reflect in the structure of convex cones inside these $\mathbb{Q}$-vector spaces and the most natural setting for convex geometry is working over $\Bbbk = \mathbb{R}$ because of completeness properties. (For example, a proper closed convex cone in $N^1(X)_\mathbb{Q}$ may have no extremal rays, while a closed convex cone in $N^1(X)_\mathbb{R}$ is always generated by its extremal rays if it contains no lines.) When working with polynomial vanishing sets on $N^1(X)_\mathbb{R}$ or with spectral properties of vector space endomorphisms (as in Chapter 3), it is also useful to work over $\Bbbk = \mathbb{C}$, but this much less natural as it loses the possibility of applying convex geometry, because we are no longer working over an ordered field. We remark that it may also be of use in some cases to work over other fields, but in this thesis we will only be concerned with $\Bbbk = \mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. Throughout this thesis, when we denote $N^1(X)_\Bbbk$ or $N_1(X)_\Bbbk$, then $\Bbbk$ is always assumed to be a field of characteristic zero and can be thought of as $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. For appropriate distinction we may sometimes use the notation $N^1(X)_\mathbb{Z} := N^1(X)$ and $N_1(X)_\mathbb{Z} := N_1(X)$ to stress that we are considering the free abelian groups. Note that there are natural inclusions

$$N^1(X)_\mathbb{Z} \subset N^1(X)_\mathbb{Q} \subset N^1(X)_\mathbb{R} \subset N^1(X)_\mathbb{C}$$

and similarly for $N_1(X)$.

**2.1.2. Preliminaries from Convex Geometry.** On the finite-dim. $\mathbb{R}$-vector spaces $N^1(X)_\mathbb{R}$ and $N_1(X)_\mathbb{R}$ we always consider the Euclidean topology. As the convex geometry of these vector spaces is important, we specify some notational conventions here for any finite dimensional $\mathbb{R}$-vector space $V$: By a *linear cone* in $V$ we mean a subset which is closed under scalar multiplication, whereas by *convex cone* we mean a subset of $V$ closed under addition and multiplication by positive scalars. Note that a convex cone is a linear cone only when it is a linear subspace. A *quadric* cone, *cubic* cone etc. is a nontrivial(!) linear cone cut out by an irreducible homogeneous polynomial of the corresponding degree.

A *ray* in $V$ is a convex cone of the form $\mathbb{R}_{\geq 0}v$ for some non-zero $v \in V$ and we refer to $\mathbb{R}_{>0}v$ as the corresponding *non-closed ray*. A convex subcone $\mathcal{C}'$ of a convex cone $\mathcal{C}$ is called *extremal*, or a *face* of $\mathcal{C}$, if $v + w \in \mathcal{C}'$ for any $v, w \in \mathcal{C}$ implies $v, w \in \mathcal{C}'$. Note that when $\mathcal{C}$ and $\mathcal{C}'$ are closed, this is equivalent to saying that $\mathcal{C}'$ is the intersection of $\mathcal{C}$ with a hyperplane in $V$. We say that a hyperplane $H$ in $V$ is a *supporting hyperplane* for a closed convex cone $\mathcal{C}$ if $H \cap \mathcal{C}$ is an extremal subcone.

In particular, the above defines the notion of an *extremal ray* of a convex cone. We say that a set $\mathcal{M}$ of rays in $V$ *converge (resp. accumulate) towards* a given ray $\mathcal{R}$ if any open cone containing $\mathcal{R} \setminus \{0\}$ contains almost all (resp. finitely many) of the non-closed rays in $\mathcal{M}$.

A convex cone $\mathcal{C} \subset V$ has dimension $k$ if any subspace of $V$ containing $\mathcal{C}$ is of dimension at least $k$ and $\mathcal{C}$ is called *full-dimensional* if its dimension equals $\dim V$. The dual cone $\mathcal{C}^\vee \subset V^\vee$ is defined as

$$\mathcal{C}^\vee := \{\ell \in V^\vee \mid \ell \geq 0 \text{ on } \mathcal{C}\}.$$

It can be shown that a closed convex cone is full-dimensional if and only its dual cone contains no lines.

When $V$ is any finite-dimensional $\Bbbk$-vector space (recall $\operatorname{char}\Bbbk = 0$), then we call a free $\mathbb{Z}$-submodule $F \subset V$ of rank $\dim_{\Bbbk} V$ an *integral structure* on $V$. (In particular, $N^1(X)_{\mathbb{Z}}$ is an integral structure on $N^1(X)_{\Bbbk}$, and similarly for $N_1(X)$.) Note that we can then identify $V = F \otimes_{\mathbb{Z}} \Bbbk$ and we have a natural $\mathbb{Q}$-linear subspace $F_{\mathbb{Q}} := F \otimes_{\mathbb{Z}} \mathbb{Q} \subset V$. With respect to a fixed integral structure $F$ on $V$, we say that a point $v \in V$ is *integral* if $v \in F$ and *rational* if $v \in F_{\mathbb{Q}}$. A linear subspace $W \subset V$ is called *rational* if it is spanned by $W \cap F_{\mathbb{Q}}$ (or equivalently, if it the kernel of a rational linear form $h \in F_{\mathbb{Q}}^{\vee} \subset V^{\vee}$).

If $\Bbbk = \mathbb{R}$, then a convex cone $\mathcal{C} \subset V$ is called *rational* if $\mathcal{C} = \operatorname{conv}(\mathcal{C} \cap F_{\mathbb{Q}})$, where conv denotes the convex hull of a subset in $V$. If $\mathcal{C} \cap F_{\mathbb{Q}} = 0$, then $\mathcal{C}$ is called *irrational*. It is easily observed that a convex cone not containing lines is rational if and only if it does not contain irrational extremal rays.

In the following we recall the most important convex cones in $N^1(X)_{\mathbb{R}}$ for the study of a smooth projective variety $X$. The *ample cone* $\operatorname{Amp}(X)$ is the open convex cone spanned by ample classes in $N^1(X)_{\mathbb{R}}$. The closure of $\operatorname{Amp}(X)$ is the *nef cone* $\operatorname{Nef}(X)$, which does not need to be rational. The *effective cone* $\operatorname{Eff}(X)$ is the rational convex cone spanned by effective divisor classes. Its interior is the *big cone* $\operatorname{Big}(X)$ and its closure is the *pseudo-effective cone* $\operatorname{Psef}(X)$. All of the mentioned cones are full-dimensional and contain no lines, see [**Ogu14**, Proposition 2.2].

A note on visualization: For a convex cone $\mathcal{C}$ not containing lines, there exists an affine hyperplane not passing through 0 which intersects $\mathcal{C}$ in a bounded convex set and it may be useful to just visualize this intersection representing the cone. We will use this convention in many Figures throughout this thesis without further notice.

## 2.2. Calabi–Yau threefolds

We define the main objects under consideration in this thesis:

**Definition 2.1.** A **variety of Calabi–Yau type** is a smooth projective variety $X$ over $\mathbb{C}$ whose canonical divisor is numerically trivial, i.e. $K_X = 0$ in $N^1(X)$. We call $X$ a **Calabi-Yau variety** if additionally $H^1(X, \mathcal{O}_X) = 0$ and $K_X = 0$ holds in $\operatorname{Pic}(X)$.

**Remark 2.2.** Calabi–Yau varieties form an important class of varieties in Algebraic Geometry and are also of high interest in Theoretical Physics. The fact that in this field there has been extensive study with approaches from different points of view has unfortunately lead to many competing (non-equivalent) definitions of Calabi–Yau varieties. The reader should be aware that in the literature conventions differing from the above are in use: Sometimes, for example, a variety of Calabi–Yau type (in our definition) is just referred to as a Calabi–Yau variety, in which case the varieties additionally satisfying $H^1(X, \mathcal{O}_X) = 0$ are often referred to as strict Calabi–Yau varieties. Some authors however use the denomination "strict Calabi–Yau" for the stronger requirement $H^i(X, \mathcal{O}_X) = 0$ for all $i \in \{1, \ldots, \dim X - 1\}$. We will stick to Definition 2.1.

**Example 2.3.** The following construction produces many examples of Calabi–Yau varieties of dimension $n \geq 3$ and Picard number $m$: In the product of projective spaces

$$\mathbb{P} := \mathbb{P}^{k_1} \times \ldots \times \mathbb{P}^{k_m}$$

with $\sum_{i=1}^{m} k_i = n + r$ for some $r \geq 1$, we can consider the complete intersection

$$X := H_1 \cap \cdots \cap H_r,$$

where each $H_j \subset \mathbb{P}$ is a general hypersurfaces of multidegree $(a_{1j}, \ldots, a_{mj})$ such that we have $\sum_{j=1}^{r} a_{ij} = k_i + 1$ for all $i$. The latter condition guarantees $K_X = 0$ and it follows from the Lefschetz Hyperplane Theorem that $\rho(X) = m$ and $H^1(X, \mathcal{O}_X) = 0$ (see e.g. [**Laz04**, Example 3.1.25]).

**Definition 2.4.** We say that a smooth variety is **simply–connected** if it is simply–connected considered as a real manifold.

**Remark 2.5.** It is known that a simply–connected smooth projective variety $X$ satisfies $H^1(X, \mathcal{O}_X) = 0$. This follows from $H^1(X, \mathbb{Z}) = 0$ and the Hodge Decomposition

$$H^1(X, \mathbb{C}) = H^1(X, \mathcal{O}_X) \oplus H^0(X, \Omega_X^1).$$

Note that Calabi–Yau varieties are a generalization of K3 surfaces to higher dimensions. In this thesis, the main focus lies on Calabi–Yau threefolds. Apart from being a diverse class of varieties which is interesting for itself, they have received particular attention from their importance in Theoretical Physics.

**Remark 2.6.** If $X$ is a Calabi–Yau threefold, then the property that $H^1(X, \mathcal{O}_X) = 0$ implies by Serre Duality that $H^2(X, \mathcal{O}_X) = 0$.

In particular, the exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^\times \to 1$$

shows that $\mathrm{Pic}(X) = H^1(X, \mathcal{O}_X^\times) \to H^2(X, \mathbb{Z})$ is an isomorphism. This implies that

$$\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong N^1(X)_{\mathbb{Q}}$$

for Calabi–Yau threefolds.

## 2.3. The Cone Conjecture

In this section, we give a precise formulation of Morrison's and Kawamata's Cone Conjectures and compare the two versions. We will also formulate a weaker version. The Cone Conjecture aims to answer the following question.

**Question 2.7.** *Let $X$ be a variety of Calabi–Yau type. How does the nef cone* $\mathrm{Nef}(X) \subset N^1(X)_{\mathbb{R}}$ *look like?*

For Fano varieties, i.e. smooth projective varieties $X$ such that $-K_X$ is ample, it is a consequence of Mori's Cone Theorem that $\mathrm{Nef}(X)$ is a rational polyhedral cone. Here we recall the following definition from Convex Geometry:

**Definition 2.8.** A convex cone $C$ in a finite-dimensional $\mathbb{R}$-vector space $V$ is **polyhedral** if it is spanned by finitely many extremal rays, i.e.

$$C = \sum_{i=1}^{k} \mathbb{R}_{\geq 0} v_i$$

for some $v_1, \ldots, v_k \in V$.

For an open subset $U \subset V$, a convex cone $C \subset V$ is **locally polyhedral** in $U$ if for every point $p \in U$ there is a compact neighborhood $K$ of $p$ such that $C \cap K$ is a polytope (i.e. the convex hull of finitely many points).

For varieties of Calabi–Yau type however, we cannot expect a similar description of $\mathrm{Nef}(X)$, as there are examples where the nef cone is not rational or not finitely generated. For instance, [**Tot12**, Section 4.1] provides examples of surfaces of Calabi–Yau type whose nef cone is a round cone (and in particular not rational) or a rational cone with infinitely many extremal rays.

The Cone Conjecture predicts that this failure of the nef cone to be rational polyhedral is explained by the action of the automorphism group $\mathrm{Aut}(X)$ on $\mathrm{Nef}(X)$. To make this precise, we need the following definition of a fundamental domain:

**Definition 2.9.** Let $V$ be a finite-dimensional vector space over $\mathbb{R}$ and let $G \to \mathrm{GL}(V)$ be a linear group action on $V$. If $\mathcal{C}$ is a subset of $V$ such that $G \cdot \mathcal{C} = \mathcal{C}$, then a **fundamental domain** for the action of $G$ on $\mathcal{C}$ is a subset $\Pi \subset \mathcal{C}$ such that

(i) $G \cdot \Pi = \mathcal{C}$, i.e. $\bigcup_{g \in G} g(\Pi) = \mathcal{C}$, and
(ii) If $g(\Pi) \cap \Pi$ has non-empty interior for some $g \in G$, then $g \in \ker(G \to \mathrm{GL}(V))$.

Certainly, this is not a very well-suited definition for arbitrary sets $\mathcal{C} \subset V$, e.g. if $\mathcal{C}$ has empty interior. However, for us, $\mathcal{C}$ will always be a full-dimensional convex cone, and the above definition permits us to construct fundamental domains which are closed.

The hope is now that the action of $\mathrm{Aut}(X)$ on the nef cone $\mathrm{Nef}(X)$ permits a rational polyhedral cone as a fundamental domain. But this cannot be true if $\mathrm{Nef}(X)$ has an irrational extremal ray as in one of the examples of [**Tot12**, Section 4.1]. Indeed, such a ray cannot be contained in $\bigcup_{f \in \mathrm{Aut}(X)} f^*(\Pi)$ for a rational polyhedral cone $\Pi$. Hence, we must pay closer attention to the boundary of $\mathrm{Nef}(X)$.

**Definition 2.10.** Let $X$ be a smooth projective variety. We define

$$\mathrm{Nef}^+(X) := \mathrm{conv}(\mathrm{Nef}(X) \cap N^1(X)_{\mathbb{Q}})$$

and

$$\mathrm{Nef}^{\mathrm{e}}(X) := \mathrm{Nef}(X) \cap \mathrm{Eff}(X).$$

Moreover, we define the group $\mathcal{A}(X)$ to be the image of the natural group homomorphism

$$\mathrm{Aut}(X)^{\mathrm{op}} \to \mathrm{GL}(N^1(X)_{\mathbb{R}}),\ f \mapsto f^*$$

given by pulling back divisor classes by an automorphism of $X$.

**Remark 2.11.** Note that $\mathrm{Amp}(X) \subset \mathrm{Nef}^+(X) \subset \mathrm{Nef}(X)$ and $\mathrm{Amp}(X) \subset \mathrm{Nef}^{\mathrm{e}}(X) \subset \mathrm{Nef}(X)$.

The cone $\mathrm{Nef}^{\mathrm{e}}(X)$ (conjecturally) plays a particular role in the study of the geometry of a variety of Calabi–Yau type $X$: The Log Abundance Conjecture for $X$ predicts that any nef effective divisor is semiample, giving rise to a morphism $X \to Y$ with connected fibers to a variety $Y$, which may be used to study the geometry of $X$. The Log Abundance Conjecture has been confirmed for $\dim X \leq 3$ in [**KMM94**] and [**KMM04**].

The following conjecture is due to Morrison [**Mor93**], predicting that $\mathrm{Nef}^+(X)$ is up to the action of $\mathrm{Aut}(X)$ a rational polyhedral cone.

**Conjecture 2.12** (Morrison's Cone Conjecture)**.** *Let $X$ be a variety of Calabi–Yau type. Then there exists a rational polyhedral cone which is a fundamental domain for the action of* $\mathrm{Aut}(X)$ *on* $\mathrm{Nef}^+(X)$*.*

This Conjecture was inspired by Mirror Symmetry. In [**Kaw97**], Kawamata referred to Morrison's Cone Conjecture with a slightly different formulation, replacing $\mathrm{Nef}^+(X)$ by $\mathrm{Nef}^{\mathrm{e}}(X)$.

**Conjecture 2.13** (Kawamata's Cone Conjecture)**.** *Let $X$ be a variety of Calabi–Yau type. Then there exists a rational polyhedral cone which is a fundamental domain for the action of* $\mathrm{Aut}(X)$ *on* $\mathrm{Nef}^{\mathrm{e}}(X)$*.*

Regarding the relation between the two versions of the Cone Conjecture, there is the following result.

**Proposition 2.14** ([**LOP16**]). *Let $X$ be a variety of Calabi–Yau type. Then* $\mathrm{Nef}^{\mathrm{e}}(X) \subset \mathrm{Nef}^{+}(X)$.

The exact relation between Morrison's and Kawamata's version of the Cone Conjecture is revealed by the following result from Convex Geometry:

**Theorem 2.15** ([**Loo14**]). *Let $V$ be a finite-dimensional vector space over $\mathbb{R}$ equipped with an integral structure $F \subset V$ and let $\mathcal{C} \subset V$ be a full-dimensional closed convex cone containing no lines. Let $G \subset$* $\mathrm{GL}(F)$ *be a group preserving $\mathcal{C}$, i.e. $G \cdot \mathcal{C} = \mathcal{C}$.*

*If there is a rational polyhedral cone $\Pi \subset \mathcal{C}^{+}$ such that $G \cdot \Pi \supset$* $\mathrm{Int}(\mathcal{C})$*, then $G \cdot \Pi = \mathcal{C}^{+}$, where $\mathcal{C}^{+} := \mathrm{conv}(C \cap F_{\mathbb{Q}})$, and there exists a rational polyhedral cone $\Pi' \subset \mathcal{C}^{+}$ which is a fundamental domain for the action of $G$ on $\mathcal{C}^{+}$.*

This implies the following consequence, as observed in [**LOP16**]:

**Corollary 2.16.** *Let $X$ be a variety of Calabi–Yau type. Then Kawamata's Cone Conjecture holds for $X$ if and only if* $\mathrm{Nef}^{\mathrm{e}}(X) = \mathrm{Nef}^{+}(X)$ *and Morrison's Cone Conjecture holds for $X$.*

Proof. If we assume Kawamata's Cone Conjecture for $X$, then there exists a rational polyhedral cone $\Pi \subset \mathrm{Nef}^{\mathrm{e}}(X)$ such that

$$\mathcal{A}(X) \cdot \Pi = \mathrm{Nef}^{\mathrm{e}}(X) \supset \mathrm{Amp}(X).$$

By Proposition 2.14, $\Pi \subset \mathrm{Nef}^{+}(X)$, so the first part of 2.15 implies $\mathrm{Nef}^{\mathrm{e}}(X) = \mathrm{Nef}^{+}(X)$. □

A further consequence of Theorem 2.15 is the following characterization of Morrison's Cone Conjecture for varieties with finite automorphism group:

**Corollary 2.17.** *Let $X$ be a variety of Calabi–Yau type. If* $\mathrm{Nef}(X)$ *is a rational polyhedral cone, then $\mathcal{A}(X)$ is a finite group and Morrison's Cone Conjecture holds on $X$. Conversely, if $\mathcal{A}(X)$ is a finite group and we assume that Morrison's Cone Conjecture holds on $X$, then* $\mathrm{Nef}(X)$ *is a rational polyhedral cone.*

Proof. If $\mathrm{Nef}(X)$ is a rational polyhedral cone, there are only finitely many primitive integral classes on the extremal rays of $\mathrm{Nef}(X)$. Any $\varphi \in \mathcal{A}(X)$ permutes these classes and is uniquely determined by this permutation since $\mathrm{Nef}(X)$ is a full-dimensional cone. There are only finitely many of these permutations, so $\mathcal{A}(X)$ is a finite group. Applying Theorem 2.15 to the convex cone $\mathrm{Nef}(X)$ and the rational polyhedral subcone $\Pi = \mathrm{Nef}(X)$, we get a rational polyhedral cone $\Pi'$ which is a fundamental domain for the action of $\mathrm{Aut}(X)$ on $\mathrm{Nef}^+(X)$.

Conversely, if $\mathcal{A}(X)$ is a finite group and Morrison's Cone Conjecture holds on $X$, then there is a rational polyhedral cone $\Pi$ which is a fundamental domain for the action of $\mathrm{Aut}(X)$ on $\mathrm{Nef}^+(X)$. Since $\mathrm{Aut}(X)$ is finite, $\mathrm{Nef}^+(X) = \mathrm{Aut}(X) \cdot \Pi$ is also a rational polyhedral cone and therefore coincides with its closure $\mathrm{Nef}(X)$. □

This is a convenient point to recall a result of Oguiso [**Ogu14**, Proposition 2.4]:

**Proposition 2.18.** *Let $X$ be a Calabi–Yau variety. Then the kernel of*

$$\mathrm{Aut}(X)^{\mathrm{op}} \to \mathcal{A}(X),\ \ f \mapsto f^*$$

*is a finite group. In particular,* $\mathrm{Aut}(X)$ *is finite if and only if* $\mathcal{A}(X)$ *is.*

As a final consequence, we observe that the second part of Theorem 2.15 also yields the following:

**Observation 2.19.** *In order to prove Morrison's (or Kawamata's) Cone Conjecture for a variety of Calabi–Yau type $X$, it is sufficient to construct a fundamental domain for the action of a subgroup $G \subset \mathcal{A}(X)$.*

With this in mind, we can restrict all considerations to working with the following subgroup:

**Definition 2.20.** Let $X$ be a smooth projective variety. Then we define

$$\mathcal{A}_+(X) := \mathcal{A}(X) \cap \mathrm{SL}(N^1(X)_{\mathbb{R}})$$

and denote by $\mathrm{Aut}_+(X)$ its preimage under $\mathrm{Aut}(X) \to \mathcal{A}(X)$.

Note that $\mathcal{A}_+(X)$ is a subgroup of index 2 of $\mathcal{A}(X)$, as $\mathcal{A}(X) \subset \mathrm{GL}(N^1(X)_{\mathbb{Z}})$ implies $|\det \varphi| = 1$ for all $\varphi \in \mathcal{A}(X)$.

We will obtain explicit descriptions of the group $\mathcal{A}_+(X)$ for Calabi–Yau threefolds with Picard number 3 in the following chapters and will use this information to deduce results on the Cone Conjecture.

We conclude this section by formulating a more modest, weaker version of the Cone Conjecture, permitting the fundamental domain to have finitely many exceptional faces where it fails to be rational polyhedral.

**Definition 2.21.** Let $V$ be a finite-dimensional vector space over $\mathbb{R}$ equipped with an integral structure. A closed convex cone $\mathcal{C} \subset V$ is

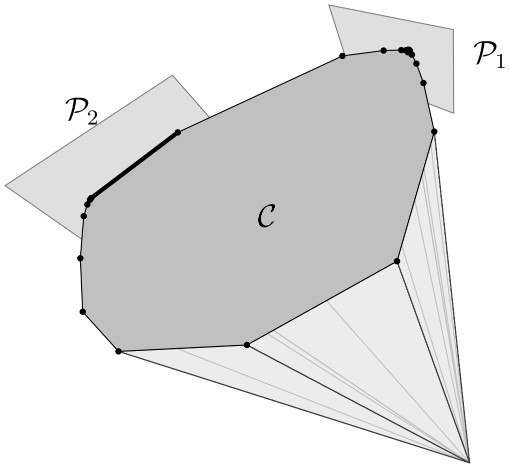


FIGURE 2.1. An almost rational polyhedral cone with two exceptional faces. The exceptional faces may be irrational and the extremal rays of $\mathcal{C}$ may accumulate towards them.

called an **almost rational polyhedral cone** if there are finitely many supporting hyperplanes $\mathcal{P}_1, \dots, \mathcal{P}_r$ of $\mathcal{C}$ such that the convex cone

$$\mathcal{C}_0 := \mathcal{C} \setminus \bigcup_{i=1}^{r} \mathcal{P}_i$$

is locally rational polyhedral inside the open set $V \setminus \bigcup_{i=1}^{r} P_i$. See Figure 2.3.

The sets $\mathcal{C} \cap \mathcal{P}_i$ are called the **exceptional faces** of $\mathcal{C}$.

**Conjecture 2.22** (Weak Cone Conjecture for $\mathrm{Nef}^+$ resp. $\mathrm{Nef}^{\mathrm{e}}$)**.** *Let $X$ be a variety of Calabi–Yau type. Then there is an almost rational polyhedral cone $\Pi \subset \mathrm{Nef}(X)$ such that $\Pi^+ := \mathrm{conv}(\Pi \cap N^1(X)_{\mathbb{Q}})$ is a fundamental domain for the action of $\mathcal{A}_+(X)$ on $\mathrm{Nef}^+(X)$ (resp. $\mathrm{Nef}^{\mathrm{e}}(X)$).*

## 2.4. Rational matrix groups

Here, we recall an important finiteness result on matrix groups due to Burnside [**Bur05**], which will facilitate the study of the group $\mathcal{A}_+(X)$.

**Theorem 2.23** ([**Bur05**])**.** *Let $V$ be a finite-dimensional vector space over $\Bbbk$ and let $G$ be a subgroup of $\mathrm{GL}(V)$. If there is a constant $N$ such that each element of $G$ is of finite order $\leq N$, then $G$ is a finite group.*

For matrix groups over $\mathbb{Z}$, we can drop the uniform bound, showing that $\mathcal{A}_+(X)$ is either a finite group or contains an element of infinite order:

**Corollary 2.24.** *Let $V$ be a finite-dimensional vector space over $\Bbbk$ with an integral structure $F \subset V$. If $G$ is a subgroup of $\mathrm{GL}(F)$ and each element of $G$ has finite order, then $G$ is a finite group.*

*In particular, for a Calabi–Yau variety $X$, either $\mathcal{A}_+(X)$ contains an element of infinite order or $\mathrm{Aut}(X)$ is finite.*

Proof. The second claim follows from the first because the kernel of $\mathrm{Aut}(X) \to \mathcal{A}(X)$ is finite by Proposition 2.18 and $\mathcal{A}_+(X)$ is a subgroup of $\mathcal{A}(X)$ of index at most 2.

To deduce the first claim from Theorem 2.23, we want to show that the order of any $\varphi \in G$ is bounded by some fixed constant. Let $\varphi \in G$ be of order $N$. Then the minimal polynomial $\mu \in \mathbb{Z}[X]$ of $\varphi$ divides $X^N - 1$, so over $\mathbb{C}$ it splits into distinct linear factors. In particular, $\psi$ diagonalizes over $\mathbb{C}$ and its eigenvalues $\xi_1, \ldots, \xi_m$ ($m \leq \dim V$) must be $N$-th roots of unity. Then $\mu(\xi_i) = 0$ for all $i$, so if $\xi_i$ is a $N_i$-th primitive root of unity, then the $N_i$-th cyclotomic polynomial $\Phi_{N_i}$ must divide the characteristic polynomial of $\varphi$, because the latter is a polynomial in $\mathbb{Z}[X]$. For degree reasons we get $\phi(N_i) \leq \dim V$, where $\phi$ denotes the Euler function. If we define $K$ to be the maximum integer such that $\phi(K) \leq \dim V$, then we get $N_i \leq K$. Certainly, $\psi^{N_1 \cdot \ldots \cdot N_m} = \mathrm{id}$, so $N \leq N_1 \cdot \ldots \cdot N_m \leq K^{\dim V}$. We deduce from Theorem 2.23 that $G$ is finite. $\square$

CHAPTER 3

# The intersection form of a variety

Since many properties of divisors like being ample or being nef and big are of numerical nature and can be expressed in terms of the intersection product, it is a natural question to investigate the intersection product.

For instance, in the case of a surface $X$ the intersection product defines a bilinear form on $N^1(X)_\mathbb{R} = N_1(X)_\mathbb{R}$ and its classification up to a $\mathbb{Z}$-basis of $N^1(X)_\mathbb{Z}$, i.e. as a lattice, is an important invariant in the study of surfaces. The classification up to a real basis of $N^1(X)_\mathbb{R}$ is much simpler, as the Hodge Index Theorem asserts that the bilinear form always has the signature $(s_+, s_-, s_0) = (1, \rho(X) - 1, 0)$.

In higher dimensions the situation is however much more involved. If $X$ is a smooth projective variety of dimension $n$, the intersection product

$$N^1(X)_\mathbb{Z} \times \cdots \times N^1(X)_\mathbb{Z} \to \mathbb{Z},\ (x_1, \ldots, x_n) \mapsto x_1 \cdot \ldots \cdot x_n,$$

extends linearly to

$$N^1(X)_\mathbb{R} \times \cdots \times N^1(X)_\mathbb{R} \to \mathbb{R},\ (x_1, \ldots, x_n) \mapsto x_1 \cdot \ldots \cdot x_n.$$

This is a symmetric $n$-linear form on the vector space $N^1(X)_\mathbb{R}$, i.e. an element of $\mathrm{Sym}^n(N^1(X)_\mathbb{R}^\vee)$. The classification of this multisymmetric form up to a basis of $N^1(X)_\mathbb{R}$ is much more diverse for $n \geq 3$ than the uniform answer in the case of surfaces.

In this chapter, we will study the intersection form by investigating its zero set given by

$$\mathcal{N} = \{\alpha \in N^1(X)_\mathbb{R} \mid \alpha^n = 0\}.$$

We start out in Section 3.1 by observing that this zero set typically reflects the properties of the intersection form up to a scalar. In Section 3.2 we give criteria for the vanishing set $\mathcal{N}$ to be reducible, splitting of a hyperplane. Finally, in Section 3.3, we use the results to obtain a classification of the intersection form for Calabi–Yau threefolds of Picard number 3.

## 3.1. The null cone of a variety

We start out by observing two basic properties of the intersection product.

**Observation 3.1.** *Let $X$ be a variety of dimension $n$ and let $f \in \mathrm{Aut}(X)$. Then*

$$f^*x_1 \cdot \ldots \cdot f^*x_n = x_1 \cdot \ldots \cdot x_n$$

*for all $x_1, \ldots, x_n \in N^1(X)_{\Bbbk}$.*

**Lemma 3.2.** *Let $X$ be a smooth projective variety of dimension $n$. Then the intersection product on $N^1(X)_{\Bbbk}$ is non-degenerate in the following sense: Let $x \in N^1(X)_{\Bbbk}$ such that*

$$x \cdot y_2 \cdot \ldots \cdot y_n = 0$$

*for all $y_2, \ldots, y_n \in N^1(X)_{\Bbbk}$. Then $x = 0$.*

Proof. We fix an integral ample class $h$ on $X$. By assumption we know that $x \cdot h^{n-2} \cdot y = 0$ for all $y \in N^1(X)_{\Bbbk}$, so $x \cdot h^{n-2} = 0$ in $N_1(X)_{\Bbbk}$ (by the definition of numerical equivalence). But the Hard Lefschetz Theorem implies that

$$N^1(X)_{\Bbbk} \to N_1(X)_{\Bbbk}, \; z \mapsto z \cdot h^{n-2}$$

is an isomorphism. From this we deduce $x = 0$. □

As a consequence of these two properties, the intersection product induces restrictions on the possible linear automorphisms in $\mathcal{A}_+(X)$. This will yield particularly strong implications on the structure of $\mathcal{A}_+(X)$ when $X$ is a threefolds with Picard number 3 (see section 3.3).

In this chapter, we abstract from the specific situation of studying the intersection product on a variety and linear automorphisms on $N^1(X)_{\mathbb{R}}$ to an abstract vector space over $\mathbb{R}$ equipped with a (non-degenerate) symmetric multilinear form. Although we may only apply the results of this section to the case of $N^1(X)_{\mathbb{R}}$ for a variety $X$, we prefer to formulate the results in the abstract setting whenever possible to highlight which results follow purely from methods of (multi-)Linear Algebra.

**Definition 3.3.** Let $V$ be a finite-dimensional $\Bbbk$-vector space and let $\omega \in \mathrm{Sym}^n(V^\vee)$ be a symmetric $n$-linear form on $V$. Then we call

$$\mathcal{N} := \{x \in V \mid \omega(x, \ldots, x) = 0\}$$

the **null cone** of $\omega$. A linear automorphism $\varphi \in \mathrm{GL}(V)$ is **compatible with** $\omega$ if

$$\omega(\varphi(x_1), \ldots, \varphi(x_n)) = \omega(x_1, \ldots, x_n)$$

holds for all $x_1, \dots, x_n \in V$. The group of such $\varphi$ is denoted by $O(\omega) \subset \mathrm{GL}(V)$. A linear automorphism $\varphi \in \mathrm{GL}(V)$ **preserves** $\mathcal{N}$ if

$$\varphi(\mathcal{N}) = \mathcal{N}.$$

It can be seen that if $V$ is a finite-dimensional vector space over an algebraically closed field, then studying a multilinear form $\omega \in \mathrm{Sym}^n(V^\vee)$ is equivalent to studying its null cone $\mathcal{N}$ in the sense that we can recover $\omega$ from $\mathcal{N}$ up to scalar and a linear automorphism in $\mathrm{GL}(V)$ preserving $\mathcal{N}$ is compatible with $\omega$ up to a scalar. (This can be seen by similar arguments as in the proof of the following Proposition below.) However, we are interested in vector spaces over $\mathbb{R}$, for which the following variant holds:

**Proposition 3.4.** *Let $V$ be a finite-dimensional $\mathbb{R}$-vector space, let $\omega \in \mathrm{Sym}^n(V^\vee)$ be a symmetric $n$-linear form on $V$ and consider its null cone $\mathcal{N}$. Assume that for some point $z \in V \setminus \mathcal{N}$ the interior of the set $W_z \subset V$ is nonempty, where*

$$W_z := \{x \in V \setminus \{z\} \mid \textit{the affine line through } z \textit{ and } x \\ \textit{intersects } \mathcal{N} \textit{ in} \geq n \textit{ points}\}.$$

*Then the following holds:*

*(i) The set $\mathcal{N}$ determines $\omega$ uniquely up to a scalar factor.*
*(ii) An automorphism $\varphi \in \mathrm{GL}(V)$ is compatible with $\omega$ up to a scalar factor if and only if it preserves $\mathcal{N}$.*

Proof. First, we note that the map $h\colon V \to \mathbb{R},\ x \mapsto \omega(x, \dots, x)$ uniquely determines $\omega$: Indeed, if $x_1, \dots, x_n \in V$, then the map

$$\mathbb{R}^n \to \mathbb{R},\ (t_1, \dots, t_n)^T \mapsto h(t_1 x_1 + \ldots + t_n x_n)$$

is given by a homogeneous polynomial of degree $n$ whose coefficient of $t_1 \dots t_n$ is precisely $n! \cdot \omega(x_1, \dots, x_n)$. Thus, the value $\omega(x_1, \dots, x_n)$ is uniquely determined by $h$.

We also observe, that $\varphi \in \mathrm{SL}(V)$ is compatible with $\omega$ if and only if $h \circ \varphi = h$: This follows from the fact that the coefficient of $t_1 \dots t_n$ of the polynomial mapping

$$\mathbb{R}^n \to \mathbb{R},\ (t_1, \dots, t_n)^T \mapsto h(\varphi(t_1 x_1 + \ldots + t_n x_n))$$

is precisely $n! \cdot \omega(\varphi(x_1), \dots, \varphi(x_n))$.

Next, we show that $\mathcal{N}$ determines $h$ up to scalar. After a choice of basis for $V$ (resp. $V^\vee$), we can view $h$ as a homogeneous polynomial of degree $n$ and $\mathcal{N}$ as its zero locus. Assume that $g$ is another homogeneous polynomial of degree $n$ with zero locus $\mathcal{N}$. Then

$$g' := h(z)g - g(z)h$$

is a homogeneous polynomial of degree $n$ vanishing at more than $n$ points along the affine line through $z$ and any $x \in W_z$. Thus, $g'$ is zero along all such lines, i.e., $g'$ vanishes at the set $W_z \subset V$. But by the assumption on $z$, the set $W_z$ has nonempty interior, implying $g' = 0$. Therefore, $g$ is a scalar multiple of $h$.

If $\varphi \in \mathrm{SL}(V)$ preserves $\mathcal{N}$, then the zero sets of $h \circ \varphi$ and $h$ coincide, so by the above argument $h \circ \varphi = \lambda h$ for some $\lambda \in \mathbb{R}$. □

## 3.2. Reducibility of the null cone

Viewing $\mathcal{N}$ as a vanishing set of a homogeneous polynomial of degree $n$, i.e. as a hypersurface of degree $n$, it is natural to ask when $\mathcal{N}$ is a reducible algebraic set. Especially, we address the following question: When does $\mathcal{N}$ split as a hyperplane and a hypersurface of degree $n-1$? We will attribute this splitting to the existence of an automorphism preserving $\mathcal{N}$ with certain spectral properties. Note that for considerations about the eigenvalues of linear automorphisms it is most natural to pass to a vector space over $\mathbb{C}$.

**Definition 3.5.** Let $V$ be a finite-dimensional vector space over $\mathbb{C}$ and let $n \in \mathbb{N}$ be any positive integer. We say that an automorphism $\varphi \in \mathrm{GL}(V)$ **has good spectral properties with respect to** $n$ if $\varphi$ has a real eigenvalue $\mu$ such that:

(i) the generalized eigenspace of $\varphi$ with respect to $\mu$ is one-dimensional,
(ii) for any $\lambda_1, \dots, \lambda_n \in \mathrm{Eig}(\varphi) \setminus \{\mu\}$ (not necessarily distinct) we have
$$\prod_{i=1}^{n} \lambda_i \neq 1$$
.

We say that $\varphi$ **has very good spectral properties with respect to** $n$ if $\mu$ can be chosen to be 1.

An automorphism $\varphi$ of a finite-dimensional vector space $V$ over $\mathbb{R}$ **has (very) good spectral properties with respect to** $n$, if $\varphi \otimes \mathbb{C} \in \mathrm{GL}(V \otimes \mathbb{C})$ does.

The main criterion for a splitting of the null cone is given by the following Lemma.

**Lemma 3.6.** *Let $V$ be a finite-dimensional $\mathbb{R}$-vector space and let $\omega \in \mathrm{Sym}^n(V^\vee)$ be a symmetric $n$-linear form on $V$. Assume that there exists an automorphism $\varphi \in \mathrm{GL}(V)$ which is compatible with $\omega$ and has good spectral properties with respect to $n$. Then the null cone*
$$\mathcal{N} := \{x \in V \mid \omega(x, \dots, x) = 0\}$$
*contains a hyperplane in $V$.*

*Explicitly, the hyperplane contained in* $\mathcal{N}$ *is given by* $\{\ell = 0\}$ *for a non-zero linear form* $\ell \in V^\vee$ *with* $\ell \circ \varphi = \mu\ell$*, where* $\mu$ *is as in Definition 3.5.*

PROOF. Let $\mu$ be the real eigenvalue of $\varphi$ as given by Definition 3.5. Since the dual automorphism $\varphi^\vee$ on the dual vector space $V^\vee$ has the same minimal polynomial as $\varphi$, there is a nonzero linear form $\ell\colon V \to \mathbb{R}$ such that $\ell \circ \varphi = \varphi^\vee(\ell) = \mu\ell$. We show that the hyperplane $\{\ell = 0\}$ is contained in $\mathcal{N}$.

Note that it suffices to consider the automorphism $\varphi_{\mathbb{C}} := \varphi \otimes \mathbb{C}$ on the complex vector space $V_{\mathbb{C}} := V \otimes \mathbb{C}$. Indeed, if we show that the hyperplane $\{\ell_{\mathbb{C}} = 0\} \subset V_{\mathbb{C}}$ defined by the linear form $\ell_{\mathbb{C}} := \ell \otimes \mathbb{C}$ is contained in

$$\mathcal{N}_{\mathbb{C}} := \{x \in V_{\mathbb{C}} \mid \omega_{\mathbb{C}}(x, \ldots, x) = 0\},$$

then the claim follows. We therefore now work over $\mathbb{C}$, but omit the subscript $\mathbb{C}$ for better readability.

We can choose a basis $x_1, \ldots, x_m$ of $V$ such that $\varphi$ has Jordan normal form with respect to this basis, i.e. for some $r \in \mathbb{N}$ there exist $\lambda_1, \ldots, \lambda_r \in \mathbb{C}$ and $1 = j_1 < \cdots < j_r < j_{r+1} = m + 1$ such that

$$\varphi(x_i) = \begin{cases} \lambda_k x_i & \text{if } i = j_k \\ \lambda_k x_i + x_{i-1} & \text{if } j_k < i < j_{k+1}. \end{cases}$$

As the generalized $\mu$-eigenspace of $\varphi$ is one-dimensional, we may assume that $\lambda_r = \mu$ and $j_k = m$, while $\lambda_i \neq \mu$ for $i < r$. In order to show that $\{\ell = 0\} \subset \mathcal{N}$, it suffices to verify the following two claims:

**Claim 1**: $\{\ell = 0\} = \langle x_1, \ldots, x_{m-1}\rangle$.

**Claim 2**: The restriction of the symmetric $n$-linear form $\omega$ to $\langle x_1, \ldots, x_{m-1}\rangle$ is trivial, i.e.

$$\omega(x_{i_1}, \ldots, x_{i_n}) = 0 \quad \text{for all } i_1, \ldots, i_n \in \{1, \ldots, m-1\}.$$

*Proof of Claim 1*: This follows from $\ell \circ \varphi = \mu\ell$. Indeed, if $i$ is the smallest index in $\{1, \ldots, m\}$ such that $\ell(x_i) \neq 0$, then

$$\mu\ell(x_i) = \ell(\varphi(x_i)) = \lambda_k \ell(x_i),$$

where $k$ is such that $j_k \leq i < j_{k+1}$. Consequently, we must have $\mu = \lambda_k$, i.e. $i = m$.

*Proof of Claim 2*: This follows from the fact that $\varphi$ is compatible with $\omega$. Assume for contradiction that $\omega$ restricted to $W := \langle x_1, \ldots, x_{m-1}\rangle$ is non-zero. Then we can inductively define $i_s$ (for $s \in \{1, \ldots, n\}$) as the smallest index in $\{1, \ldots, m-1\}$ such that the symmetric $(n-s)$-linear form $\omega(x_{i_1}, \ldots, x_{i_s}, \_, \ldots, \_)$ is non-zero on $W$.

Let $k_s \in \{1, \dots, r-1\}$ be such that $j_{k_s} \le i_s < j_{k+1}$. Note that the definition of $i_1$ guarantees

$$\omega(\varphi(x_{i_1}), \dots, \varphi(x_{i_n})) = \lambda_{k_1}\omega(x_{i_1}, \varphi(x_{i_2}), \dots, \varphi(x_{i_n})),$$

which in turn by definition of $i_2$ equals

$$\lambda_{k_1}\lambda_{k_2}\omega(x_{i_1}, x_{i_2}, \varphi(x_{i_3}), \dots, \varphi(x_{i_n})).$$

Inductively, we get

$$\omega(\varphi(x_{i_1}), \dots, \varphi(x_{i_n})) = \prod_{s=1}^{n} \lambda_{k_s} \cdot \omega(x_{i_1}, \dots, x_{i_n}).$$

As $\varphi$ is compatible with $\omega$ and $\omega(x_{i_1}, \dots, x_{i_n}) \neq 0$ by construction, we deduce that $\prod_{s=1}^{n} \lambda_{k_s} = 1$, contradicting property (ii) in Definition 3.5. □

As a first consequence, we obtain:

**Proposition 3.7.** *Let $V$ be a finite-dimensional $\mathbb{R}$-vector space and let $\omega \in \mathrm{Sym}^n(V^\vee)$ be a symmetric $n$-linear form on $V$. Assume that for some $k < n$ there is a $\varphi \in O(\omega)$ having very good spectral properties with respect to each $i \in \{k+1, \dots, n\}$. Then the null cone $\mathcal{N}$ of $\omega$ is the union of a hyperplane and the null cone of a symmetric $k$-linear form.*

*If $V$ is equipped with an integral structure $F \subset V$ and $\varphi \in \mathrm{GL}(F)$, then the hyperplane is rational.*

Proof. This follows from repeated application of Lemma 3.6: Let $\ell \in V^\vee$ be an eigenvector of $\varphi^\vee$ with eigenvalue 1 (which is unique up to scalar by Definition 3.5). Then Lemma 3.6 implies that $\{\ell = 0\} \subset \mathcal{N}$. After a choice of basis for $V$, we can view

$$h \colon V \to \mathbb{R},\ x \mapsto \omega(x, \dots, x)$$

as a homogeneous polynomial mapping of degree $n$ with vanishing set $\mathcal{N}$, so viewing $\ell$ as a linear form with respect to the chosen basis, we deduce $h = \ell \cdot h'$ for some homogeneous polynomial mapping $h' \colon V \to \mathbb{R}$ of degree $n-1$.

As $h \circ \varphi = h$ and $\ell \circ \varphi = \ell$ (by the choice of $\ell$), we get $h' \circ \varphi = h'$. Thus, if we consider the symmetric $(n-1)$-linear form $\omega' \in \mathrm{Sym}^{n-1}(V^\vee)$ associated to $h'$ by defining $\omega'(x_1, \dots, x_{n-1}) \in \mathbb{R}$ to be the $1/(n-1)!$ times the coefficient of $t_1 \dots t_{n-1}$ in the polynomial mapping

$$\mathbb{R}^{n-1} \to \mathbb{R},\ (t_1, \dots, t_{n-1})^T \mapsto h'(t_1 x_1 + \dots + t_{n-1} x_{n-1}),$$

then we see that $\varphi \in O(\omega')$. Replacing $\omega$ by $\omega'$ and $n$ by $n-1$, we may now sucessively repeat the argument $n-k$ times and deduce the claim.

If $\varphi \in \mathrm{GL}(F)$ for an integral structure $F \subset V$, then the subspace $\ker(\varphi^\vee - \mathrm{id})$ of $V^\vee$ is rational with respect to the natural integral structure $F^\vee$ on $V^\vee$. In particular, we may choose $\ell \in F^\vee$, i.e. $\{\ell = 0\}$ is a rational hyperplane in $V$. ☐

The assumption on the spectral properties of elements in $O(\omega) \cap \mathrm{SL}(F)$ is rather technical, but when $\dim V = 3$, we can deduce the following general result:

**Theorem 3.8.** *Let $V$ be a three-dimensional vector space over $\mathbb{R}$ with an integral structure $F \subset V$ and a non-degenerate $n$-linear symmetric form $\omega \in \mathrm{Sym}^n(V^\vee)$ for $n$ odd. If the group $O(\omega) \cap \mathrm{SL}(F)$ is neither trivial nor of order three, then the null cone $\mathcal{N}$ of $\omega$ splits as the union of a plane and the vanishing set of a homogeneous polynomial of degree $n-1$.*

Proof. We assume for contradiction that $\mathcal{N}$ does not contain a plane.

Let $\varphi \in O(\omega) \cap \mathrm{SL}(F)$ such that $\varphi \neq \mathrm{id}$. We claim that $\varphi$ must be of order 3. Knowing that, we can deduce that $O(\omega) \cap \mathrm{SL}(F)$ is a $p$-group for $p = 3$. By Theorem 2.23, we also know that $O(\omega) \cap \mathrm{SL}(F)$ is a finite group, so by properties of $p$-groups we deduce that the order of $O(\omega) \cap \mathrm{SL}(F)$ is a power of 3. On the other hand, it was shown by Minkowski in [**Min87**] that the order of any finite subgroup of $\mathrm{GL}(3, \mathbb{Q})$ divides 48, so $|O(\omega) \cap \mathrm{SL}(F)| = 3$, contradicting the assumption.

We will show $\operatorname{ord} \varphi = 3$ by examining the set of eigenvalues $\mathrm{Eig}_{\mathbb{C}}(\varphi) := \mathrm{Eig}(\varphi \otimes \mathbb{C})$. Note that by Lemma 3.6, $\varphi$ cannot have good spectral properties with respect to $n$. We distinguish three cases:

**Case 1**: $\varphi \otimes \mathbb{C}$ has a non-real eigenvalue.

Then $\mathrm{Eig}_{\mathbb{C}} = \{a, \lambda, \overline{\lambda}\}$ for some $a \in \mathbb{R}$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$. As $\varphi$ does not have good spectral properties with respect to $n$, we must have $\lambda^i \overline{\lambda}^{n-i} = 1$ for some $i \in \{0, \ldots, n\}$. In particular, we deduce $|\lambda| = 1$, so $\det \varphi = 1$ implies $a = 1$. This in turn shows

$$2 \cdot \mathrm{Re}(\lambda) = \lambda + \overline{\lambda} = \mathrm{Tr}(\varphi) - 1.$$

As $\varphi \in \mathrm{SL}(F)$, we have $\mathrm{Tr}(\varphi) \in \mathbb{Z}$, so $\mathrm{Re}(\lambda) \in \frac{1}{2}\mathbb{Z}$. The only elements of $\mathbb{C} \setminus \mathbb{R}$ with this property are the primitive $r$-th roots of unity for $r \in \{3, 4, 6\}$. From $1 = \lambda^i \overline{\lambda}^{n-i} = \lambda^{2i-n}$ and the fact that $n$ is odd we deduce that $\lambda$ is a primitive third root of unity. Since in a suitable basis of $V$ we have $\varphi \otimes \mathbb{C} = \mathrm{diag}(1, \lambda, \overline{\lambda})$, we see that $\varphi$ is of order 3.

**Case 2**: $\varphi$ diagonalizes (over $\mathbb{R}$).

If $\varphi$ has a multiple eigenvalue, then in a suitable basis of $V$ we have $\varphi = \mathrm{diag}(a, a, 1/a^2)$ for some $a \in \mathbb{R}$. Since $\varphi$ is not the identity, $a \neq 1$, so $a^3 \neq 1$ and $a^n \neq 1$ (as $n$ is odd). But then $\varphi$ has good spectral properties with respect to $n$ (choosing $\mu = 1/a^2$ in Definition 3.5).

Therefore, $\varphi$ has three distinct real eigenvalues $a, b, c \in \mathbb{R}$ with $abc = \det \varphi = 1$. At least two of them are not 1, so we may assume $a \neq 1$, $b \neq 1$. As the generalized eigenspace of $\varphi$ with respect to $c$ is one-dimensional, but $\varphi$ does not have good spectral properties with respect to $n$, we must have $a^i b^{n-i} = 1$ for some $i \in 0, \dots, n$. By possibly interchanging $a$ and $b$, we can assume $i$ to be odd. If $\alpha \in \mathbb{R} \setminus \{1\}$ is the unique real number with $\alpha^i = b$, then $a^i b^{n-i} = 1$ implies $a = \alpha^{i-n}$ and, thus, $c = \alpha^{n-2i}$. Then both $b$ and $c$ are odd powers of $\alpha$, so any product of $n$ numbers from $\{b, c\}$ is an odd power of $\alpha$ as well and can therefore not equal 1. This shows that $\varphi$ has good spectral properties with respect to $n$ (choosing $\mu = a$ in Definition 3.5), a contradiction.

**Case 3**: $\mathrm{Eig}_{\mathbb{C}}(\varphi) \subset \mathbb{R}$, but $\varphi$ does not diagonalize.

If $a \neq 1$ is a simple eigenvalue of $\varphi$ whose generalized eigenspace is 2-dimensional, then again choosing $\mu = 1/a^2$ in Definition 3.5 shows that $\varphi$ has good spectral properties, so necessarily $a = 1$. When the generalized eigenspace of $\varphi$ with respect to some eigenvalue $a$ is 3-dimensional, then also $a = 1$, because $1 = \det \varphi = a^3$. Therefore, choosing a suitable basis $(x, y, z)$ of $V$ such that $\varphi$ is in Jordan normal form, we are left with the cases

$$\varphi = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{or} \qquad \varphi = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

We will lead both cases to a contradiction.

In the following we use multiplicative notation for $\omega$, i.e. we denote

$$\alpha_1 \cdot \ldots \cdot \alpha_n := \omega(\alpha_1, \dots, \alpha_n)$$

and use powers $\alpha^i$ in the obvious manner. Moreover, we denote $\alpha_1 \cdot \ldots \cdot \alpha_k \equiv 0$ for some $\alpha_1, \dots \alpha_k \in V$ if

$$\alpha_1 \cdot \ldots \cdot \alpha_k \cdot \beta_1 \cdot \ldots \cdot \beta_{n-k} = 0$$

for all $\beta_1, \dots, \beta_{n-k} \in V$.

In the first of the two cases above we show that $x \equiv 0$, contradicting the non-degeneracy of $\omega$. If $x \not\equiv 0$, then we can define $i$ to be maximal in $\{1, \dots, n\}$ such that $x^i \not\equiv 0$. Then for any $j, k \in \{0, \dots, n\}$ such that $i + j + k = n$ we get:

$$\begin{aligned} x^{i-1} y^{j+1} z^k &= \varphi(x)^{i-1} \varphi(y)^{j+1} \varphi(z)^k = x^{i-1} (y + x)^{j+1} z^k \\ &= x^{i-1} y^{j+1} z^k + (j+1) \cdot x^i y^j z^k \qquad \text{(using } x^{i+1} \equiv 0\text{)}, \end{aligned}$$

implying $x^iy^jz^k = 0$, thus $x^i \equiv 0$, a contradiction.

In the second of the two cases we show that the plane $\langle x, y\rangle$ is contained in $\mathcal{N}$. For this, we show that $x^iy^jz^k = 0$ whenever $i + 2j + 3k \leq 2n$, implying in particular that for any $i \in \{1, \dots, n\}$ we have $x^iy^{n-i} = 0$, from which the claim follows.

Assume the converse and let $M \in \{n, \dots, 2n\}$ be minimal such that $x^iy^jz^k \neq 0$ for some $i + j + k = n$ with $i + 2j + 3k = M$. First, we assume that $M < 2n$. Let $i \in \{0, \dots, n\}$ be minimal such that there exist $j, k \in \{0, \dots, n\}$ with $i + j + k = n$ and $i + 2j + 3k = M$ and $x^iy^jz^k \neq 0$. Then $i \geq 1$ (because of $M < 2n$) and

$$\begin{aligned} x^{i-1}y^{j+1}z^k &= \varphi(x)^{i-1}\varphi(y)^{j+1}\varphi(z)^k \\ &= x^{i-1}(y + x)^{j+1}(z + y)^k = x^{i-1}y^{j+1}z^k + (j + 1) \cdot x^iy^jz^k. \end{aligned}$$

Here, the last step is due to minimality of $M$ and $i$. This gives a contradiction, so we must have $M = 2n$.

Now let $k$ be minimal such that there exist $i, j \in \{0, \dots, n\}$ with $i + j + k = n$, $i + 2j + 3k = 2n$ and $x^iy^jz^k \neq 0$. Note that in this case $j + 2k = n$, so the fact that $n$ is odd implies $j \geq 1$. Then

$$\begin{aligned} x^iy^{j-1}z^{k+1} &= \varphi(x)^i\varphi(y)^{j-1}\varphi(z)^{k+1} \\ &= x^i(y + x)^{j-1}(z + y)^{k+1} = x^iy^{j-1}z^{k+1} + (k + 1) \cdot x^iy^jz^k, \end{aligned}$$

using minimality of $M = 2n$ and $k$. This gives a contradiction, concluding the proof that $\langle x, y\rangle \subset \mathcal{N}$ and therefore finishing the proof that $\mathcal{N}$ contains a plane. $\square$

Since Theorem 3.8 constitutes a central result on the splitting of the null cone, we explicitly state here the consequence for varieties:

**Corollary 3.9.** *Let $X$ be a smooth projective variety of odd dimension $n$ with Picard number 3 and assume that the group $\mathcal{A}_+(X)$ is neither trivial nor of order three. Then the null cone $\mathcal{N}$ splits as the union of a plane and the vanishing set of a homogeneous polynomial of degree $n - 1$.*

*If $c_{i_1}(X) \cdot \ldots \cdot c_{i_k}(X) \neq 0$ in $N_1(X)$ for some $i_1 + \cdots + i_k = n - 1$, then the plane is given by*

$$\{x \in N^1(X)_\mathbb{R} \mid x \cdot c_{i_1}(X) \cdot \ldots \cdot c_{i_k}(X) = 0\}.$$

*(In particular, all such products $c_{i_1}(X) \cdot \ldots \cdot c_{i_k}(X)$ are proportional to each other.)*

Proof. The first part immediately follows from Theorem 3.8 using Observation 3.1 and Lemma 3.2.

For the second claim, let $c_{i_1}(X) \cdot \ldots \cdot c_{i_k}(X) \neq 0$ for some $i_1 + \cdots + i_k = n-1$. As in the beginning of the proof of Theorem 3.8, the fact that $\mathcal{A}_+(X)$ is neither trivial nor of order 3 implies that there exists $\varphi \in \mathcal{A}_+(X)$ not of order 3. As any automorphism preserves $c_{i_1}(X) \cdot \ldots \cdot c_{i_k}(X)$, we see that $1 \in \mathrm{Eig}(\varphi^\vee) = \mathrm{Eig}(\varphi)$. If $\varphi$ has good spectral properties, then it must in fact have very good spectral properties, so $\{\ell = 0\} \subset N$ according to Lemma 3.6, where $\ell$ is an eigenvector of $\varphi^\vee$ with eigenvalue 1, i.e. $\ell$ coincides with $c_{i_1}(X) \ldots c_{i_k}(X)$ up to scalar. (Note that the eigenspace of $\varphi$ of the eigenvalue 1 is one-dimensional.)

If $\varphi$ does not have good spectral properties, then the case distinction in the proof of Theorem 3.8 shows that for some basis $x, y, z \in N^1(X)_\mathbb{R}$ we have $\varphi(x) = x$, $\varphi(y) = y + x$, $\varphi(z) = z + y$ and the plane contained in $\mathcal{N}$ is $\langle x, y \rangle$. Then

$$c_{i_1}(X) \cdot \ldots \cdot c_{i_k}(X) \cdot y = c_{i_1}(X) \cdot \ldots \cdot c_{i_k}(X) \cdot \varphi(y)$$

and

$$c_{i_1}(X) \cdot \ldots \cdot c_{i_k}(X) \cdot z = c_{i_1}(X) \cdot \ldots \cdot c_{i_k}(X) \cdot \varphi(z)$$

imply that $\langle x, y \rangle = \{c_{i_1}(X) \cdot \ldots \cdot c_{i_k}(X) = 0\}$. □

In particular, for Calabi–Yau threefolds with Picard number 3, we deduce:

**Corollary 3.10.** *Let $X$ be a simply connected Calabi–Yau threefold of Picard number 3 and assume that the group $\mathcal{A}_+(X)$ is neither trivial nor of order 3. Then*

$$\mathcal{N} = \{c_2(X) = 0\} \cup \{q = 0\}$$

*for some quadratic form $q$ on $N^1(X)_\mathbb{R}$.*

Proof. This follows from the above by the fact that $c_2(X) \neq 0$ for simply connected Calabi–Yau threefolds, see [**Kob87**, Corollary IV.4.15] or [**LOP16**, Proposition 2.1]. □

## 3.3. Classification for threefolds with Picard number 3

In this section, we classify the null cone (and by that the intersection form, according to Proposition 3.4) for threefolds $X$ with Picard number $\rho(X) = 3$ and non-trivial group $\mathcal{A}_+(X)$. This is inspired by [**LOP13**, Propositions 3.2 and 4.3], where such a classification was shown for the case that $c_2(X) \neq 0$ and $\mathcal{A}_+(X)$ is an infinite group. The classification result below will extend their result by getting rid of these two assumptions.

We will deduce the classification result from Corollary 3.9. It will be an easy consequence after establishing the following result [**KW14**, Proposition 4.1], which shows how the Hodge Index Theorem imposes restrictions on a splitting intersection form for threefolds.

**Lemma 3.11** ([**KW14**]). *Let $X$ be a threefold and assume that its null cone $\mathcal{N} \subset N^1(X)_{\mathbb{R}}$ splits as the union of a hyperplane $P$ and the zero set of a quadratic form $q$ on $N^1(X)_{\mathbb{R}}$. Assume that the sign of $q$ is chosen in such a way that $q(\mathrm{Amp}(X)) > 0$ and denote its associated bilinear form by $(\_,\_)_q$. Then:*

*(i) The radical of $(\_,\_)_q$, defined by*

$$\mathcal{R} := \{x \in N^1(X)_{\mathbb{R}} \mid (x,y)_q = 0 \text{ for all } y \in N^1(X)_{\mathbb{R}}\},$$

*intersects the hyperplane $P$ trivially.*

*(ii) The bilinear form $(\_,\_)_q$ has signature $(s_+, s_-, s_0) = (1, \rho(X) - 1, 0)$ or $(1, \rho(X) - 2, 1)$ or $(2, \rho(X) - 2, 0)$.*

Proof. We loosely follow the proof in [**KW14**].

(i) Assume for contradiction that there exists a non-zero $u \in \mathcal{R} \cap P$. Then we can extend this to a basis $u, v, w$ of $N^1(X)_{\mathbb{R}}$ and we denote by $U, V, W \in N^1(X)_{\mathbb{R}}^{\vee}$ its dual basis.

Let $\ell \in N^1(X)_{\mathbb{R}}^{\vee}$ be a non-zero linear form vanishing on $P$. Then we can naturally view $\ell$ and $q$ as polynomials in $\mathbb{R}[U,V,W]$ of degrees 1 and 2. Since $\ell(u) = 0$ and $u \in \mathbb{R}$, we must have $\ell, q \in \mathbb{R}[V,W] \subset \mathbb{R}[U,V,W]$. We know from the proof of Proposition 3.4 that the intersection product $x \cdot y \cdot z$ for $x, y, z \in N^1(X)_{\mathbb{R}}$ is given up to a factor of 6 by the coefficient of $t_1 t_2 t_3$ in the polynomial mapping

$$h \colon \mathbb{R}^3 \to \mathbb{R}, \ (t_1, t_2, t_3)^T \mapsto (t_1 x + t_2 y + t_3 z)^3.$$

Observing that $\mathbb{N}^1(X)_{\mathbb{R}} \to \mathbb{R}, \ \alpha \to \alpha^3$ coincides up to scalar with $\ell \cdot q$, we may view $h$ as an element of $\mathbb{R}[V,W]$. This shows that $u \cdot y \cdot z = 0$ for all $y, z \in N^1(X)_{\mathbb{R}}$, contradicting Lemma 3.2.

(ii) Let $H$ be a very ample divisor on $X$, which we may view as a smooth surface contained in $X$.

**Step 1**: The bilinear form $(\_,\_)_H$ on $N^1(X)_{\mathbb{R}}$ given by

$$(x,y)_H := Hxy \quad \text{for } x, y \in N^1(X)_{\mathbb{R}}$$

has signature $(1, \rho(X) - 1, 0)$.

Because of $(H,H)_H = H^3 > 0$, it suffices to show that for any non-zero $x \in N^1(X)_{\mathbb{R}}$ the implication

$$(x,H)_H = 0 \ \Rightarrow \ (x,x)_H < 0$$

holds. But by the Hodge Index Theorem on $H$, the equality $0 = (x,H)_H = x|_H \cdot H|_H$ implies $(x,x)_H = x|_H \cdot x|_H < 0$ unless $x|_H = 0$. But the Lefschetz Hyperplane Theorem implies that

$$N^1(X) \to N^1(H), \ x \mapsto x|_H$$

is injective, which concludes the proof of Step 1.

**Step 2**: Let $\ell$ be the non-zero linear form on $N^1(X)_{\mathbb{R}}$ such that $x^3 = \ell(x) \cdot q(x)$ for all $x \in N^1(X)_{\mathbb{R}}$. Then there are positive real numbers $a, b, c$ such that

$$\begin{aligned}(x, y)_H = a \cdot (x, y)_q &+ b \cdot (H, x)_H \cdot \ell(y) \\ &+ b \cdot (H, y)_H \cdot \ell(x) - c \cdot \ell(x) \cdot \ell(y)\end{aligned}$$

holds for all $x, y \in N^1(X)_{\mathbb{R}}$.

Here, we use again the fact that the intersection product $x \cdot y \cdot H$ is given up to a factor of 6 by the coefficient of $t_1 t_2 t_3$ in the polynomial mapping

$$\mathbb{R}^3 \to \mathbb{R},\ (t_1, t_2, t_3)^T \mapsto (t_1 x + t_2 y t_3 H)^3.$$

Expanding out $(\ell \cdot q)(t_1 x + t_2 y + t_3 H)$ gives

$$6(x, y)_H = 2(x, y)_q \cdot \ell(H) + 2(H, x)_q \cdot \ell(y) + 2(H, y)_q \cdot \ell(x).$$

For $y = H$ we obtain

$$(H, x)_q = \frac{3}{2\ell(H)}(H, x)_H - \frac{q(H)}{2\ell(H)}\ell(x)$$

and by plugging this into the above formula, we obtain the claimed expression for

$$a := \frac{\ell(H)}{3}, b := \frac{1}{2\ell(H)}, c := \frac{q(H)}{3\ell(H)}.$$

Note that $\ell(H) > 0$, as $H^3 > 0$ and by assumption on $q$ also $q(H) > 0$, so these are all positive.

**Step 3**: If the linear forms $\ell$ and $(H, \_)_H$ are proportional, then $(\_, \_)_q$ has signature $(1, \rho(X) - 1, 0)$.

Since $(\_, \_)_H$ is non-degenerate and $(H, H)_H = H^3 > 0$, there is a $(\_, \_)_H$-orthogonal basis of the form $x_1 = H, x_2, \ldots, x_{\rho(X)}$. By Step 2 we have $(x_i, x_j)_H = a \cdot (x_i, x_j)_q$ for all $i \geq 1, j > 1$, in particular this basis is also $(\_, \_)_q$-orthogonal. By Step 1, we know that $(H, H)_H > 0$ implies $(x_i, x_i)_q = 1/a \cdot (x_i, x_i)_H < 0$ for $i > 1$. Since $(H, H)_q = q(H) > 0$, this shows the claim.

**Step 4**: If the linear forms $\ell$ and $(H, \_)_H$ are linearly independent, then $(\_, \_)_q$ has signature $(2, \rho(X) - 2, 0)$ or $(1, \rho(X) - 2, 1)$ or $(1, \rho(X) - 1, 0)$.

Since $(\_, \_)_H$ is non-degenerate we have $N^1(X)_{\mathbb{R}} = \langle H \rangle \oplus Y$, where $Y$ is the $(\_, \_)_H$-orthogonal complement of $\langle H \rangle$. By assumption, the linear form $\ell|_Y$ on $Y$ is non-zero, so by non-degeneracy of $(\_, \_)_H$ there exists a unique non-zero $x_2 \in Y$ such that $\ell|_Y = (x_2, \_)_H$ on $Y$.

From the signature of $(\_,\_)_H$ and $(H,H)_H = H^3 > 0$ and $(x_2,H)_H = 0$, we know that $(x_2,x_2)_H < 0$. We can therefore extend $x_1 := H, x_2$ to a $(\_,\_)_H$-orthogonal basis $x_1,\dots,x_{\rho(X)} \in N^1(X)_{\mathbb{R}}$. From Step 2 it follows that $(x_i,x_j)_H = a \cdot (x_i,x_j)_q$ holds for all $i \geq 1, j > 2$. In particular, $(\_,\_)_q$ is negative definite on $\langle x_3,\dots,x_{\rho(X)}\rangle$. On the other hand, we know that $(H,H)_q = q(H) > 0$. From this, the claim on the signature of $(\_,\_)_q$ follows.

□

Now, we classify the null cone of threefolds with Picard number 3 and nontrivial $\mathcal{A}_+(X)$. Note that we do *not* need to require $X$ to be of Calabi–Yau type.

**Theorem 3.12.** *Let $X$ be a threefold of Picard number 3 such that the group $\mathcal{A}_+(X)$ is nontrivial. Then the null cone $\mathcal{N} \subset N^1(X)_{\mathbb{R}}$ takes one of the following forms (see Figure 3.1):*

- *Type I: $\mathcal{N}$ is the union of three distinct planes whose common intersection is trivial.*
- *Type II: $\mathcal{N}$ is the union of a plane and a quadric cone intersecting at two lines.*
- *Type III: $\mathcal{N}$ is the union of a plane and a quadric cone whose intersection is one line.*
- *Type IV: $\mathcal{N}$ is the union of a plane and a quadric cone intersecting trivially.*
- *Type V: $\mathcal{N}$ is an irreducible cubic cone, which is invariant under the action of a linear automorphism $\varphi \in \mathrm{GL}(N^1(X)_{\mathbb{R}})$ with order 3.*

Proof. First, we consider the case that the intersection form splits, i.e. the null cone $\mathcal{N}$ is the union of a plane $P$ and the vanishing set $\{q = 0\}$ for a quadratic form $q$ on $N^1(X)_{\mathbb{R}}$. By Lemma 3.11, the bilinear form $(\_,\_)_q$ associated to $q$ has signature $(s_+,s_-,s_0) = (1,2,0)$ or $(2,1,0)$ or $(1,1,1)$.

In the first two cases the vanishing set of $q$ is a quadric cone and depending on the way this cone intersects the plane $P$, the null cone is of Type II, III or IV. In the third case the vanishing set of $q$ is a union of two distinct planes whose intersection is the radical $\mathcal{R}$ of $(\_,\_)_q$. By Lemma 3.11, we have $\mathcal{R} \cap P = 0$, so the null cone is of Type I.

It remains to consider the case that $\mathcal{N}$ does not split. Then Corollary 3.9 implies that $\mathcal{A}_+(X)$ is generated by an element $\varphi$ of order 3. Certainly, $\varphi$ leaves $\mathcal{N}$ invariant, so the null cone is of Type V. This concludes the proof. □

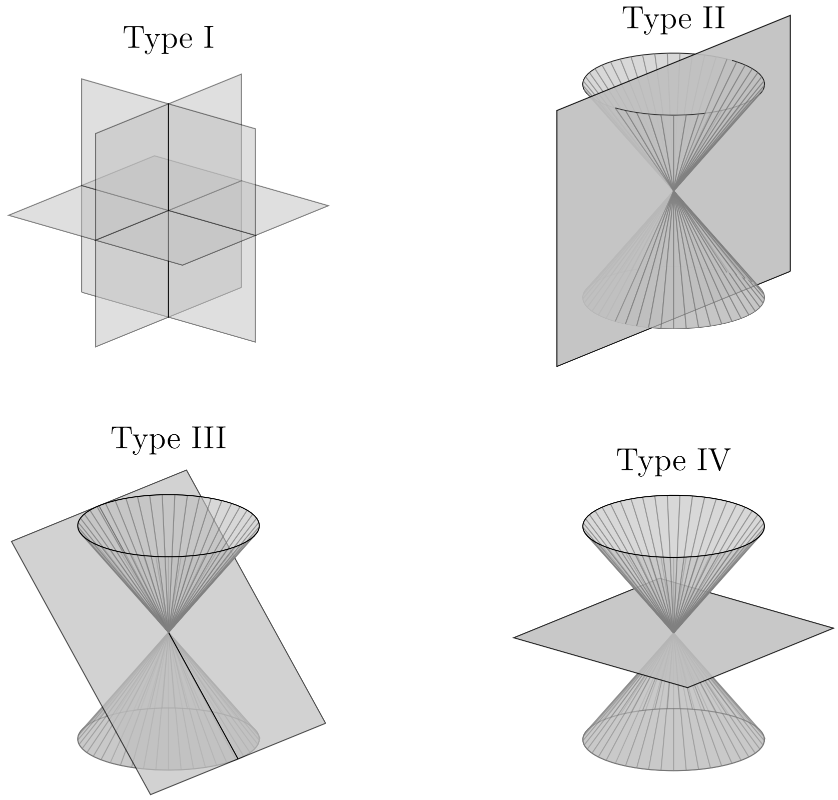


FIGURE 3.1. Illustration of Types I to IV

**Definition 3.13.** In view of Theorem 3.12 we say that a threefold $X$ is a **threefold of Type I** if it has Picard number 3, its group $\mathcal{A}_+(X)$ is non-trivial and its null cone is of Type I. Similarly for the Types II–V.

**Remark 3.14.** It is unknown to the author whether all of these types do in fact occur. The reader is challenged to try and construct examples of such threefolds.

**Remark 3.15.** In [**LOP13**] only the types I, II and III are mentioned. Indeed, we will see later on that the cases Type IV and Type V can only occur when $\mathcal{A}_+(X)$ is finite. The last case Type V requires that $\mathcal{A}_+(X)$ is a group of order 3.

**Warning 3.16.** For visualization purposes it is tempting to consider some affine plane $\mathcal{P}$ and to draw a picture of $\mathcal{N} \cap \mathcal{P}$. However, one must be aware of the position of the nef cone in order to choose the plane $\mathcal{P}$ in such a way that $\mathrm{Nef}(X) \cap \mathcal{P}$ is a non-empty compact set. Otherwise, the picture could be very misleading, as it could even happen that $\mathrm{Nef}(X) \cap \mathcal{P} = \emptyset$. It may be necessary to choose different planes $\mathcal{P}$ for different positions of the nef cone.

CHAPTER 4

# The structure of Types I to V

In this chapter, we will use the classification of the null cone for threefolds of Picard number three (see Theorem 3.12) to investigate the different types of threefolds with Picard number 3. For each type, we will establish a structure theorem, explicitly describing the group $\mathcal{A}_+(X)$ and giving more detailed information in each case. The following table summarizes the descriptions of $\mathcal{A}_+(X)$ to be established in Propositions 4.16, 4.23, 4.26, 4.30 and 4.31.

| TYPE | $\mathcal{A}_+(X)$ | $\mathcal{N}$ |
|---|---|---|
| I | $\left\langle \begin{pmatrix} \frac{1}{\lambda} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle \cong \mathbb{Z}, \lambda > 1$ or $\left\langle \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\rangle \cong \mathbb{Z}/3$ | $\{U = 0\} \cup \{V = 0\} \cup \{W = 0\}$ |
| II | $\left\langle \begin{pmatrix} \frac{1}{\lambda} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle \cong \mathbb{Z},\ \lambda > 1$ | $\{UV - W^2 = 0\} \cup \{W = 0\}$ |
| III | $\left\langle \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle \cong \mathbb{Z}$ | $\{2UW - V^2 = 0\} \cup \{W = 0\}$ |
| IV | $\left\langle \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle \cong \mathbb{Z}/k,$ $\alpha = 2\pi/k,\ k \in \{2, 3, 4, 6\}$ | $\{W^2 - U^2 - V^2 = 0\} \cup \{W = 0\}$ |
| V | $\left\langle \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle \cong \mathbb{Z}/3,$ $\alpha = 2\pi/3$ | $\{V(V^2 - 3U^2) + aW(U^2 + V^2) + W^3 = 0\},$ $a \in \mathbb{R}$ |

FIGURE 4.1. Classification of the groups $\mathcal{A}_+(X)$ and the null cones $\mathcal{N}$ for threefolds with $c_1(X)^2 \neq 0$ or $c_2(X) \neq 0$. The entries refer to a suitable basis $u, v, w \in N^1(X)_{\mathbb{R}}$ and its dual basis $U, V, W \in N^1(X)_{\mathbb{R}}^{\vee}$.

As an immediate consequence, we obtain the following result:

**Theorem 4.1.** *Let $X$ be a threefold with Picard number 3 with $c_1(X)^2 \neq 0$ or $c_2(X) \neq 0$. Then $\mathcal{A}_+(X)$ is either trivial or a cyclic group.*

We will use the obtained explicit descriptions of $\mathcal{A}_+(X)$ to deduce results towards the Cone Conjecture and the existence of rational curves. We will focus on investigating threefolds of Type I.

## 4.1. General observations

In this section we collect some tools for the study of the Types I–V.

**4.1.1. The importance of non-trivial Chern classes.** The following simple observation shows that the existence of non-trivial Chern classes on $X$ imply a restriction on the group $\mathcal{A}_+(X)$. Essentially, we have already encountered this argument in the proof of Corollary 3.9, but we state it here explicitly for easier reference.

**Lemma 4.2.** *Let $X$ be a smooth projective variety of dimension $n$ and let $\varphi \in \mathcal{A}_+(X)$. If $c_{i_1}(X) \cdot \ldots \cdot c_{i_k}(X) \neq 0$ in $N_1(X)_{\mathbb{R}}$ for some $i_1 + \cdots + i_k = n-1$, then $\varphi$ has an integral eigenvector of eigenvalue 1.*

Proof. Let $\varphi = f^*$ for some $f \in \mathrm{Aut}(X)$. The action of $f$ on $N_1(X)_{\mathbb{R}}$ induced by the push-forward $f_*\colon N_1(X)_{\mathbb{Z}} \to N_1(X)_{\mathbb{Z}}$ coincides with the dual action $\varphi^\vee \in \mathrm{GL}(N_1(X)_{\mathbb{R}})$ of $\varphi$ on $N_1(X)_{\mathbb{R}} = (N^1(X)_{\mathbb{R}})^\vee$. Thus,

$$f_*(c_{i_1}(X) \cdot \ldots \cdot c_{i_k}(X)) = c_{i_1}(X) \cdot \ldots \cdot c_{i_k}(X)$$

in $N_1(X)_{\mathbb{Z}}$ shows that $\varphi^\vee$ has an integral eigenvector of eigenvalue 1 and therefore $\varphi$ does, too. □

For Calabi–Yau threefolds, this turns out to be a very useful property, as $c_2(X) \neq 0$ holds apart from a well-studied case [**OS01**]:

**Theorem 4.3.** *Let $X$ be a Calabi–Yau threefold. If $c_2(X) = 0$ in $N_1(X)_{\mathbb{R}}$, then $\mathrm{Nef}(X)$ is a rational polyhedral cone and $\mathrm{Nef}^{\mathrm{e}}(X) = \mathrm{Nef}^{+}(X)$. In particular, $\mathrm{Aut}(X)$ is finite and Kawamata's Cone Conjecture holds for $X$.*

Proof. The first part is [**OS01**, Theorem (0.1).IV] and the second part follows from Corollary 2.17. □

Moreover, there is the following important non-negativity result for the second Chern class.

**Proposition 4.4** ([**Miy87**], [**Kob87**])**.** *Let $X$ be a threefold of Calabi–Yau type. Then $x \cdot c_2(X) \geq 0$ for all nef classes $x$.*

*Moreover, if $X$ is a simply-connected Calabi–Yau threefold, then $c_2(X) \neq 0$ in $N_1(X)$.*

Proof. The non-negativity of $c_2(X)$ was shown in [**Miy87**]. The non-triviality follows from [**Kob87**, Corollary IV.4.15]. □

**4.1.2. The local structure of the nef cone.** While for a variety of Calabi–Yau type $X$ the precise structure of its nef cone is subject to the Cone Conjecture, the local structure of $\mathrm{Nef}(X)$ inside the big cone is well-understood due to the following important result by Kawamata [**Kaw88**, Theorem 5.7], which is an easy consequence of the Log Cone Theorem.

**Proposition 4.5.** *Let $X$ be a variety of Calabi–Yau type. Then* $\mathrm{Nef}(X) \cap \mathrm{Big}(X)$ *is a locally rational polyhedral convex subcone of* $\mathrm{Big}(X)$.

Another way to phrase this is by considering the positive component $\mathcal{U}$ of $X$, defined as follows.

**Definition 4.6.** Let $X$ be a smooth projective variety. The connected component $\mathcal{U}$ of $N^1(X)_{\mathbb{R}} \setminus \mathcal{N}$ containing $\mathrm{Amp}(X)$ is called the **positive component** of $X$.

**Remark 4.7.** In the literature this is sometimes referred to as the *positive cone*. This is a reasonable denomination for surfaces, as the Hodge Index Theorem shows that it is in fact a round convex cone. We would like to point out however, that in higher dimensions the positive component may be non-convex.

**Lemma 4.8.** *Let $X$ be a smooth projective variety of dimension $n$. Then a nef class $x \in \mathrm{Nef}(X)$ is big if and only if $x^n > 0$. In other words,*

$$\mathrm{Nef}(X) \cap \mathrm{Big}(X) = \mathrm{Nef}(X) \cap \mathcal{U},$$

*where $\mathcal{U}$ denotes the positive component.*

*In particular,* $\mathrm{Nef}(X) \cap \mathcal{U}$ *is locally rational polyhedral in $\mathcal{U}$, where $\mathcal{U}$ is the positive component.*

Proof. The first claim is shown in [**Laz04**, Theorem 2.2.16] as a consequence of Siu's criterion for bigness, which we formulate below in Proposition 4.10. Alternatively, it follows immediately from the asymptotic Riemann–Roch Theorem for nef divisors [**Laz04**, Corollary 1.4.41].

The equivalence with $\mathrm{Nef}(X) \cap \mathrm{Big}(X) = \mathrm{Nef}(X) \cap \mathcal{U}$ follows from the observation that $\mathrm{Nef}(X)$ is contained in the closure of $\mathcal{U}$, so a nef class $x$ lies in the positive component $\mathcal{U}$ if and only if $x^n > 0$.

The last claim is then just a reformulation of Lemma 4.5. □

An easy consequence is the following result:

**Proposition 4.9.** *Let $X$ be a variety of Calabi–Yau type and assume that all non-zero nef classes are big, i.e.* $\mathrm{Nef}(X) \setminus \{0\}$ *is contained in the positive component. Then* $\mathrm{Nef}(X)$ *is a rational polyhedral cone and Kawamata's Cone Conjecture holds true on $X$.*

Proof. Since each non-zero nef class is big and therefore lies in $\mathrm{Eff}(X)$, we have $\mathrm{Nef}(X) = \mathrm{Nef}^{\mathrm{e}}(X)$. Hence, by Corollary 2.17, it suffices to show that $\mathrm{Nef}(X)$ is a rational polyhedral cone.

Since $\mathrm{Nef}(X)$ contains no lines, we can choose a rational affine hyperplane $\mathcal{P}$ in $N^1(X)_{\mathbb{R}}$ not passing through 0 such that $\mathrm{Nef}(X) \cap \mathcal{P}$ is a compact (convex) set. By Lemma 4.5, each point $x \in \mathrm{Nef}(X) \cap \mathcal{P}$ permits a compact neighborhood $K_x$ such that $\mathrm{Nef}(X) \cap K_x$ and hence also $\mathrm{Nef}(X) \cap L \cap K_x$ is a rational polytope. By compactness of $\mathrm{Nef}(X) \cap L$, we see that the convex set $\mathrm{Nef}(X) \cap L$ is a finite union of rational polytopes, so it is a rational polytope itself. Since $\mathrm{Nef}(X)$ is the closed convex cone generated by $\mathrm{Nef}(X) \cap L$, we deduce that $\mathrm{Nef}(X)$ is rational polyhedral. □

**4.1.3. Criteria for big and effective.** We conclude the section on basic tools for the study of Calabi–Yau threefolds with providing criteria for bigness and effectiveness of divisors classes.

First, we recall a basic criterion for big classes due to Siu [**Siu93**, Corollary 1.2], see also [**Laz04**, Theorem 2.2.15].

**Proposition 4.10.** *Let $X$ be a projective variety of dimension $n$. Let $x, y \in \mathrm{Nef}(X)$ and assume that*

$$x^n > nx^{n-1}y.$$

*Then $x - y$ is big.*

We will use this criterion to study the big cone in some of the Types I–V. Regarding the effective cone, we present the following result which is an adaption of an argument in [**HBW92**, p. 53–54].

**Lemma 4.11.** *Let $X$ be a threefold of Calabi–Yau type. Let $E$ is an integral divisor with $E \cdot H^2 > 0$ and $E^2 \cdot H > 0$ for some ample divisor $H$. Then there are finitely many integral curves $C_1, \ldots, C_r \subset X$ such that an integral class $x \in N^1(X)_{\mathbb{Z}}$ in*

$$(\mathbb{R}_{>0}E + \mathrm{Nef}(X)) \cap \{x \in N^1(X)_{\mathbb{R}} \mid x \cdot C_i \geq 0 \ \forall i\}$$

*lies in* $\mathrm{Eff}(X)$ *if* $c_2(X) \cdot x + 2x^3 > 0$.

**Remark 4.12.** The argument given in [**HBW92**, p. 53–54] shows the claim only for $x$ in

$$(\mathbb{R}_{>0}E + \mathrm{Amp}(X)) \cap \{x \in N^1(X)_{\mathbb{R}} \mid x \cdot C_i > 0 \ \forall i\}.$$

Our improvement will come from using the Kawamata–Viehweg Vanishing Theorem instead of the Kodaira Vanishing Theorem. Later, when applying Lemma 4.11 in the proof of 5.7, it will in fact be crucial to have the stronger version of this Lemma.

Before proving this lemma, we show how this result may also be suitable for excluding certain irrational rays from $\mathrm{Nef}(X) \cap \mathcal{N}$ by arithmetic arguments. Here, we work out a suggestion by Wilson [**Wil97**, p. 390].

**Proposition 4.13.** *Let $X$ be a threefold of Calabi–Yau type and let $\mathbb{R}_{\geq 0}x \subset \mathcal{N}$ be an irrational ray inside the null cone with $x \cdot c_2(X) > 0$ and $x^2 \neq 0$ in $N_1(X)_{\mathbb{R}}$. Assume that there exists a sequence of rational rays $\mathcal{R}_i \subset N^1(X)_{\mathbb{R}}$ converging towards the ray $\mathbb{R}_{\geq 0}x$ with the property that*

*(i) $x \in \mathrm{conv}(\mathrm{Amp}(X) \cup \mathcal{R}_i)$ for all $i$.*
*(ii) On each of the rational rays $\mathcal{R}_i$ there exists an integral class $y_i$ such that*
$$y_i \cdot c_2(X) + 2y_i^3 > 0.$$

*Then $x \notin \mathrm{Nef}(X)$.*

First, we show how Proposition 4.13 follows from the Lemma.

Proof of 4.13. Assume for contradiction that $x \in \mathrm{Nef}(X)$. Since $x^2 \neq 0$ in $N_1(X)_{\mathbb{R}}$, there must exist an integral ample class $h$ such that $x^2 \cdot h > 0$ and $x \cdot h^2 > 0$. Hence, there exists a neighborhood $U$ of $x$ such that $y^2 \cdot h > 0$, $y \cdot h^2 > 0$ for all $y \in U$. Let $e$ be a rational class in $U$ such that $\mathbb{R}_{>0}x$ is contained in $\mathbb{R}_{>0}e + \mathrm{Amp}(X)$. Let $C_1, \dots, C_r \in N_1(X)$ be curve classes as in Lemma 4.11.

Then the fact that $x$ is nef and the ray $\mathbb{R}_{\geq 0}x$ is irrational imply that $\mathbb{R}_{>0}x$ lies in the interior of the set
$$W := (\mathbb{R}_{>0}E + \mathrm{Nef}(X)) \cap \{x \in N^1(X)_{\mathbb{R}} \mid x \cdot C_i \geq 0 \ \forall i\}.$$
Hence, the rays $\mathcal{R}_i = \mathbb{R}_{>0}y_i$ lie in $W$ for $i \gg 0$, so from Lemma 4.11 we deduce $y_i \in \mathrm{Eff}(X)$ for $i \gg 0$, because $y_i \cdot c_2(X) + 2y_i^3 > 0$ by assumption.

Because of $x \in \mathrm{conv}(\mathrm{Amp}(X) \cup \mathcal{R}_{\geq 0}y_i)$, this shows $x \in \mathrm{Big}(X)$, contradicting Lemma 4.8. □

Now, we turn to the proof of the Lemma.

Proof of Lemma 4.11. By replacing $H$ with a sufficiently large multiple, we may assume that $H$ is very ample and by Bertini's Theorem we can view $H$ as a smooth surface contained in $X$.

**Step 1**: $E|_H \in \mathrm{Eff}(H)$.

It follows from the Riemann–Roch Theorem on $H$ that

$$h^0(H, kE|_H) + h^0(H, K_H - kE|_H) > 0$$

for $k \gg 0$, since ${E|_H}^2 = E^2 \cdot H > 0$. On the other hand, $H|_H \cdot E|_H = H^2 \cdot E > 0$ implies that $H|_H \cdot (K_H - kE|_H) < 0$ for $k \gg 0$, so $K_H - kE|_H \notin \mathrm{Eff}(H)$. We deduce that $kE|_H$ is linearly equivalent to an effective divisor for $k \gg 0$ and in particular $E|_H \in \mathrm{Eff}(H)$. In the following, we may assume that $E|_H$ itself is effective.

**Step 2**: If $C_1, \ldots, C_r$ are the fixed curves of the linear system $|E|_H|$, then

$$W := (\mathbb{R}_{>0}E + \mathrm{Nef}(X)) \cap \{x \in N^1(X)_{\mathbb{R}} \mid x \cdot C_i \geq 0 \ \forall i\}$$

maps into $\mathrm{Nef}(H) \cap \mathrm{Big}(H)$ under the restriction $N^1(X)_{\mathbb{R}} \to N^1(H)_{\mathbb{R}}$.

Let $x \in W$, i.e. $x = \lambda E + \mu v$, where $v \in \mathrm{Nef}(X)$ and $\lambda > 0$, $\mu \geq 0$. In order to deduce that $x|_H \in \mathrm{Nef}(H)$ we need to show that $C \cdot x|_H \geq 0$ for all curves $C$ in $H$.

So, let $C$ be a curve in $H$. If $C$ is a component of $\mathrm{Fix}\,|E|_H|$, then $C \cdot x|_H = C \cdot x \geq 0$ by assumption on $x$. Otherwise we have $C \cdot E|_H \geq 0$, because $E|_H$ is effective. As $C \cdot v|_H \geq 0$, it follows that $C \cdot x|_H \geq 0$. Hence, $x|_H \in \mathrm{Nef}(H)$.

Now

$$\begin{aligned} {x|_H}^2 &= x^2 \cdot H = \lambda^2 E^2 \cdot H + \lambda\mu E|_H \cdot v|_H + \mu^2 v^2 \cdot H \\ &\geq \lambda^2 E^2 \cdot H > 0, \end{aligned}$$

so $x|_H \in \mathrm{Big}(H)$ by 4.8.

**Step 3**: For any divisor $D \in \mathrm{Pic}(X)$ whose class lies in $W$, we have

$$h^0(X, \mathcal{O}_X(D)) = \frac{1}{6}D^3 + \frac{1}{12}c_2(X) \cdot D,$$

possibly after replacing $D$ with some multiple of itself.

The Riemann–Roch Theorem on $X$ shows

$$h^0(\mathcal{O}_X(D)) + h^2(\mathcal{O}_X(D)) \geq \frac{1}{6}D^3 + \frac{1}{12}c_2(X) \cdot D$$

and this quantity is positive if $c_2(X) \cdot D + 2D^3 > 0$, so it suffices to show that $h^2(\mathcal{O}_X(D)) = 0$.

For this, we consider the short exact sequence

$$0 \to \mathcal{O}_X(K_X - D - (k+1)H) \to \mathcal{O}_X(K_X - D - kH) \to \mathcal{O}_H(K_X - D - kH) \to 0$$

for all $k \geq 0$. Since $D|_H$ is nef and big by Step 2, the same is true for $(-K_X + D + kH)|_H$ for all $k \geq 0$ (note that $K_X$ is numerically trivial, so it does not influence numerical properties). This implies

$$h^i(\mathcal{O}_H(K_X - D - kH)) = 0 \text{ for } i = 1, 2$$

by the Kawamata–Viehweg Vanishing Theorem. Hence, it follows that

$$h^1(\mathcal{O}_X(K_X - D - (k+1)H)) = h^1(\mathcal{O}_X(K_X - D - kH)) \quad \text{for all } k \geq 0.$$

But ampleness of $H$ implies that

$$h^1(\mathcal{O}_X(K_X - D - kH)) = h^2(\mathcal{O}_X(D + kH)) = 0$$

for $k \gg 0$, so we deduce that $h^2(\mathcal{O}_X(D)) = h^1(\mathcal{O}_X(K_X - D)) = 0$. □

As an immediate corollary we deduce:

**Proposition 4.14.** *Let $X$ be a threefold of Calabi–Yau type. Let $x \in \mathrm{Nef}(X) \cap N^1(X)_\mathbb{Z}$ be a rational nef class such that $x \cdot c_2(X) \neq 0$ and $x^2 \neq 0$ in $N_1(X)_\mathbb{R}$. Then $x \in \mathrm{Eff}(X)$.*

Proof. If $x^3 > 0$, then $x$ is big by Proposition 4.8, so in particular it lies in $\mathrm{Eff}(X)$.

Otherwise we have $x^3 = 0$. We may assume that $x$ is integral. Then we can apply Lemma 4.11 by choosing $E$ in the Lemma to be a divisor whose class is $x$. Indeed, the fact that $x^2 \neq 0$ in $N_1(X)_\mathbb{R}$ implies that $x^2 \cdot h > 0$ and $x \cdot h^2 > 0$ for some ample class $h$. Then $x$ lies in the set $W$ of the Lemma and it satisfies

$$x \cdot c_2(X) + 2x^3 = x \cdot c_2(X) > 0,$$

so $x \in \mathrm{Eff}(X)$ follows. □

This result can be refined: It can be deduced from the Log Abundance Theorem in dimension 3 that an effective nef divisor on a variety of Calabi–Yau type is semiample. Moreover, Oguiso proved that the condition $x^2 \neq 0$ in the above Corollary is in fact not necessary for simply-connected Calabi–Yau threefolds, showing the following result, [**Ogu93**, Proposition 2.7].

**Proposition 4.15.** *Let $X$ be a simply-connected Calabi–Yau threefold and let $D \in \mathrm{Pic}(X)$ be an integral nef divisor on $X$ such that $D \cdot c_2(X) \neq 0$. Then $D$ is semiample.*

## 4.2. Threefolds of Type I

**4.2.1. The structure theorem.** In the following, we determine the structure of $\mathcal{A}_+(X)$ for Calabi–Yau threefolds of Type I, naturally extending the results of [**LOP13**].

**Proposition 4.16.** *Let $X$ be a threefold of Type I and assume one of the following:*

*(a) $X$ is a Calabi–Yau variety or*
*(b) $c_2(X) \neq 0$ in $N_1(X)$ or*
*(c) $c_1(X)^2 \neq 0$ in $N_1(X)$.*

*Then one of the following two cases must occur:*

*(1) $\mathcal{A}_+(X) \cong \mathbb{Z}$ and there exists a basis $(u, v, w)$ of $N^1(X)_\mathbb{R}$ such that the following holds:*
  - *(i) $\mathcal{A}_+(X)$ is generated by $\varphi$ with $\varphi(w) = w$, $\varphi(v) = \lambda v$ and $\varphi(u) = (1/\lambda)u$ for some $\lambda > 0$, $\lambda \neq 1$.*
  - *(ii) The null cone is given by $\mathcal{N} = \langle u, v\rangle \cup \langle v, w\rangle \cup \langle w, u\rangle$ and the positive component is $\mathbb{R}_{>0}u + \mathbb{R}_{>0}v + \mathbb{R}_{>0}w$.*
  - *(iii) The classes $u, v \in N^1(X)_\mathbb{R}$ are nef and $w$ is an integral class, i.e. $w \in N^1(X)_\mathbb{Z}$.*
  - *(iv) The rays $\mathbb{R}_{\geq 0}u$ and $\mathbb{R}_{\geq 0}v$ are irrational, but $\langle u, v\rangle$ is a rational plane.*

*(2) $\mathcal{A}_+(X) \cong \mathbb{Z}/3$ and there exists a basis $(u, v, w)$ of $N^1(X)_\mathbb{R}$ such that the following holds:*
  - *(i) $\mathcal{A}_+(X)$ is generated by $\varphi$ with $\varphi(u) = v$, $\varphi(v) = w$ and $\varphi(w) = v$.*
  - *(ii) The null cone is given by $\mathcal{N} = \langle u, v\rangle \cup \langle v, w\rangle \cup \langle w, u\rangle$ and the positive component is $\mathbb{R}_{>0}u + \mathbb{R}_{>0}v + \mathbb{R}_{>0}w$.*
  - *(iii) The subspaces $\langle u\rangle$, $\langle v\rangle$, $\langle w\rangle$, $\langle u, v\rangle$, $\langle v, w\rangle$, $\langle w, u\rangle$ are either all rational or all irrational.*

PROOF. Since $X$ is of Type I, we have

$$\mathcal{N} = \langle u, v\rangle \cup \langle v, w\rangle \cup \langle w, u\rangle$$

for some $u, v, w \in N^1(X)_\mathbb{R}$. After possibly changing the signs of $u, v, w$, we may assume that $\mathrm{Nef}(X) \subset \mathbb{R}_{\geq 0}u + \mathbb{R}_{\geq 0}v + \mathbb{R}_{\geq 0}w$. This shows (ii).

Any linear automorphism in $\mathcal{A}_+(X)$ preserves $\mathcal{N}$ and must therefore permute the rays $\mathbb{R}_{\geq 0}u$, $\mathbb{R}_{\geq 0}v$ and $\mathbb{R}_{\geq 0}w$. Note that $\mathcal{A}_+(X) \subset \mathrm{SL}(N^1(X)_\mathbb{R})$ forces this permutation to have signum 1. We distinguish two cases.

**Case 1**: All non-trivial elements in $\mathcal{A}_+(X)$ have order 3.

In this case $\mathcal{A}_+(X)$ is a $p$-group for $p = 3$. By 2.23, we know that $\mathcal{A}_+(X)$ is a finite group, so by properties of $p$-groups we deduce that the order of $\mathcal{A}_+(X)$ is a power of 3. On the other hand, it was shown by Minkowski in [**Min87**] that the order of any finite subgroup of $\mathrm{GL}(3, \mathbb{Q})$ divides 48, so $|\mathcal{A}_+(X)| = 3$.

Then a generator $\varphi$ of $\mathcal{A}_+(X)$ cannot leave the rays $\mathbb{R}_{\geq 0}u$, $\mathbb{R}_{\geq 0}v$ and $\mathbb{R}_{\geq 0}w$ invariant, but must permute them. We may assume that $\varphi u = av$ $\varphi v = bw$, $\varphi w = cu$ for some $a, b, c > 0$ with $abc = 1$. Rescaling $u, v, w$, we can assume that $a = b = c = 1$. The claims about rationality immediately follow from the symmetry induced by $\varphi$.

Now let $\mathcal{A}_+(X)$ be not of order 3. As before, this implies the existence of a $\varphi \in \mathcal{A}_+(X)$ such that $\psi := \varphi^3$ is not the identity. Note that $\psi$ leaves the rays $\mathbb{R}_{\geq 0}u$, $\mathbb{R}_{\geq 0}v$ and $\mathbb{R}_{\geq 0}w$ fixed, so it is of infinite order and diagonalizes with respect to the basis $u, v, w$.

If $X$ is a Calabi–Yau variety (i.e. assumption (a)), then the fact that $\mathcal{A}_+(X)$ is an infinite group implies $c_2(X) \neq 0$ in $N^1(X)$, by Theorem 4.3. Therefore, in any case we can define $\ell \in N_1(X) \setminus \{0\}$ by $\ell := c_2(X)$ (in the case of assumption (a) or (b)) or $\ell := c_1(X)^2$ (for assumption (c)). In particular, by Proposition 4.2, we see that 1 is an eigenvalue of $\psi$. We may assume that $\psi(w) = w$, while $\psi(v) = \lambda v$ and $\psi(u) = (1/\lambda)u$ for some $\lambda > 1$.

Since $\mathbb{R}w$ is the eigenspace of $\varphi$ with respect to eigenvalue 1 and $\varphi \in \mathrm{SL}(N^1(X)_{\mathbb{Z}})$ acts on the integral structure, we may assume $w \in N^1(X)_{\mathbb{Z}}$. Meanwhile, the line $\mathbb{R}v$ cannot be rational, as otherwise $\psi$ would act on the discrete subgroup $N^1(X)_{\mathbb{Z}} \cap \mathbb{R}v$, contradicting $\psi^{-K}(v) \to 0$, as $K \to \infty$. Analogously, $\mathbb{R}u$ is not a rational line either. Note however, that we have $\{\ell = 0\} = \langle u, w\rangle$, as $\ell(w) = \ell(\varphi(w)) = \lambda\ell(w)$ (and similarly for $u$). So, this is a rational plane. If $\alpha \in N^1(X)_{\mathbb{R}}$ is an ample class, then $(1/\lambda)^K \cdot \psi^K(\alpha) \to v$, so $v$ is nef and similarly for $w$.

Finally, we observe that irrationality of the rays $\mathbb{R}_{\geq 0}u$ and $\mathbb{R}_{\geq 0}v$ as opposed to rationality of the ray $\mathbb{R}_{\geq 0}w$ implies that any $\varphi \in \mathcal{A}_+(X)$ must leave the ray $\mathbb{R}_{\geq 0}w$ invariant. Then $\det\varphi = 1$ also implies invariance of the rays $\mathbb{R}_{\geq 0}u$ and $\mathbb{R}_{\geq 0}v$. Because of Proposition 4.2, the eigenspace of $\varphi$ with respect to 1, being a rational line, must coincide with $\mathbb{R}w$. In other words, any $\varphi \in \mathcal{A}_+(X)$ is given by $\varphi w = w$, $\varphi v = \lambda_\varphi v$, $\varphi u = (1/\lambda_\varphi)u$ for some $\lambda_\varphi > 0$, so we can view $\mathcal{A}_+(X)$ as a subgroup of the multiplicative group $\mathbb{R}^*$. Since $\mathcal{A}_+(X)$ acts on the integral structure $N^1(X)_{\mathbb{Z}}$ and discrete subgroups of the multiplicative group $\mathbb{R}^*$ are cyclic, this implies that $\mathcal{A}_+(X)$ is an infinite cyclic group. □

**4.2.2. Type I with infinite automorphism group.** Using Proposition 4.10, we deduce the following about the big cone on $X$:

**Proposition 4.17.** *Let $X$ be a threefold of Type I with $\mathcal{A}_+(X) \cong \mathbb{Z}$. Then* $\mathrm{Big}(X)$ *coincides with the positive component.*

Proof. Let $\varphi \in \mathcal{A}_+(X)$ and $u, v, w \in N^1(X)_{\mathbb{R}}$ be as in Proposition 4.16, so the positive component is $\mathbb{R}_{>0}u + \mathbb{R}_{>0}v + \mathbb{R}_{>0}w$. We proceed in two steps.

**Step 1**: $\operatorname{Big}(X) \supset \mathbb{R}_{>0}u + \mathbb{R}_{>0}v + \mathbb{R}_{>0}w$.

To show this inclusion, we first notice that $v^2 = 0$ in $N_1(X)_{\mathbb{R}}$. To see this, we observe that $(v + Ku)^3 = 0$ for all $K \in \mathbb{R}$ (since it is contained in $\mathcal{N}$), which in particular implies $v^2u = 0$. In the same way we can show $v^2w = 0$. Since we also know $v^3 = 0$ and $u, v, w$ form a basis of $N^1(X)_{\mathbb{R}}$, this shows $v^2 = 0$ in $N_1(X)_{\mathbb{R}}$.

Together with $v^3 = 0$ this implies, that

$$(x - Kv)^3 = x^3 - 3x^2(Kv)$$

for any $x \in N^1(X)_{\mathbb{R}}$ and any $K \in \mathbb{R}$. Proposition 4.10 implies that $x - Kv$ is big whenever $x$ is nef and $(x - Kv)^3 > 0$ for some $K \geq 0$. We deduce that the interior of the set

$$Z := \{y \in N^1(X)_{\mathbb{R}} \mid y^3 \geq 0, y + Kv \text{ is nef for some } K \geq 0\}$$

(which is non-empty, since it contains $\operatorname{Amp}(X)$) is contained in $\operatorname{Big}(X)$. We show that $Z \supset \mathbb{R}_{>0}u + \mathbb{R}_{\geq 0}v + \mathbb{R}_{>0}w$.

It is immediate to note that $Z$ is a convex cone with $\varphi(Z) = Z$ for any $\varphi \in \mathcal{A}_+(X)$. Moreover, $Z = \operatorname{conv}(\mathbb{R}_{\geq 0}v \cup Z')$, where

$$Z' := Z \cap (\mathbb{R}_{\geq 0}u + \mathbb{R}_{\geq 0}w),$$

because $y \in Z$ implies $y = au + bv + cw$ for $a, b, c \geq 0$ and $y - bv \in Z'$. It therefore suffices to show that $Z' \supset \mathbb{R}_{>0}u + \mathbb{R}_{>0}w$.

Note that $Z'$ is a convex subcone of $\mathbb{R}_{\geq 0}u + \mathbb{R}_{\geq 0}w$ such that $\varphi(Z') = Z'$ for any $\varphi \in \mathcal{A}_+(X)$. Since $Z$ has non-empty interior, it follows from $Z = \operatorname{conv}(\mathbb{R}_{\geq 0}v \cup Z')$ that $Z'$ contains a point $p = au + bw$ with $a, b > 0$. Then writing $\mathcal{A}_+(X) = \mathbb{Z}\varphi$ as given in Proposition 4.16, we obtain $\varphi^K(p) \to bw$ and $(1/\lambda)^K \cdot \varphi^{-K}(p) \to au$ as $K \to \infty$. Thus, $u, w$ are in the closure of $Z'$, so $Z' \supset \mathbb{R}_{>0}u + \mathbb{R}_{>0}w$. This concludes the proof of Step 1.

**Step 2**: $\operatorname{Big}(X) \subset \mathbb{R}_{>0}u + \mathbb{R}_{>0}v + \mathbb{R}_{>0}w$.

Since the pseudo-effective cone $\operatorname{Psef}(X)$ is the closure of $\operatorname{Big}(X)$, it suffices to show that

$$\operatorname{Psef}(X) \subset \mathbb{R}_{\geq 0}u + \mathbb{R}_{\geq 0}v + \mathbb{R}_{\geq 0}w.$$

For contradiction, assume that there exists $x = au + bv + cw \in \operatorname{Psef}(X)$ where at least one of $a, b, c \in \mathbb{R}$ is negative.

If $c < 0$, then $\mathbb{R}_{>0}u + \mathbb{R}_{>0}v$ is contained in the interior of

$$\operatorname{conv}(\{x\} \cup (\mathbb{R}_{>0}u + \mathbb{R}_{>0}v + \mathbb{R}_{>0}w)).$$

But the latter set is contained in the closed convex cone $\mathrm{Psef}(X)$. In particular, $\mathbb{R}_{>0}u+\mathbb{R}_{>0}v$ lies in the interior of $\mathrm{Psef}(X)$, which is $\mathrm{Big}(X)$. In particular, $u+v$ is big, but by Proposition 4.16 it is also nef and lies in the null cone. This contradicts Lemma 4.8.

We are left with the case that $c \geq 0$ and at least one of $a, b \in \mathbb{R}$ is negative. We may assume that $b < 0$, as we can otherwise interchange $u$ and $v$, replacing $\varphi$ by $\varphi^{-1}$. Since the pseudoeffective classes $(1/\lambda)^K \varphi^K(x)$ converge to $bv$ as $K \to \infty$, this implies that the line $\mathbb{R}v$ is contained in $\mathrm{Psef}(X)$. This is a contradiction, since the cone $\mathrm{Psef}(X)$ contains no lines. This finishes the proof of Step 2. □

Our main result regarding the Cone Conjecture is the following:

**Theorem 4.18.** *Let $X$ be a threefold of Calabi–Yau type which is of Type I with $\mathcal{A}_+(X) \cong \mathbb{Z}$.*

*Then Morrison's Cone Conjecture holds true for $X$. Moreover, Kawamata's Cone Conjecture holds true for $X$ if and only if there exists an effective divisor $D$ with $c_2(X) \cdot D = 0$.*

Note that by Theorem 4.3, the condition $c_2(X) \neq 0$ is automatically fulfilled when $X$ is a Calabi–Yau threefold of Type I with $\mathcal{A}_+(X) \cong \mathbb{Z}$.

Proof. We adopt the notation from Proposition 4.16. We distinguish two cases:

**Case 1**: There exists $x \in \mathrm{Nef}(X) \cap \mathcal{N} \setminus \langle u, v\rangle$.

After rescaling $x$ we have $x = w+\alpha u$ or $x = w+\alpha v$ for some $\alpha \geq 0$. Then $\varphi^m(x) \to w$ as $m \to \infty$ resp. $m \to -\infty$. As $\mathrm{Nef}(X)$ is closed, we deduce that $w \in \mathrm{Nef}(X)$, so

$$\mathrm{Nef}(X) = \mathbb{R}_{\geq 0}u + \mathbb{R}_{\geq 0}v + \mathbb{R}_{\geq 0}w.$$

The plane $\langle u, v\rangle$ is rational, so there exists an integral class $z_0 = a_0u+b_0v$ for some $a_0 > 0$, $b_0 > 0$. We claim that the rational polyhedral convex cone

$$\Pi := \mathbb{R}_{\geq 0}w + \mathbb{R}_{\geq 0}z_0 + \mathbb{R}_{\geq 0}\varphi(z_0)$$

is a fundamental domain for the action of $\mathcal{A}_+(X)$ on $\mathrm{Nef}^+(X)$ (see Figure 4.2.2).

Since $\mathbb{R}_{\geq 0}u$ and $\mathbb{R}_{\geq 0}v$ are irrational rays and $w$ is integral, we get

$$\mathrm{Nef}^+(X) = \mathbb{R}_{>0}u + \mathbb{R}_{>0}v + \mathbb{R}_{\geq 0}w.$$

We consider the map

$$g\colon \mathbb{R}_{>0}u + \mathbb{R}_{>0}v + \mathbb{R}_{\geq 0}w \to (0,\infty), (au + bv + cw) \mapsto b/a.$$

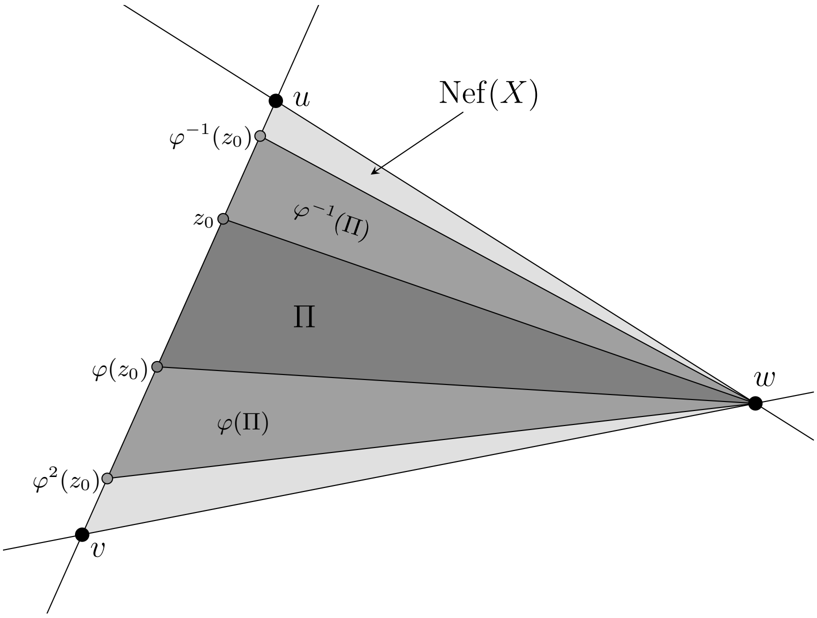


FIGURE 4.2. The fundamental domain for $\mathcal{A}_+(X)$ in Case 1

Then $g(\varphi(x)) = \tilde{\varphi}(g(x))$, where $\tilde{\varphi}\colon (0,\infty) \to (0,\infty)$ is the multiplication by $\lambda^2$. As $[g(z_0), \lambda^2 g(z_0)]$ is a fundamental domain for the action of $\tilde{\varphi}$ on $(0,\infty)$, its preimage under $g$ is a fundamental domain for the action of $\varphi$. To finish, we note that $g^{-1}([g(z_0), g(\varphi(z_0))]) = \Pi$.

**Case 2**: $\mathrm{Nef}(X) \cap \mathcal{N} = \mathbb{R}_{\geq 0} u + \mathbb{R}_{\geq 0} v$.

By Lemma 4.8, $\mathrm{Nef}(X)$ is locally rational polyhedral inside

$$\mathcal{U} := \mathbb{R}_{>0} u + \mathbb{R}_{>0} v + \mathbb{R}_{>0} w.$$

In particular, each extremal ray of $\mathrm{Nef}(X)$ in $\mathcal{U}$ is rational and these extremal rays are locally discrete in $\mathcal{U}$. Pick one such extremal ray and let $z_0 \in \mathrm{Nef}(X) \cap \mathcal{U}$ be an nonzero integral class on the rational ray. After possibly rescaling $z_0$, the affine line through $w$ and $z_0$ intersects $\mathbb{R}_{\geq 0} u + \mathbb{R}_{\geq 0} v$ in a point $z_1$. As $\langle u, v \rangle$ is a rational plane and $w$ and $z_0$ are integral points, $z_1$ is rational.

Define $g$ and $\tilde{\varphi}$ as in Case 1. Then $[g(z_0), \lambda^2 g(z_0)]$ is a fundamental domain for the action of $\tilde{\varphi}$ on $(0,\infty)$, so its preimage

$$\Pi := g^{-1}([g(z_0), g(\varphi(z_0))]) \cap \mathrm{Nef}(X)$$

is a fundamental domain for the action of $\varphi$ on $\mathrm{Nef}^+(X)$. Note that

$$g^{-1}([g(z_0), g(\varphi(z_0))]) = \mathbb{R}_{\geq 0} z_1 + \mathbb{R}_{\geq 0} \varphi(z_1) + \mathbb{R}_{\geq 0} w$$

(compare Figure 4.2.2). As the extremal rays of $\mathrm{Nef}(X)$ in $\mathcal{U}$ are locally discrete, only finitely many of them lie in $\mathbb{R}_{\geq 0} z_0 + \mathbb{R}_{\geq 0} \varphi(z_0) + \mathbb{R}_{\geq 0} w$,

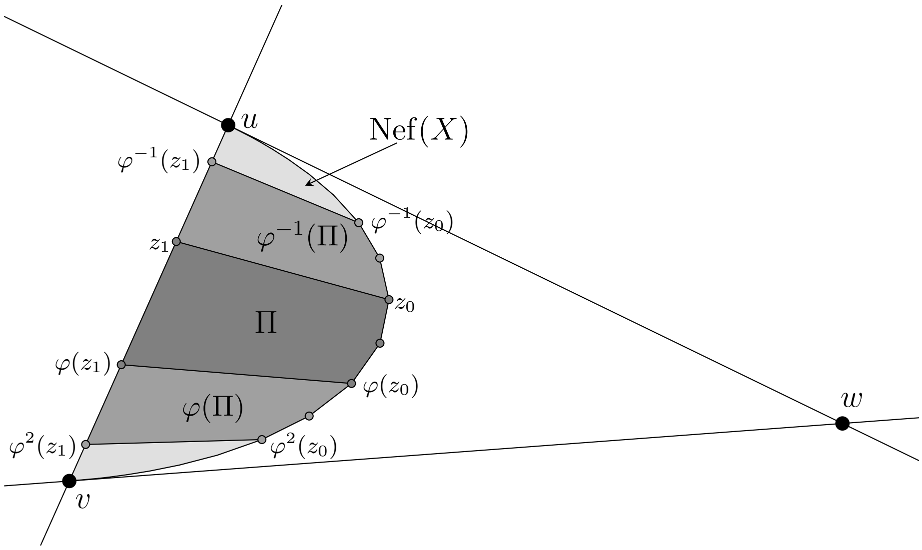


FIGURE 4.3. The fundamental domain for $\mathcal{A}_+(X)$ in Case 2

let's say $\mathbb{R}_{\geq 0}x_1, \ldots, \mathbb{R}_{\geq 0}x_m$. Then the fundamental domain

$$\Pi = \operatorname{Nef}(X) \cap (\mathbb{R}_{\geq 0}z_1 + \mathbb{R}_{\geq 0}\varphi(z_1) + \mathbb{R}_{\geq 0}w) = \sum_{i=1}^{m} \mathbb{R}_{\geq 0}x_i + \mathbb{R}_{\geq 0}z_1 + \mathbb{R}_{\geq 0}\varphi(z_1)$$

is a rational polyhedral convex cone. □

We can also deduce the existence of rational curves on Calabi–Yau threefolds of Type I with infinite automorphism group, building on the following result by Oguiso [**Ogu93**, Theorem 5.1].

**Proposition 4.19** ([**Ogu93**])**.** *Let $X$ be a simply connected Calabi–Yau threefold such that one of the following holds:*

*(i) There exists a non-zero $x \in \operatorname{Eff}(X) \cap N^1(X)_{\mathbb{Q}} \setminus \operatorname{Amp}(X)$.*
*(ii) There exists a non-zero $x \in \partial \operatorname{Nef}(X) \cap N^1(X)_{\mathbb{Q}}$ such that $x \cdot c_2(X) \neq 0$ or $x^2 = 0$ in $N_1(X)_{\mathbb{R}}$.*

*Then $X$ contains a rational curve.*

**Theorem 4.20.** *Let $X$ be a simply connected Calabi–Yau threefold of Type I with $\mathcal{A}_+(X) \cong \mathbb{Z}$. Then $X$ contains a rational curve.*

PROOF. We use the notation from Proposition 4.16. If $\operatorname{Nef}(X) = \mathbb{R}_{\geq 0}u + \mathbb{R}_{\geq 0}v + \mathbb{R}_{\geq 0}w$, then $w \in \partial \operatorname{Amp}(X)$ satisfies condition (ii) from Proposition 4.19. Otherwise any non-zero rational $x \in \mathbb{R}_{>0}u + \mathbb{R}_{>0}v + \mathbb{R}_{>0}w \setminus \operatorname{Amp}(X)$ satisfies condition (i) by Proposition 4.17. □

**4.2.3. Type I with finite automorphism group.** The second case, according to Proposition 4.16, is that $X$ is a Calabi–Yau threefold of Type I with $\mathcal{A}_+(X) \cong \mathbb{Z}/3$. In this case, Corollary 2.17 shows that

Kawamata's Cone Conjecture holds for $X$ if and only if $\mathrm{Nef}(X)$ is a rational polyhedral cone and $\mathrm{Nef}^{\mathrm{e}}(X) = \mathrm{Nef}^{+}(X)$. We confirm the second part of this assertion:

**Proposition 4.21.** *Let $X$ be a simply-connected Calabi–Yau threefold of Type I with $\mathcal{A}_{+}(X) \cong \mathbb{Z}/3$. Then*

$$\mathrm{Nef}^{\mathrm{e}}(X) = \mathrm{Nef}^{+}(X).$$

Proof. Theorem 4.3 shows this when $c_2(X) = 0$, so we may assume from now on that $c_2(X) \neq 0$.

By Proposition 2.14, we always have $\mathrm{Nef}^{\mathrm{e}}(X) \subset \mathrm{Nef}^{+}(X)$. To show the reverse inclusion we use that each rational nef class $x$ with $x \cdot c_2(X) > 0$ is effective, by Proposition 4.15. Hence, it is enough to show that $x \cdot c_2(X) > 0$ for all non-zero nef classes $x$.

Adopting the notation from Proposition 4.16, we have

$$v \cdot c_2(X) = \varphi(u) \cdot c_2(X) = u \cdot c_2(X),$$
$$w \cdot c_2(X) = \varphi^2(u) \cdot c_2(X) = u \cdot c_2(X).$$

Since $u, v, w$ form a basis of $N^1(X)_{\mathbb{R}}$ and $c_2(X) \neq 0$ in $N_1(X)_{\mathbb{R}}$, we must have $u \cdot c_2(X) \neq 0$. If $x = au + bv + cw$ is a nef class, then $a, b, c \geq 0$ by the description of the positive component in Proposition 4.16, so $a + b + c > 0$ if $x \neq 0$. We deduce that

$$x \cdot c_2(X) = (a + b + c) \cdot (u \cdot c_2(X)) \neq 0.$$

By Proposition 4.4, we deduce that $x \cdot c_2(X) > 0$. This finishes the proof. ☐

**Theorem 4.22.** *The Weak Cone Conjecture for* $\mathrm{Nef}^{+}$ *holds on threefolds of Calabi–Yau type which are of Type I and have a finite automorphism group.*

*Here, the almost rational polyhedral fundamental domain can be chosen to have at most two exceptional faces.*

Proof. We use the notation established in Proposition 4.16. We will distinguish several cases, but first we remark that in all cases the class $h = u + v + w$ is a rational ample class. Indeed, if $x = au + bv + cw$ with $a, b, c > 0$ is any integral ample class, then

$$x + \varphi(x) + \varphi^2(x) = (a + b + c) \cdot h \in N^1(X)_{\mathbb{Z}}$$

must be ample as well, which implies $h \in \mathrm{Amp}(X) \cap N^1(X)_{\mathbb{Q}}$.

**Case 1**: $u$ is nef.

In this case also $v = \varphi(u)$ and $w = \varphi^2(u)$ are nef, so $\mathrm{Nef}(X) = \mathbb{R}_{\geq 0} u + \mathbb{R}_{\geq 0} v + \mathbb{R}_{\geq 0} w$.

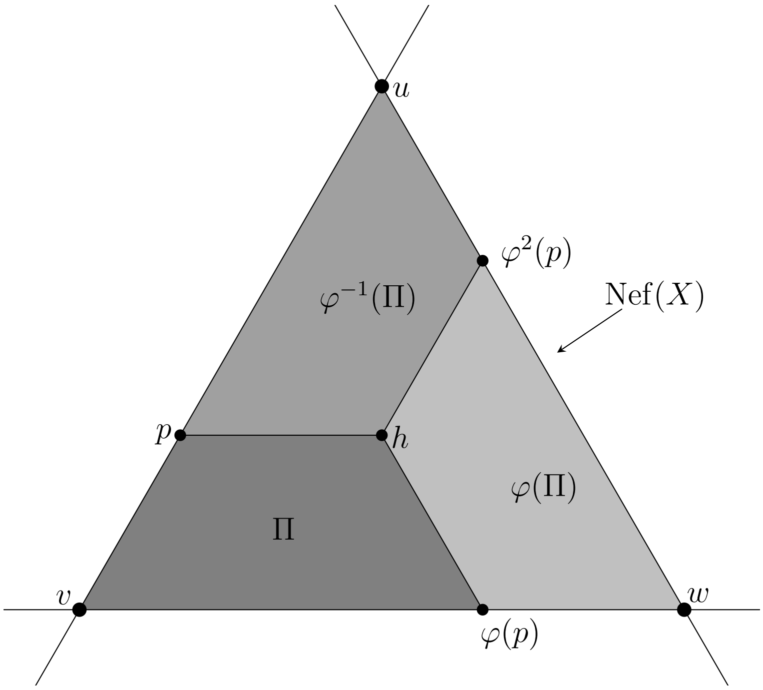


FIGURE 4.4. The fundamental domain for $\mathcal{A}_+(X)$ in Case 1

According to Proposition 4.16, there are two possibilities: Either the non-trivial subspaces of $\mathcal{N}$ are all rational or they are all irrational.

When $\mathcal{N}$ has only rational subspaces, the rays $\mathbb{R}_{\geq 0}u$, $\mathbb{R}_{\geq 0}v$ and $\mathbb{R}_{\geq 0}w$ are all rational, so $\mathrm{Nef}(X)$ is a rational polyhedral cone. This implies by Corollary 2.17 the existence of a fundamental domain for the action of $\mathrm{Aut}(X)$ on $\mathrm{Nef}^+(X)$ which is a rational polyhedral cone (i.e. without any exceptional faces). Explicitly, $\Pi = \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}u + \mathbb{R}_{\geq 0}v$ is easily seen to be a fundamendal domain for the action of $\mathcal{A}_+(X)$ on $\mathrm{Nef}^e(X)$.

We now consider the case that the subspaces of $\mathcal{N}$ are irrational.[1] Then we choose a rational plane $A$ containing the line $\mathbb{R}h$ and intersecting $\mathbb{R}_{>0}u + \mathbb{R}_{>0}v$ along a ray $\mathbb{R}_{\geq 0}p$ for some $p = au + bv$, $a, b > 0$. We claim that the convex cone

$$\Pi := \mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}p + \mathbb{R}_{\geq 0}v + \mathbb{R}_{\geq 0}\varphi(p)$$

provides the desired fundamental domain for $\mathcal{A}_+(X)$, see Figure 4.2.3.

First, we consider the supporting planes $\mathcal{P}_1 := \langle u, v\rangle$ and $\mathcal{P}_2 := \langle v, w\rangle$ of $\Pi$ and observe that

$$\Pi_0 := \Pi \setminus (\mathcal{P}_1 \cup \mathcal{P}_2)$$

[1]Note that, by Corollary 2.17, Morrison's Cone Conjecture predicts that this case should in fact not occur.

is locally rational polyhedral in $N^1(X)_{\mathbb{R}} \setminus (\mathcal{P}_1 \cup \mathcal{P}_2)$. Indeed, the points $x \in \Pi_0$ not contained in the interior of $\Pi_0$ are contained in the rational faces $\Pi \cap A$, $\Pi \cap \varphi(A)$ or their common intersection $\mathbb{R}_{\geq 0}h$, so locally around $x$ the cone $\Pi_0$ is locally rational polyhedral. (Note that we use the rationality of the plane $A$ at this point.) This shows that $\Pi$ is an almost rational polyhedral cone with two exceptional faces.

Second, we observe that $\Pi$ is a fundamental domain for the action of $\mathcal{A}_+(X)$ on $\mathrm{Nef}(X)$. Note that $\Pi$ and $\varphi(\Pi)$ lie on different sides of the plane $A$, so their interiors don't intersect and similarly for $\Pi$ and $\varphi^2(\Pi)$. In order to show

$$\mathrm{Nef}(X) = \Pi \cup \varphi(\Pi) \cup \varphi^2(\Pi),$$

we consider an arbitrary $x \in \mathrm{Nef}(X)$. Then $x' := x - \lambda h \in \partial\,\mathrm{Nef}(X)$ for precisely one $\lambda \geq 0$. Up to replacing $x$ by $\varphi(x)$ or $\varphi^2(x)$, we may assume that $x' \in \mathbb{R}_{\geq 0}u + \mathbb{R}_{\geq 0}v$. We have $x' \in \Pi$ if $x' \in \mathbb{R}_{\geq 0}u + \mathbb{R}_{\geq 0}p$ and $x' \in \varphi^2(\Pi)$ if $x' \in \mathbb{R}_{\geq 0}v + \mathbb{R}_{\geq 0}p$, so we deduce $x \in \Pi \cup \varphi^2(\Pi)$.

Finally, we show that $\Pi^+$ is a fundamental domain for the action of $\mathcal{A}_+(X)$ on $\mathrm{Nef}^{\mathrm{e}}(X) = \mathrm{Nef}^+(X)$. (We showed the latter equality in Proposition 4.21.) This follows from the above if we show that any irrational ray in $\mathrm{Nef}^+(X)$ is still contained in $\varphi^i(\Pi^+)$ for some $i \in \{0, 1, 2\}$. As the irrational rays in $\mathrm{Nef}^+(X)$ cannot lie on the boundary of $\mathrm{Nef}^+(X)$, they are contained in $\mathrm{Amp}(X)$, so it suffices to show that

$$\Pi^+ \cap \mathrm{Amp}(X) = \Pi \cap \mathrm{Amp}(X).$$

But this is the case, since the convex cone $\Pi \cap \mathrm{Amp}(X)$ is rational by the rationality of the plane $A$.

**Case 2**: $u$ is not nef.

In this case

$$\mathrm{Nef}(X) \subsetneq \mathbb{R}_{\geq 0}u + \mathbb{R}_{\geq 0}v + \mathbb{R}_{\geq 0}w,$$

so the boundary of $\mathrm{Nef}(X)$ must intersect the positive component $\mathcal{U}$. By Lemma 4.8, $\mathrm{Nef}(X)$ is locally rational polyhedral inside the positive component, so there must be a rational class $p \in \partial\,\mathrm{Nef}(X) \cap \mathcal{U}$. Note that $p$ does not lie on the ray $\mathbb{R}_{\geq 0}h$ by ampleness of $h$.

Since $h$ and $p$ are rational, there exists a non-zero rational class $\gamma \in N_1(X)_{\mathbb{Q}}$ such that $h \cdot \gamma = 0$ and $p \cdot \gamma = 0$. We claim that

$$\Pi := \{x \in \mathrm{Nef}(X) \mid x \cdot \gamma \geq 0, \varphi^{-1}(x) \cdot \ \gamma \leq 0\}$$

provides the desired fundamental domain, see Figure 4.2.3

First, we observe that $\Pi$ is a fundamental domain for the action of $\mathcal{A}_+(X)$ on $\mathrm{Nef}(X)$. Indeed, it follows from the definition of $\Pi$ that the

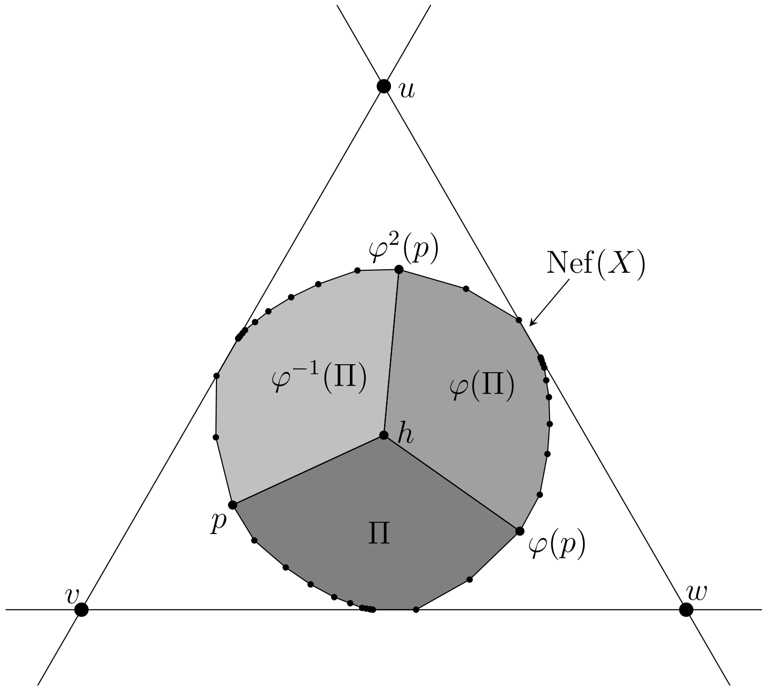


FIGURE 4.5. The fundamental domain for $\mathcal{A}_+(X)$ in Case 2

interiors of $\Pi$ and $\varphi(\Pi)$ don't intersect, and the same holds for $\Pi$ and $\varphi^2(\Pi)$. On the other hand,

$$\mathrm{Nef}(X) = \Pi \cup \varphi(\Pi) \cup \varphi^2(\Pi),$$

because any nef class $x = au + bv + cw$ satisfies

$$(x + \varphi(x) + \varphi^2(x)) \cdot \gamma = (a + b + c) \cdot h \cdot \gamma = 0,$$

which implies $\varphi^i(x) \cdot \gamma \geq 0$, $\varphi^{i-1}(x) \cdot \gamma \leq 0$ for some $i \in \{0, 1, 2\}$.

From this we deduce that $\Pi^+$ is a fundamental domain for the action of $\mathcal{A}_+(X)$ on $\mathrm{Nef}^e(X) = \mathrm{Nef}^+(X)$. As in Case 1, it suffices to show that the convex cone $\Pi \cap \mathrm{Amp}(X)$ is rational. This follows from the fact that the planes

$$\{x \in N^1(X)_\mathbb{R} \mid x \cdot \gamma = 0\} \quad \text{and} \quad \{x \in N^1(X)_\mathbb{R} \mid \varphi^{-1}(x) \cdot \ \gamma = 0$$

are rational, since they are spanned by $h$ and $p$, resp. $h$ and $\varphi(p)$.

Finally, we show that $\Pi$ is an almost rational polyhedral cone with at most one exceptional face. From the fact that $\mathrm{Nef}(X)$ is locally rational polyhedral in the positive component $\mathcal{U}$ and the rationality of the just mentioned planes, it follows that

$$\Pi \setminus (\mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3)$$

is locally rational polyhedral inside $\mathbb{R} \setminus (\mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3)$, where

$$\mathcal{P}_1 := \langle u, v \rangle, \ \mathcal{P}_2 := \langle v, w \rangle \text{ and } \mathcal{P}_3 := \langle w, u \rangle.$$

So $\Pi$ is an almost rational polyhedral cone with at most three exceptional faces.

To see that it has in fact at most one exceptional face, it suffices to show that $\Pi$ intersects at most one of the three planes $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$ non-trivially. Assume for contradiction that $\Pi$ contains non-zero points $x \in \mathcal{P}_1$ and $y \in \mathcal{P}_2$. (We may assume this by possibly replacing $\Pi$ by $\varphi(\Pi)$ or $\varphi^2(\Pi)$.) Then both $\varphi(x)$ and $y$ are non-zero nef classes in $\mathcal{P}_2$ such that $\varphi(x) \in \varphi(\Pi)$, while $y \in \Pi$. It follows that there exists a non-zero nef class $z$ in $\mathcal{P}_2 \cap \Pi \cap \varphi(\Pi)$. But $\Pi \cap \varphi(\Pi)$ is by definition contained in $\mathbb{R}_{\geq 0}h + \mathbb{R}_{\geq 0}p$ and, hence, lies in the positive component $\mathcal{U}$. But $z \in \mathcal{P}_2$ implies $z \notin \mathcal{P}_2$, a contradiction.

This shows that $\Pi$ has at most one exceptional face in Case 2, finishing the proof of the Theorem. □

## 4.3. Threefolds of Type II

**Proposition 4.23.** *Let $X$ be a threefold of Type II. Then $\mathcal{A}_+(X) \cong \mathbb{Z}$ and there exists a basis $(u, v, w)$ of $N^1(X)_{\mathbb{R}}$ such that the following holds:*

- *(i) The null cone is given by $\mathcal{N} = L \cup Q$, where $L = \langle u, v\rangle$ and $Q$ is the vanishing set of the quadratic polynomial $q = UV - W^2$ (where $U, V, W \in N^1(X)^\vee_{\mathbb{R}}$ is the dual basis of $u, v, w$).*
- *(ii) The positive component is either*

$$\mathcal{U} = \{UV - W^2 > 0 \text{ and } U, V, W > 0\}$$

  *or*

$$\mathcal{U} = \{UV - W^2 < 0 \text{ and } W > 0\}.$$

- *(iii) $\mathcal{A}_+(X)$ is generated by $\varphi$ with $\varphi(w) = w$, $\varphi(v) = \lambda v$ and $\varphi(u) = (1/\lambda)u$ for some $\lambda > 0$, $\lambda \neq 1$.*
- *(iv) The ray $\mathbb{R}_{\geq 0}w$ is rational and the rays $\mathbb{R}_{\geq 0}u$ and $\mathbb{R}_{\geq 0}v$ are irrational. The plane $L$ and the quadric cone $Q$ are rational (i.e., with respect to a* rational *basis of $N^1(X)_{\mathbb{R}}$ we can describe $Q$ as the vanishing set of a rational polynomial).*
- *(v) The classes $u$ is nef and either $v$ or $-v$ is nef.*

**Remark 4.24.** The above basis is useful because of the simplicity of the description of $\mathcal{A}_+(X)$. However, for picturing the null cone $\mathcal{N}$ and the positive component $\mathcal{U}$, it may be worth noting that in the basis $(x, y, z)$ with $x := u + v$, $y := u - v$, $z := w$ we have

$$\mathcal{N} = \{X^2 - Y^2 - Z^2 = 0\} \cup \{Z = 0\},$$

$$\mathcal{U} = \{X^2 - Y^2 - Z^2 > 0, X, Y, Z > 0\} \text{ or } \{X^2 - Y^2 - Z^2 < 0, Z > 0\},$$

where $(X, Y, Z)$ is the dual basis of $(x, y, z)$.

Proof. We proceed mostly in the order given in the formulation of the Proposition.

(i) Recall that by definition, $\mathcal{A}_+(X) \neq 0$ and $\mathcal{N} = Q \cup L$, where $L \subset N^1(X)_{\mathbb{R}}$ is a plane and $Q \subset N^1(X)_{\mathbb{R}}$ is a quadric cone intersecting $L$ along two lines.

There is a quadratic form $q\colon N^1(X)_{\mathbb{R}} \to \mathbb{R}$ with vanishing set $Q$. After possibly changing the sign of $q$ we can assume that the associated bilinear form $(\_,\_)_q$ on $N^1(X)_{\mathbb{R}}$ has signature $(s_+, s_-, s_0) = (1, 2, 0)$. Note that the property that $Q \cap L$ consists of two distinct lines translates to the fact that the restriction of this bilinear form to $L$ has signature $(1, 1, 0)$.

Let $u$ and $v$ be non-zero classes spanning the two lines in $Q \cap L$. Since $(\_,\_)_q$ and its restriction to $L$ are non-degenerate bilinear forms, we can find a non-zero $w \in N^1(X)_{\mathbb{R}}$ which is orthogonal to $\langle u, v\rangle$ with respect to $(\_,\_)_q$. According to the signature of the bilinear form, we must have $q(w) < 0$. After rescaling the basis, we can assume that $(\_,\_)_q$ is given with respect to the basis $(u, v, w)$ by the matrix

$$\begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

In particular, we have $q = UV - W^2$ and this shows the claim on the description of $\mathcal{N}$.

Regarding the positive component $\mathcal{U}$, we observe for now that we can assume that $\mathrm{Amp}(X)$ lies in the half-space $\{W > 0\}$ by possibly replacing $w$ by its negative. Here, we use that $\mathrm{Amp}(X) \cap \{W = 0\} = \emptyset$, since $\{W = 0\} = L$ lies in the null cone.

(ii) We consider any $\varphi \in \mathcal{A}_+(X)$, $\varphi \neq \mathrm{id}$. Then $\varphi$ must preserve the sets $Q$ and $L$, so there must exist $\gamma, \gamma' \in \mathbb{R}^\times$ such that $W \circ \varphi = \gamma W$ and $q \circ \varphi = \gamma' q$. In particular, $\varphi$ preserves the bilinear form $(\_,\_)_q$ up to a factor of $\gamma'$. Hence, $\varphi(w)$ is orthogonal to $\varphi(L) = L$ with respect to $(\_,\_)_q$, implying that $\varphi(w)$ is a multiple of $w$. In fact, $\varphi(w) = \gamma w$ because of

$$W(\varphi(w)) = (W \circ \varphi)(w) = \gamma W(w).$$

Since $\varphi$ preserves $\mathrm{Amp}(X)$ and $\mathrm{Amp}(X) \subset \{W > 0\}$, we further know $\gamma > 0$. Hence, with respect to the basis $(u, v, w)$ we have

$$\varphi = \begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{$\varphi|_L$}} & 0 \\ & & 0 \\ 0 & 0 & \gamma \end{pmatrix}.$$

Note that $\varphi$ preserves the cubic form $N^1(X)_{\mathbb{R}} \to \mathbb{R},\ x \mapsto x^3$, which is up to a scalar given by $q \cdot W$. This implies $\gamma' = 1/\gamma > 0$. Since $q(u) = q(v) = 0$, necessarily $\varphi|_L$ must act on $\mathbb{R}u \cup \mathbb{R}v$. If $\varphi|_L$ were of the form

$$\varphi|_L = \begin{pmatrix} 0 & \beta \\ \alpha & 0 \end{pmatrix}$$

for some $\alpha, \beta \in \mathbb{R}^\times$, then $(\varphi(u), \varphi(v))_q = \gamma' \cdot (u, v)_q$ implies $\alpha\beta = \gamma' > 0$. But this contradicts $\det \varphi = 1$.

Therefore, $\varphi|_L$ must satisfy $\varphi(u) = \alpha u$, $\varphi(v) = \beta v$ for some $\alpha, \beta \in \mathbb{R}^\times$, i.e. $\varphi = \operatorname{diag}(\alpha, \beta, \gamma)$ with respect to the basis $(u, v, w)$. As $\varphi$ preserves $q \cdot W$, we have

$$(UV - W^2) \cdot W = (\alpha U \cdot \beta V - \gamma^2 W^2) \cdot \gamma W,$$

which implies $\gamma = 1$ and $\alpha\beta = 1$.

We have now shown that any element of $\mathcal{A}_+(X)$ must be of the form $\operatorname{diag}(1/\beta, \beta, 1)$ with respect to the basis $(u, v, w)$. Since $\mathcal{A}_+(X) \subset \mathrm{SL}(N^1(X)_{\mathbb{Z}})$ is a non-trivial discrete group, we deduce the existence of $\lambda \in \mathbb{R}$, $|\lambda| > 1$ such that $\mathcal{A}_+(X)$ is the infinite cyclic group generated by $\varphi = \operatorname{diag}(1/\lambda, \lambda, 1)$. We will see in a moment that in fact $\lambda$ is positive.

(iii) Let $h = au + bv + cw$ be an ample class with $a, b \neq 0$. Then $\lambda^{-k}\varphi^k(h) \to bv$ and $\lambda^k \varphi^{-k}(h) \to au$, as $k \to \infty$. Therefore, $u, v \in \partial \operatorname{Amp}(X)$ are nef up to sign. Without changing $q$, we may change the sign of both simultaneously to guarantee that $u$ is nef. In particular, $\lambda$ must be positive, as otherwise both $u$ and $-u$ were nef, but $\operatorname{Nef}(X)$ contains no lines.

(iv) The line $\mathbb{R}w$ is the eigenspace of $\varphi$ for the eigenvalue 1, so it is rational. Since for any $a > 0$ the classes $\varphi^{-k}(av)$ accumulate towards zero as $k \to \infty$, we see that the line $\mathbb{R}v$ cannot contain an integral class. In the same way we show the irrationality of $\mathbb{R}u$.

The plane $L$ is given by $\{W = 0\}$, where $W \in N^1(X)^\vee_{\mathbb{R}}$ is an eigenvector of the dual morphism $\varphi^\vee$ for the eigenvalue 1. After replacing $W$ with a suitable multiple of itself, $W$ is a rational linear form on $N^1(X)_{\mathbb{R}}$, so $L$ is rational. Moreover, the cubic form

$$h\colon N^1(X)_{\mathbb{R}} \to \mathbb{R},\ x \mapsto x^3$$

is given up to scalar by $q \cdot W$, so $q$ is up to scalar the rational quadratic form $h/W$. This shows the rationality of $Q$.

(v) Finally, we establish the description of the positive component. We already asserted that $\mathrm{Amp}(X)$ lies in the half-plane $\{W > 0\}$. The connected components of $\{W > 0\} \setminus \{q = 0\}$ are

$$\{UV - W^2 > 0, U, V, W > 0\},$$
$$\{UV - W^2 > 0, W > 0, U, V < 0\} \text{ and}$$
$$\{UV - W^2 < 0, W > 0\}.$$

(The reader may find the advice from Remark 4.24 useful to illustrate this.) Since $u$ is nef, it must lie in the closure of the positive component, so this rules out the possibility $\{UV - W^2 > 0, W > 0, U, V < 0\}$. This concludes the proof.

□

**Theorem 4.25.** *Let $X$ be a threefold of Calabi–Yau type of Type II. We adopt the notation from Proposition 4.26 and assume that the positive component of $X$ is*

$$\mathcal{U} = \{UV - W^2 > 0 \textit{ and } U, V, W > 0\}.$$

*Then:*

*(i) If up to the action of $\mathrm{Aut}(X)$ there are only finitely many extremal rays of $\mathrm{Nef}(X)$ lying on the null cone $\mathcal{N}$, then the Weak Cone Conjecture for $\mathrm{Nef}^+(X)$ holds.*
*(ii) If the intersection $\mathrm{Nef}(X) \cap \mathcal{N}$ only consists of $\mathbb{R}_{\geq 0}u + \mathbb{R}_{\geq 0}v$, then Morrison's Cone Conjecture holds on $X$.*

Proof. We give a rough sketch of the proof, as the ideas are very similar to the proof of 4.22.

Let $\ell \in N^1(X)^\vee_{\mathbb{Q}}$ be a non-zero rational linear passing through the line $\mathbb{R}w$ and intersecting $\mathrm{Amp}(X)$. Then the closed convex cone

$$M := \{x \in \overline{\mathcal{U}} \mid \ell(x) > 0, \ell(\varphi^{-1}(x)) < 0\}$$

is easily seen to be a fundamental domain for the action of $\mathcal{A}_+(X)$ on $\overline{\mathcal{U}}^+$.

We define $\Pi := M \cap \mathrm{Nef}(X)$, see Figure 4.3. Then it follows that $\Pi^+ := \mathrm{conv}(\Pi \cap N^1(X)_{\mathbb{Q}})$ is a fundamental domain for the action of $\mathcal{A}_+(X)$ on $\mathrm{Nef}^+(X)$. By Lemma 4.8, $\Pi$ is locally rational polyhedral at each point inside $\mathcal{U}$.

In (i), we know by assumption that the intersection $\Pi \cap \mathcal{N}$ consists only of finitely many extremal faces of $\mathrm{Nef}(X)$ which may possibly form exceptional faces of $\Pi$. This shows that $\Pi$ is an almost polyhedral fundamental domain for the action of $\mathcal{A}_+(X)$ on $\mathrm{Nef}^+(X)$.

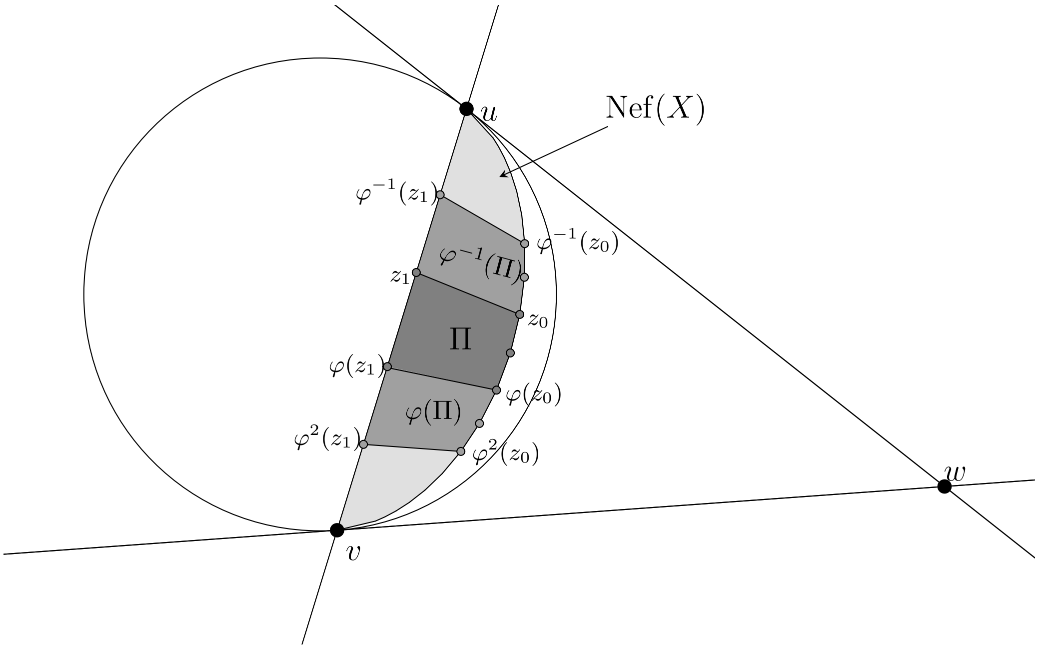


FIGURE 4.6. The fundamental domain for the action of $\mathcal{A}_+(X)$

In (ii), Π can be seen to have no exceptional faces, showing that under the assumption in (ii), the fundamental domain Π is a rational polyhedral cone, showing Morrison's Cone Conjecture. □

## 4.4. Threefolds of Type III

**Proposition 4.26.** *Let $X$ be a threefold of Type III. Then $\mathcal{A}_+(X) \cong \mathbb{Z}$ and there exists a basis $(u, v, w)$ of $N^1(X)_\mathbb{R}$ such that the following holds:*

*(i) The null cone is given by $\mathcal{N} = L \cup Q$, where $L = \langle u, v\rangle$ and $Q$ is the vanishing set of the quadratic polynomial $q = 2UW - V^2$ (where $U, V, W \in N^1(X)^\vee_\mathbb{R}$ is the dual basis of $u, v, w$).*

*(ii) With respect to the basis $(u, v, w)$, the group $\mathcal{A}_+(X)$ is generated by*

$$\varphi = \begin{pmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

*for some $\lambda > 0$.*

*(iii) The class $u \in N^1(X)_\mathbb{R}$ is nef and the ray $\mathbb{R}_{\geq 0}u$ is rational.*

*(iv) The plane $L$ and the quadric cone $Q$ are rational.*

*(v) The positive component is either*

$$\mathcal{U} = \{2UW - V^2 > 0, W > 0\} \text{ or } \{2UW - V^2 < 0, W < 0\}.$$

**Remark 4.27.** As in Remark 4.24, we observe that for a more intuitive illustration it is worth noting that

$$\begin{aligned} \mathcal{N} &= \{X^2 - Y^2 - Z^2 = 0\} \cup \{X - Y = 0\} \text{ and} \\ \mathcal{U} &= \{X^2 - Y^2 - Z^2 > 0, X - Y > 0\} \\ &\quad \text{or } \{X^2 - Y^2 - Z^2 < 0, X - Y > 0\}, \end{aligned}$$

where $X, Y, Z \in \mathbb{N}^1(X)^\vee_{\mathbb{R}}$ is the dual basis corresponding to $x := \frac{u+w}{\sqrt{2}}$, $y := \frac{u-w}{\sqrt{2}}$, $z := v$.

Proof. (i) Recall that by definition, $\mathcal{A}_+(X) \neq 0$ and $\mathcal{N} = Q \cup L$, where $L \subset N^1(X)_{\mathbb{R}}$ is a plane and $Q \subset N^1(X)_{\mathbb{R}}$ is a quadric cone intersecting $L$ along one line.

We consider a quadratic form $q\colon N^1(X)_{\mathbb{R}} \to \mathbb{R}$ with vanishing set $Q$ such that the associated bilinear form $(\_,\_)_q$ on $N^1(X)_{\mathbb{R}}$ has signature $(s_+, s_-, s_0) = (1, 2, 0)$. The property that $Q \cap L$ consists of one line translates to the fact that the restriction of $(\_,\_)_q$ to $L$ has signature $(0, 1, 1)$.

Let $u$ be a class spanning the line $L \cap Q$. Then $q(u) = 0$, so $u$ lies in the radical of the bilinear form on $L$ associated to $q|_L$, because $q|_L$ is negative semi-definite. We can extend this to a basis $(u, v)$ of $L$ such that $q(v) = -1$. Let $L'$ be the orthogonal complement of $v$ with respect to $(\_,\_)_q$. Then $L'$ is two-dimensional and, since $q|_{L'}$ has signature $(1, 1, 0)$, the intersection $L \cap Q$ consists of two lines. One of them is $\mathbb{R}u$ and let $w$ be a class spanning the other line in $L \cap Q$. Then $(u, v, w)$ form a basis of $N^1(X)_{\mathbb{R}}$. After rescaling $w$, we can assume that $(u, w)_q = 1$, so that the bilinear form $(\_,\_)_q$ is given with respect to the basis $(u, v, w)$ by the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

This shows $q = 2UW - V^2$ with respect to the chosen basis.

(ii) Let $\varphi \in \mathcal{A}_+(X)$, $\varphi \neq \mathrm{id}$. Since $\varphi$ preserves the intersection form

$$h\colon N^1(X)_{\mathbb{R}} \to \mathbb{R},\ x \mapsto x^3,$$

which is up to scalar given by $q \cdot W$, there exists $\gamma \in \mathbb{R}^\times$ such that $W \circ \varphi = \gamma W$ and $q \circ \varphi = (1/\gamma) \cdot q$.

Since the line $\mathbb{R}u$ is the intersection of $L$ and $Q$, it is preserved under $\varphi$, i.e. $\varphi(u) = \alpha u$ for some $\alpha \neq 0$. We deduce from these two obsverations that

$$\varphi = \begin{pmatrix} \alpha & c & b \\ 0 & \beta & a \\ 0 & 0 & \gamma \end{pmatrix}$$

holds with respect to the basis $(u, v, w)$ for some $\beta, a, b, c \in \mathbb{R}$. Note that $\det \varphi = 1$ implies $\alpha\beta\gamma = 1$.

Together with

$$1 = (u, w)_q = \gamma \cdot (\varphi(u), \varphi(w))_q = \alpha\gamma^2$$

and

$$-1 = (v, v)_q = \gamma \cdot (\varphi(v), \varphi(v))_q = -\beta^2\gamma,$$

this implies $\alpha = \beta = \gamma = 1$.

Now, we obtain

$$0 = (v, w)_q = \gamma \cdot (\varphi(v), \varphi(w))_q = c - a$$

and

$$0 = (w, w)_q = \gamma \cdot (\varphi(w), \varphi(w))_q = 2b - a^2,$$

so

$$\varphi = \begin{pmatrix} 1 & a & \frac{a^2}{2} \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}.$$

If we denote such a matrix by $M_a$ for any $a \in \mathbb{R}$, then one easily verifies $M_a \cdot M_b = M_{a+b}$ for all $a, b \in \mathbb{R}$. Hence, we can consider the discrete group $\mathcal{A}_+(X) \subset \mathrm{SL}(N^1(X)_\mathbb{R})$ as a subgroup of $\mathbb{R}$. Since discrete subgroups of $\mathbb{R}$ are cyclic, it follows that $\mathcal{A}_+(X) = \langle M_\lambda \rangle$ for some $\lambda > 0$.

By replacing $u$ with $\lambda^2 u$ and $v$ with $\lambda v$, the linear form $W$ and the quadratic form $q = 2UW - V^2$ remain unchanged, while we obtain $\mathcal{A}_+(X) = \langle \varphi \rangle$ with $\varphi = M_1$.

(iii) Let $h = au + bv + cw$ be an ample class with $c \neq 0$. Then $\frac{2}{k^2} \cdot \varphi^k(h) \to cu$ as $k \to \infty$, so $u$ or $-u$ is nef. By changing the sign of $u$, $v$ and $w$ simultaneously we do not change the description of $\mathcal{N}$ or $\varphi$, so we may assume that $u$ is nef. The line $\mathbb{R}u$ is the eigenspace of $\varphi$ for the eigenvalue 1, so it is rational, since $\varphi$ acts on the integral structure $N^1(X)_\mathbb{Z}$.

(iv) The linear form $W \in N^1(X)_\mathbb{R}^\vee$ is an eigenvector of $\varphi^\vee$ for the eigenvalue 1, so up to rescaling $W$ is rational. Hence, the line $L = \{W = 0\}$ is rational. Since the cubic form

$$h \colon N^1(X)_\mathbb{R} \to \mathbb{R},\ x \mapsto x^3$$

is rational and coincides with $q \cdot W$ up to scalar, $Q$ is given by the vanishing of the rational quadratic form $h/W$. Hence, $Q$ is rational.

(v) The only connected components of $N^1(X)_{\mathbb{R}} \setminus \mathcal{N}$ containing the nef class $u$ in their closure are

$$\{2UW - V^2 > 0, W > 0\},$$
$$\{2UW - V^2 < 0, W > 0\} \text{ and}$$
$$\{2UW - V^2 < 0, W < 0\},$$

so $\mathcal{U}$ must be one of them.

We consider ample class $h = au + bv + cw$ with $c \neq 0$. Then

$$\frac{1}{k^2}\varphi^k(h) \to \frac{c}{2}u \quad \text{as } k \to \infty.$$

But this implies that $cu$ is nef, so $c > 0$. On the other hand, $u + \epsilon h$ must be ample for $\epsilon \ll 1$ and

$$(2UW - V^2)(u + \epsilon h) = 2\epsilon c + \epsilon^2(2ac - b) > 0$$

for $\epsilon \ll 1$. Hence, $\mathcal{U} = \{2UW - V^2 > 0, W > 0\}$.

□

**Theorem 4.28.** *Let $X$ be a threefold of Calabi–Yau type of Type III. We adopt the notation from Proposition 4.26.*

*(i) If up to the action of* $\mathrm{Aut}(X)$ *there are only finitely many extremal rays of* $\mathrm{Nef}(X)$ *lying on the null cone* $\mathcal{N}$*, then the Weak Cone Conjecture for* $\mathrm{Nef}^+(X)$ *holds.*
*(ii) If the intersection* $\mathrm{Nef}(X) \cap \mathcal{N}$ *only consists of the ray* $\mathbb{R}_{\geq 0}u$*, then Morrison's Cone Conjecture holds on* $X$*.*

Proof. We sketch the proof, as it is very similar to the proof of 4.22.

We fix a non-zero rational linear form $\ell \in N^1(X)^\vee_{\mathbb{Q}}$ not proportional to $W$ passing through the line $\mathbb{R}u$. Then the closed convex cone

$$M := \{x \in \overline{\mathcal{U}} \mid \ell(x) > 0, \ell(\varphi^{-1}(x)) < 0\}$$

is easily seen to be a fundamental domain for the action of $\mathcal{A}_+(X)$ on $\overline{\mathcal{U}}$, see Figure 4.4.

Defining $\Pi := M \cap \mathrm{Nef}(X)$, it follows that $\Pi^+ := \mathrm{conv}(\Pi \cap N^1(X)_{\mathbb{Q}})$ is a fundamental domain for the action of $\mathcal{A}_+(X)$ on $\mathrm{Nef}^+(X)$. By Lemma 4.8, $\Pi$ is locally rational polyhedral at each point inside $\mathcal{U}$.

In (i), we know by assumption that the intersection $\Pi \cap \mathcal{N}$ consists only of finitely many extremal rays which may possibly form exceptional faces of $\Pi$. This shows that $\Pi$ is an almost polyhedral fundamental domain for the action of $\mathcal{A}_+(X)$ on $\mathrm{Nef}^+(X)$.

In (ii), this construction has no exceptional faces when we note the following: The ray $\mathbb{R}_{\geq 0}u$ is rational (by Proposition 4.26) and there

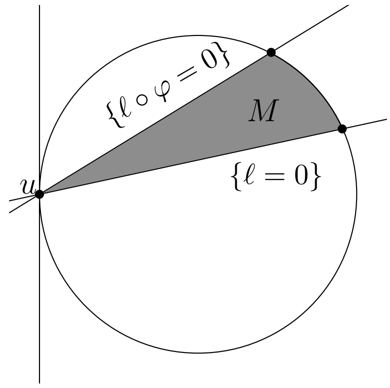


FIGURE 4.7. The fundamental domain for the action of $\mathcal{A}_+(X)$ on the positive cone

exists an open cone $\mathcal{C}$ containing the ray $\mathbb{R}_{>0}u$ such that

$$\overline{\mathcal{C}} \cap M \subset \mathrm{Nef}(X).$$

For this, we have to see that the planes $\{\ell = 0\}$ and $\{\ell \circ \varphi = 0\}$ intersect $\mathrm{Nef}(X)$ outside the ray $\mathbb{R}_{\geq 0}u$. This is easy to show by examining the action of $\varphi$, but the easiest way to establish this is to choose $\ell$ from the beginning in such a way that $\{\ell = 0\}$ intersects $\mathrm{Amp}(X)$.

Hence, under the assumption in (ii), the fundamental domain $\Pi$ is a rational polyhedral cone, showing Morrison's Cone Conjecture. □

Regarding Kawamata's Cone Conjecture, by Corollary 2.16, the question is about the difference between $\mathrm{Nef}^+$ and $\mathrm{Nef}^{\mathrm{e}}$. We obtain the following result:

**Proposition 4.29.** *Let $X$ be a simply-connected Calabi–Yau threefold of Type III. Then $\mathrm{Nef}^{\mathrm{e}}(X) = \mathrm{Nef}^+(X)$ if and only if there exists an effective divisor $D$ such that $D \cdot c_2(X) = 0$.*

PROOF. First, we observe that $\mathrm{Nef}^{\mathrm{e}}(X) = \mathrm{Nef}^+(X)$ is equivalent to $\mathbb{R}_{\geq 0}u \subset \mathrm{Eff}(X)$. Indeed, we know from Proposition 4.15 and Proposition 2.14 that $\mathrm{Nef}^{\mathrm{e}}(X)$ and $\mathrm{Nef}^+(X)$ coincide away from the plane $\{c_2(X) = 0\}$. Note that by Theorem 4.3 we must have $c_2(X) \neq 0$ because of $\mathcal{A}_+(X) \cong \mathbb{Z}$. The intersection of $\overline{\mathcal{U}}$ with $\{c_2(X) = 0\}$ is just the ray $\mathbb{R}_{\geq 0}u$and in fact $\mathbb{R}_{\geq 0}u \subset \mathrm{Nef}^+(X)$ by Proposition 4.26. Hence, $\mathrm{Nef}^e(X) = \mathrm{Nef}^+(X)$ holds if and only if $u \in \mathrm{Eff}(X)$.

To finish the proof, we have to notice that any effective class in $\{c_2(X) = 0\}$ must lie on the ray $\mathbb{R}_{\geq 0}u$. Indeed, if $au + bv$ is an effective class, then $1/k \cdot \varphi^k(au + bv) \to bu$, as $|k| \to \infty$. Considering this for $k \to \infty$ and $k \to -\infty$ shows that $bu$ and $-bu$ are both pseudo-effective. Since $\mathrm{Psef}(X)$ contains no lines, we deduce $b = 0$. □

## 4.5. Threefolds of Type IV

**Proposition 4.30.** *Let $X$ be a threefold of Type IV with $c_2(X) \neq 0$ or $c_1(X)^2 \neq 0$ in $N_1(X)$. Then $\mathcal{A}_+(X) \cong \mathbb{Z}/k$ for a $k \in \{2,3,4,6\}$ and there exists a basis $(u,v,w)$ of $N^1(X)_\mathbb{R}$ such that the following holds:*

*(i) The null cone is given by $\mathcal{N} = L \cup Q$, where $L = \langle u, v\rangle$ and $Q$ is the vanishing set of the quadratic polynomial $q = W^2 - U^2 - V^2$ (where $U, V, W \in N^1(X)^\vee_\mathbb{R}$ is the dual basis of $u, v, w$).*

*(ii) With respect to the basis $(u,v,w)$, the group $\mathcal{A}_+(X)$ is generated by*

$$\varphi = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

*for $\alpha = 2\pi/k$.*

*(iii) The class $w \in N^1(X)_\mathbb{R}$ is ample and the ray $\mathbb{R}_{\geq 0} w$ is rational.*

*(iv) The positive component is*

$$\mathcal{U} = \{W^2 - U^2 - V^2 > 0, W > 0\}.$$

*(v) The plane $L$ and the quadric cone $Q$ are rational.*

Proof. (i) Recall that by definition, $\mathcal{A}_+(X) \neq 0$ and $\mathcal{N} = Q \cup L$, where $L \subset N^1(X)_\mathbb{R}$ is a plane and $Q \subset N^1(X)_\mathbb{R}$ is a quadric cone intersecting $L$ trivially.

We consider a quadratic form $q\colon N^1(X)_\mathbb{R} \to \mathbb{R}$ with vanishing set $Q$ such that the associated bilinear form $(\_,\_)_q$ on $N^1(X)_\mathbb{R}$ has signature $(s_+, s_-, s_0) = (1,2,0)$. The property that $Q \cap L = \{0\}$ translates to the fact that the restriction of $(\_,\_)_q$ to $L$ has signature $(0,2,0)$.

Let $u, v$ be an orthogonal basis of $L$ such that $q(u) = (u,u)_q = -1$ and $q(v) = (v,v)_q = -1$. We can extend this to a basis $(u,v,w)$ which is orthogonal with respect to $(\_,\_)_q$ and such that $q(w) = (w,w)_q = 1$. With respect to this basis, we have

$$q = W^2 - U^2 - V^2$$

and $L = \{W = 0\}$.

(ii) By Corollary 3.9, we have $L = \{\ell = 0\}$, where $\ell \in N_1(X) \setminus \{0\}$ is given by $\ell = c_2(X)$ or $\ell = c_1(X)^2$. Since the linear forms $\ell, W \in N^1(X)^\vee_\mathbb{R}$ are both non-zero and vanish on $L$, they must agree up to a scalar multiple and hence, as $\ell \circ \varphi = \ell$, we have $W \circ \varphi = W$. Since the cubic form

$$h\colon N^1(X)_\mathbb{R} \to \mathbb{R},\ x \mapsto x^3$$

is preserved by $\varphi$ and is given up to scalar by $W \cdot q$, we also know $q \circ \varphi = q$, i.e. $\varphi$ is compatible with the bilinear form $(\_,\_)_q$.

The line $\mathbb{R}w$ is the orthogonal complement of $L$ with respect to $(\_,\_)_q$ and $\varphi(L) = L$, so $w$ is an eigenvector of $\varphi$. Because of

$$W(\varphi(w)) = (W \circ \varphi)(w) = W(w) = 1,$$

we see that $\varphi(w) = w$. Therefore, the fact that $\varphi$ is compatible with $(\_,\_)_q$ implies that with respect to the basis $(u, v)$ of $L$ we have

$$\varphi\big|_L \in \mathrm{SO}(2, \mathbb{R}).$$

This shows that each $\varphi \in \mathcal{A}_+(X)$ is with respect to the basis $(u, v, w)$ of the form

$$\varphi = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for some $\alpha \in \mathbb{R}/(2\pi\mathbb{Z})$. If we denote such a matrix by $M_\alpha$, then for any $\alpha, \beta \in \mathbb{R}/(2\pi\mathbb{Z})$ we have $M_\alpha \cdot M_\beta = M_{\alpha+\beta}$, so we can consider the discrete group $\mathcal{A}_+(X)$ as a subgroup of $\mathbb{R}/(2\pi\mathbb{Z})$. But discrete subgroups of $\mathbb{R}/(2\pi\mathbb{Z})$ are finite cyclic groups of the form $\langle \frac{2\pi}{k} \rangle$ for $k > 0$. This shows $\mathcal{A}_+(X) = \langle \varphi \rangle$ with $\varphi = M_{2\pi/k}$.

It remains to be shown that $k \in \{2, 3, 4, 6\}$. Since $\mathcal{A}_+(X)$ is non-trivial, we have $k > 1$. Note that $\varphi\big|_L$ preserves the integral structure $N^1(X)_\mathbb{Z} \cap L$ (note that $L = \{\ell = 0\}$ is rational), so we can consider its minimal polynomial over $\mathbb{Q}$, which is the $k$-th cyclotomic polynomial. For $k = 5$ or $k > 7$, the $k$-th cyclotomic polynomial has degree $\geq 4$, but on the other hand the characteristic polynomial of $\varphi\big|_L$ is of degree $\dim L = 2$. Hence, we must have $k \in \{2, 3, 4, 6\}$.

(iii) Since the minimal polynomial of $\varphi\big|_L$ is the $k$-th cyclotomic polynomial, we have in particular

$$\sum_{i=0}^{k-1} \varphi^i\Big|_L = 0.$$

In particular, if $h = au + bv + cw$ is an integral ample class with $c \neq 0$, then

$$\sum_{i=0}^{k-1} \varphi^i(h) = k \cdot cw,$$

so $w$ or $-w$ is ample and the line $\mathbb{R}w$ is rational. We may change the sign of $w$ without affecting the description of $\mathcal{N}$ or $\varphi$, so we can assume that indeed $w$ is ample.

(iv) The connected component of $N^1(X)_\mathbb{R} \setminus (L \cup Q)$ given by

$$\{W^2 - U^2 - V^2 > 0, W > 0\}$$

contains the ample class $w$, so it is the positive component.

(v) The plane $L = \{\ell = 0\}$ is rational, because $\ell \in N_1(X)_{\mathbb{Z}}$. The cubic form

$$h\colon N^1(X)_{\mathbb{R}} \to \mathbb{R},\ x \mapsto x^3$$

is rational and factors up to a scalar as $\ell \cdot q$. Hence, the quadric cone $Q$ is given by the rational quadratic form $h/\ell$.

□

## 4.6. Threefolds of Type V

**Proposition 4.31.** *Let $X$ be a threefold of Type V. Then $\mathcal{A}_+(X) \cong \mathbb{Z}/3$ and there exists a basis $(u, v, w)$ of $N^1(X)_{\mathbb{R}}$ such that the following holds:*

*(i) With respect to the basis $(u, v, w)$, the group $\mathcal{A}_+(X)$ is generated by*

$$\varphi = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

*for $\alpha = 2\pi/3$.*

*(ii) The class $w \in N^1(X)_{\mathbb{R}}$ is ample and the ray $\mathbb{R}_{\geq 0} w$ is rational.*

*(iii) The plane $\langle u, v\rangle$ is rational.*

*(iv) The null cone is given by the vanishing set*

$$\{V(V^2 - 3U^2) + aW(U^2 + V^2) + W^3 = 0\}$$

*for some $a \in \mathbb{R}$ (where $U, V, W \in N^1(X)^\vee_{\mathbb{R}}$ is the dual basis of $u, v, w$).*

Proof. (i) By Corollary 3.9, the group $\mathcal{A}_+(X)$ must be of order 3. The characteristic polynomial of $\varphi$ is $T^3 - 1 = (T-1)(T^2 + T + 1)$, implying that with respect to a suitable basis $u, v, w$ of $N^1(X)_{\mathbb{R}}$ the automorphism $\varphi$ is given by

$$\varphi = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\alpha = 2\pi/3$, i.e. $\varphi$ acts by rotation by $2\pi/3$ on the plane spanned by $u$ and $v$.

(ii) If $h = au + bv + cw$ is any integral ample class with $c \neq 0$, then

$$3cw = h + \varphi(h) + \varphi^2(h)$$

is also ample and integral. Hence, either $w$ or $-w$ is ample. By possibly changing the sign of $w$, we may assume that $w$ is ample.

(iii) The plane $\langle u,v\rangle = \{W=0\}$ is rational because $\mathbb{R}W \subset N^1(X)_\mathbb{R}$ is the eigenspace of the eigenvalue 1 of $\varphi^\vee$ and $\varphi$ is defined over the integral structure $N^1(X)_\mathbb{Z}$.

(iv) Denote by $U,V,W$ the dual basis associated to $u,v,w$ and let $h \in \mathbb{R}[U,V,W]$ be the homogeneous polynomial of degree 3 corresponding to the polynomial mapping

$$N^1(X)_\mathbb{R} \to \mathbb{R},\ \alpha \mapsto \alpha^3.$$

Note that $h$ is irreducible, as $\mathcal{N}$ does not split.

If we write

$$h = h_3 + W \cdot h_2 + W^2 \cdot h_1 + \gamma W^3$$

with $\gamma \in \mathbb{R}$, $h_i \in \mathbb{R}[U,V]$ homogeneous of degree $i$, then $h_i \circ \varphi|_{\langle u,v\rangle} = h_i$ (for $i = 1,2,3$).

The vanishing set of $h_3$ in $\langle u,v\rangle$ is the union of one, two or three lines, because $h_3 \neq 0$ by irreducibility of $h$. This vanishing set can therefore only be preserved by rotation by $2\pi/3$ if it is the union of three lines given by $h_3 = \alpha V(V^2 - 3U^2)$ for some $\alpha \in \mathbb{R}^\times$ (after rotating the basis $u,v$ accordingly).

The vanishing set of $h_1$ and $h_2$ in $\langle u,v\rangle$ cannot consist of one or two lines, as this would not be preserved by $\varphi|_{\langle u,v\rangle}$. This shows $h_1 = 0$ and either $h_2 = 0$ or $h_2$ is a homogeneous quadratic polynomial in $\mathbb{R}[U,V]$ without real zeroes. For $\varphi|_{\langle u,v\rangle}$ to preserve $h_2$, we must have $h_2 = \beta(U^2+V^2)$ for some $\beta \in \mathbb{R}$. Thus,

$$h = \alpha V(V^2 - 3U^2) + \beta W(U^2+V^2) + \gamma W^3).$$

Because of $h(w) > 0$ (as $w$ is ample), we must have $\gamma > 0$ and after rescaling $w$, we can assume $\gamma = 1$. Similarly, we can rescale $u$ and $w$ by a common factor to guarantee $\alpha = 1$. This shows the claim.

□

CHAPTER 5

# Diophantine Approximation problems for Nef($X$)

In this final chapter, we will provide an instance how arithmetic arguments may provide restrictions on the structure of the nef cone. The main idea behind this is to possibly exclude that the nef cone contains certain irrational extremal rays which are too well approximable by rational rays in the sense of simultaneous Diophantine Approximations. In fact, we have already seen in Proposition 4.13 an indication how rational approximations of an irrational rays may guarantee that the irrational ray cannot be contained in $\mathrm{Nef}(X)$.

We will raise a natural question in the context of Diophantine Approximations (Question 5.2) and we will see how it would imply the existence of rational curves on all simply-connected Calabi–Yau threefolds of Type I, extending Theorem 4.20.

The treatment will be based on Lemma 4.11 together with the following finiteness result by Szendrői [**Sze99**].

**Theorem 5.1** ([**Sze99**]). *Let $X$ be a Calabi–Yau threefold and fix an integer $K$. Then there are only finitely many ample divisors $H$ with $H^3 \leq K$ up to the action of* $\mathrm{Aut}(X)$.

We raise the following question about Diophantine Approximations:

**Question 5.2.** *Let $\alpha_1, \alpha_2 \in \mathbb{R}$ be real numbers. Given any $w_1, w_2 \in \mathbb{R}$, are there infinitely many $(p_1, p_2, q) \in \mathbb{Z}^3$ with $q > 0$ satisfying the following inequalities:*

$$\left|\frac{p_1}{q} - \alpha_1\right| \leq \frac{1}{q^{3/2}},$$
$$\left|\frac{p_2}{q} - \alpha_2\right| \leq \frac{1}{q^{3/2}},$$
$$w_1 \cdot \left(\frac{p_1}{q} - \alpha_1\right) + w_2 \cdot \left(\frac{p_2}{q} - \alpha_2\right) \geq 0?$$

**Remark 5.3.** Without the last condition (i.e. for $w_1 = w_2 = 0$) this is the well-known Dirichlet's Approximation Theorem in two dimensions, see for example [**HW08**, Theorem 200]. We also observe that the question has a trivially positive answer if $\alpha_1 = \frac{a_1}{b_1}$ and $\alpha_2 = \frac{a_2}{b_2}$ are rational: The points $(ka_1b_2, ka_2b_1, kb_1b_2) \in \mathbb{Z}^3$ are solutions for all $k > 0$.

Note that in the case of "one-dimensional" Diophantine approximation, an analogous statement holds true:

**Proposition 5.4.** *Let $\alpha \in \mathbb{R}$ be a real number. Then there are infinitely many $(p, q) \in \mathbb{Z}^2$ with $q > 0$ satisfying*

$$\frac{p}{q} - \alpha \leq \frac{1}{q^2}.$$

*Similarly, there are infinitely many $(p, q) \in \mathbb{Z}^2$ with $q > 0$ satisfying*

$$\frac{p}{q} - \alpha \geq \frac{1}{q^2}.$$

PROOF. This is remarked in [**HW08**, p. 202]. It follows from the fact that for irrational $\alpha$ the partial continued fractions of $\alpha$ give corresponding rational approximations, [**HW08**, Theorem 171], and alternate between approximating $\alpha$ from below and from above, [**HW08**, Theorem 167]. □

From this, we can confirm Question 5.2 in a simple case:

**Proposition 5.5.** *Question 5.2 has an affirmative answer for those $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $1, \alpha_1, \alpha_2$ are linearly dependent over $\mathbb{Q}$.*

PROOF. If both $\alpha_1$ and $\alpha_2$ are rational, then this is trivial as observed in Remark 5.3.

We may now assume that $\alpha_1 \in \mathbb{R} \setminus \mathbb{Q}$. By the linear dependence, there exist integers $a, b, c$ such that

$$a\alpha_2 = b\alpha_1 + c$$

and we may assume that $a > 0$.

For any points $(p_0, q_0) \in \mathbb{Z}^2$ with $q > 0$ satisfying

$$\left|\frac{p_0}{q_0} - \alpha_1\right| \leq \frac{1}{q_0^2}$$

we can consider $q := aq_0$, $p_1 := ap_0$ and $p_2 := bp_0 + cq_0$.

Then

$$\left|\frac{p_1}{q} - \alpha_1\right| = \left|\frac{p_0}{q_0} - \alpha_1\right| \leq \frac{1}{q_0^2}.$$

and

$$\left|\frac{p_2}{q} - \alpha_2\right| = \left|\frac{b}{a} \cdot \left(\frac{p_0}{q_0} - \alpha_1\right)\right| \leq \frac{|b|}{aq_0^2}.$$

Moreover,

$$w_1 \cdot \left(\frac{p_1}{q} - \alpha_1\right) + w_2 \cdot \left(\frac{p_2}{q} - \alpha_2\right) = \left(\frac{aw_1 + bw_2}{a}\right) \cdot \left(\frac{p_0}{q_0} - \alpha_1\right).$$

By Proposition 5.4, there are infinitely many $(p_0, q_0) \in \mathbb{Z}^2$ such that the sign of $\frac{p_0}{q_0} - \alpha_1$ is such that

$$\left(\frac{aw_1 + bw_2}{a}\right) \cdot \left(\frac{p_0}{q_0} - \alpha_1\right) \geq 0.$$

Moreover, we can exclude the finite number of $(p_0, q_0)$ such that $q_0 < a^3$ or $q_0 < a^3 b^2$. Then for the corresponding $(p_1, p_2, q)$, the above identities give

$$\left|\frac{p_1}{q} - \alpha_1\right| \leq \frac{1}{q_0^2} \leq \frac{1}{q^{3/2}},$$

$$\left|\frac{p_2}{q} - \alpha_2\right| \leq \frac{|b|}{aq_0^2} \leq \frac{1}{q^{3/2}},$$

$$w_1 \cdot \left(\frac{p_1}{q} - \alpha_1\right) + w_2 \cdot \left(\frac{p_2}{q} - \alpha_2\right) = \left(\frac{aw_1 + bw_2}{a}\right) \cdot \left(\frac{p_0}{q_0} - \alpha_1\right) \geq 0.$$

This concludes the proof. □

**Proposition 5.6.** *If Question 5.2 has an affirmative answer, then any simply-connected Calabi–Yau threefold of Type I contains a rational curve.*

Proof. Let $X$ be a simply-connected Calabi–Yau threefold of Type I. If $\mathcal{A}_+(X) \cong \mathbb{Z}$, then $X$ contains a rational curve by Theorem 4.20, so we may assume $\mathcal{A}_+(X) \cong \mathbb{Z}/3$. We use the notation from Proposition 4.16.

If

$$\operatorname{Nef}(X) \subsetneq \mathbb{R}_{\geq 0}u + \mathbb{R}_{\geq 0}v + \mathbb{R}_{\geq 0}w,$$

then $\partial \operatorname{Nef}(X)$ contains a rational big divisor according to Proposition 4.5. This implies the existence of rational curves by Proposition 4.19.

The other case to consider is

$$\operatorname{Nef}(X) = \mathbb{R}_{\geq 0}u + \mathbb{R}_{\geq 0}v + \mathbb{R}_{\geq 0}w.$$

According to Proposition 4.16, either the non-trivial subspaces of $\mathcal{N}$ are all rational or all irrational. When they are rational, then in particular some non-zero multiple of $u$ is rational and lies in $\partial \operatorname{Nef}(X)$. As seen in the proof of Proposition 4.21 we have $u \cdot c_2(X) > 0$, so Proposition 4.19 implies the existence of rational curves on $X$ in this case.

We are left with the case that $\mathrm{Nef}(X) = \mathbb{R}_{\geq 0}u + \mathbb{R}_{\geq 0}v + \mathbb{R}_{\geq 0}w$ and the subspaces of $\mathcal{N}$ are irrational. We show that this case does not occur in the following Lemma, concluding the proof. $\square$

**Lemma 5.7.** *Let $X$ be a Calabi–Yau threefold of Type I such that the non-trivial subspaces of $\mathcal{N}$ are irrational. If Question 5.2 has an affirmative answer, then*

$$\mathrm{Nef}(X) \neq \mathbb{R}_{\geq 0}u + \mathbb{R}_{\geq 0}v + \mathbb{R}_{\geq 0}w$$

*with $u, v, w$ as in Proposition 4.16.*

Proof. Assume for contradiction that $\mathrm{Nef}(X) = \mathbb{R}_{\geq 0}u + \mathbb{R}_{\geq 0}v + \mathbb{R}_{\geq 0}w$, so in particular $u$ is nef.

We will roughly proceed as follows: Using the assumption on Question 5.2, we construct an infinite sequence $(\alpha_i)$ of integral classes such that the rays $\mathbb{R}_{\geq 0}\alpha_i$ accumulate towards the ray $\mathbb{R}_{\geq 0}u$ and such that $|\alpha_i^3|$ is bounded by a constant. From this we deduce either the existence of infinitely many integral ample classes with bounded self-intersection or the existence of an effective divisor outside of the positive component by using Lemma 4.11. In the latter case, we get a contradiction to $\mathrm{Big}(X) = \mathrm{Amp}(X)$, and in the former case, we get a contradiction to Theorem 5.1.

We observe, that by Proposition 4.16 and Theorem 4.3, we are necessarily in the case $\mathcal{A}_+(X) = \mathbb{Z}/3$ and $c_2(X) \neq 0$. We proceed in several steps.

**Step 1**: Construction of the sequence $(\alpha_i)$ in $N^1(X)_\mathbb{Z}$.

We choose an integral basis $x, y, z \in N^1(X)_\mathbb{Z}$ such that $x$ and $y$ span the plane $\{c_2(X) = 0\}$. Up to rescaling $u$, we may assume $u \cdot c_2(X) = 1$, so that there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ with

$$u = \lambda_1 x + \lambda_2 y + z.$$

To the basis $(u, v, w)$ of $N^1(X)_\mathbb{R}$ we associate the dual basis $(U, V, W)$ of $N^1(X)_\mathbb{R}^\vee = N_1(X)_\mathbb{R}$ and similarly, we denote by $X, Y, Z \in N^1(X)_\mathbb{R}^\vee$ the dual basis to $x, y, z \in N^1(X)_\mathbb{R}$.

We choose $a, b > 0$ and consider the plane $\{\ell = 0\}$ for $\ell = aV + bW$. If possible, we choose $a, b > 0$ such that this plane is rational. (Note that there is at most one such rational plane, as there cannot be two distinct rational planes passing through the irrational line $\mathbb{R}u$.) Then there are $w_1, w_2, w_3 \in \mathbb{R}$ such that

$$\ell w_1 X + w_2 Y + w_3 Z.$$

Since we assume Question 5.2 to have a positive answer, there exists an infinite sequence $(p_1^{(i)}, p_2^{(i)}, q^{(i)})_i$ of tuples in $\mathbb{Z}^3$ with $q^{(i)} > 0$ such that

$$\left|\frac{p_1^{(i)}}{q^{(i)}} - \lambda_1\right| \le \frac{1}{(q^{(i)})^{3/2}},$$
$$\left|\frac{p_2^{(i)}}{q^{(i)}} - \lambda_2\right| \le \frac{1}{(q^{(i)})^{3/2}},$$
$$w_1 \cdot \left(\frac{p_1^{(i)}}{q^{(i)}} - \lambda_1\right) + w_2 \cdot \left(\frac{p_2^{(i)}}{q^{(i)}} - \lambda_2\right) \ge 0$$

We define $\alpha_i := p_1^{(i)}x + p_2^{(i)}y + q^{(i)}z \in N^1(X)_{\mathbb{Z}}$, which defines an infinite sequence of integral classes. Some simple observations about this sequence are:

(1) As $i \to \infty$, the rays $\mathbb{R}_{\ge 0}\alpha_i$ accumulate towards the ray $\mathbb{R}_{\ge 0}u$ and $q^{(i)} \to \infty$.
(2) The classes $\alpha_i$ lie in the half-space $\{\ell \ge 0\}$.

Property (1) is an immediate consequence of the first two inequalities from above, as the points $\alpha_i/q^{(i)}$ converge to $u = \lambda_1 x + \lambda_2 y + z$. Property (2) follows from

$$\begin{aligned}\ell(\alpha_i) &= w_1 p_1^{(i)} + w_2 p_2^{(i)} + w_3 q^{(i)}) \\ &\ge q^{(i)} \cdot (w_1\lambda_1 + w_2\lambda_2 + w_3) \\ &= q^{(i)} \cdot \ell(u) = 0.\end{aligned}$$

**Step 2**: There exists a constant $C > 0$ such that $|\alpha_i^3| \le C$ for all $i \ge 0$.

By the description of $\mathcal{N}$, we know that there is $\mu > 0$ such that

$$\alpha_i^3 = \mu \cdot U(\alpha_i) \cdot V(\alpha_i) \cdot W(\alpha_i).$$

The set $A := [-1,1] \cdot x + [-1,1] \cdot y \subset N^1(X)_{\mathbb{R}}$ is compact, so there exists a constant $C_0 > 0$ such that

$$|W(\delta)| \le C_0, \ |V(\delta)| \le C_0$$

for all $\delta \in A$. By construction,

$$(q^{(i)})^{3/2} \cdot \left(\frac{\alpha_i}{q^{(i)}} - u\right) \in A$$

holds for all $i$, so using $V(u) = W(u) = 0$ we deduce

$$\begin{aligned}|\alpha_i^3| &= \mu \cdot \left|U\left(\frac{\alpha_i}{q^{(i)}}\right)\right| \cdot \left|V\left((q^{(i)})^{3/2}\left(\frac{\alpha_i}{q^{(i)}} - u\right)\right)\right| \cdot \left|W\left((q^{(i)})^{3/2}\left(\frac{\alpha_i}{q^{(i)}} - u\right)\right)\right| \\ &\le \mu C_0^2 \cdot \left|U\left(\frac{\alpha_i}{q^{(i)}}\right)\right|.\end{aligned}$$

Since $\frac{\alpha_i}{q^{(i)}} \to u$ as $i \to \infty$ and $U(u) = 1$, this shows that $|\alpha_i^3|$ is bounded by some constant $C$.

**Step 3**: We can assume that $\alpha_i \in \mathbb{R}_{>0}u + \mathbb{R}_{>0}v + \mathbb{R}_{<0}w$.

Since the classes $\alpha_i$ are integral and $|\alpha_i^3| \le C$, Theorem 5.1 tells us that at most finitely many $\alpha_i$ lie in $\mathrm{Amp}(X) = \mathbb{R}_{>0}u + \mathbb{R}_{>0}v + \mathbb{R}_{>0}w$. We may exclude these $\alpha_i$ from our sequence.

Since $\frac{\alpha_i}{q^{(i)}}$ converge to $u$, we have $U(\alpha_i) > 0$ for all $i \gg 0$. We constructed $\alpha_i$ such that $\ell(\alpha_i) \ge 0$ for $\ell = aV + bW$ with $a, b > 0$, so for $i \gg 0$ we have

$$\alpha_i \in \mathbb{R}_{>0}u + \mathbb{R}_{\le 0}v + \mathbb{R}_{\le 0}w \cup \mathbb{R}_{>0}u + \mathbb{R}_{\le 0}v + \mathbb{R}_{\ge 0}w.$$

We may assume that infinitely many $\alpha_i$ lie in $\mathbb{R}_{>0}u + \mathbb{R}_{\le 0}v + \mathbb{R}_{\le 0}w$ and we can restrict to this subsequence.

Finally, the planes $\{V = 0\}$ and $\{W = 0\}$ are by assumption irrational, so they contain each at most one rational line, which must be distinct from the irrational line $\mathbb{R}u$. Hence, after possibly excluding finitely many elements of the sequence $(\alpha_i)$ we can assume that all $\alpha_i$ lie in

$$\mathbb{R}_{>0}u + \mathbb{R}_{>0}v + \mathbb{R}_{<0}w.$$

**Step 4**: For any class $\delta \in \mathbb{R}_{>0}u + \mathbb{R}_{>0}v + \mathbb{R}_{<0}w$, there exists an integral ample class $h$ such that $\delta \cdot h^2 > 0$, $\delta^2 \cdot h > 0$.

First, we observe that by continuity and rescaling it suffices to show the existence of a real ample class $h$ with the desired property. Writing $\delta = ru + sv - tw$ with $r, s, t > 0$, we can choose $h := \epsilon u + \epsilon v + w$ for some $\epsilon \ll 1$. Then

$$\begin{aligned}\delta^2 \cdot h &= 2rs - 2\epsilon st - 2\epsilon tr > 0, \\ \delta \cdot h^2 &= \epsilon r + \epsilon s - \epsilon^2 t > 0.\end{aligned}$$

**Step 5**: The classes $\alpha_i$ are effective for $i \gg 0$ and this gives a contradiction.

Choose any integral class $\delta \in \mathbb{R}_{>0}u + \mathbb{R}_{>0}v + \mathbb{R}_{<0}w$ with $\ell(\delta) \le 0$ and choose an integral ample class $h$ as given by Step 4. Then by

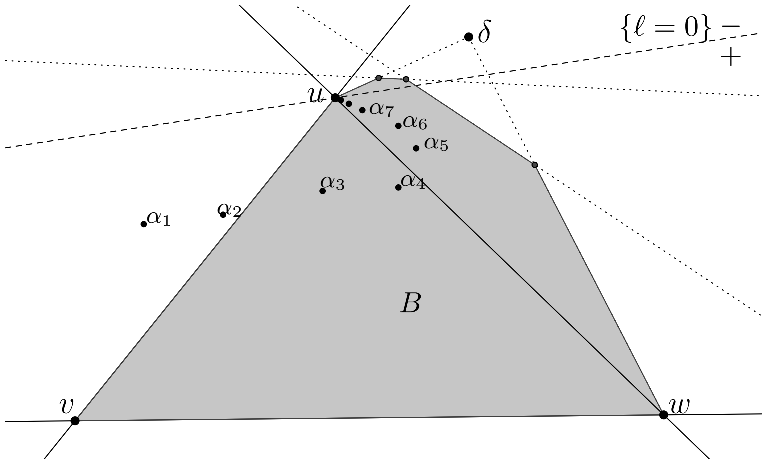


FIGURE 5.1. Illustration of the proof of Lemma 5.7

Lemma 4.11, there are finitely many integral curve classes $\gamma_1, \dots, \gamma_r \in N_1(X)_{\mathbb{Z}}$ such that any class $\delta'$ lying in

$$B := (\mathbb{R}_{>0}\delta + \mathrm{Nef}(X)) \cap \{x \in N^1(X)_{\mathbb{R}} \mid x \cdot \gamma_j \geq 0 \ \forall j\}$$

is effective if it satisfies $\delta' \cdot c_2(X) + 2\delta'^3 > 0$. See Figure 5 for an illustration.

Note that any class $\delta' \in \mathbb{R}_{>0}u + \mathbb{R}_{>0}v + \mathbb{R}_{<0}w$ with $\ell(\delta') \geq 0$ lies in $\mathbb{R}_{>0}\delta + \mathrm{Nef}(X)$ if the ray $\mathbb{R}_{\geq 0}\delta'$ is sufficiently close to $\mathbb{R}_{\geq 0}u$. In particular, $\alpha_i \in \mathbb{R}_{>0}\delta + \mathrm{Nef}(X)$ for $i \gg 0$.

Note that $\mathrm{Nef}(X)$ lies in the half-plane $\{\gamma_j \geq 0\}$. Since the ray $\mathbb{R}_{\geq 0}u$ is irrational, at most one of the rational planes $\{\gamma_j = 0\}$ can pass through $\mathbb{R}u$ and if it does, it must coincide with $\ell$ by the choice of $\ell$. This shows that for $i \gg 0$ we have $\alpha_i \cdot \gamma_j \geq 0$ for all $j$. We conclude that $\alpha_i \in B$ for $i \gg 0$.

Finally, we have $\frac{\alpha_i}{q^{(i)}} \cdot c_2(X) \to 1$ as $i \to \infty$. Hence, $\alpha_i \cdot c_2(X) \to \infty$ as $i \to \infty$, but $|\alpha_i^3|$ is bounded by Step 2. Hence,

$$\alpha_i \cdot c_2(X) + 2\alpha_i^3 > 0$$

for $i \gg 0$. This implies $\alpha_i \in \mathrm{Eff}(X)$ for $i \gg 0$.

However, $\mathrm{Big}(X)$ is the interior of $\mathrm{Eff}(X)$ and $\mathbb{R}_{>0}u + \mathbb{R}_{>0}v + \mathbb{R}_{>0}w = \mathrm{Amp}(X)$ lies in $\mathrm{Big}(X)$, so $\alpha_i \in \mathrm{Eff}(X)$ implies that $\mathbb{R}_{>0}u + \mathbb{R}_{>0}w$ is contained in $\mathrm{Big}(X)$. But $\mathbb{R}_{>0}u + \mathbb{R}_{>0}w$ also lies in $\mathrm{Nef}(X)$ and in $\mathcal{N}$. This is a contradiction due to Lemma 4.8. This concludes the proof. ☐

We also point out the following: A closer look at the proof of Theorem 4.22 shows that the constructed fundamental domain for the Weak Cone Conjecture for Type I has two exceptional faces only when the ray $\mathbb{R}_{\geq 0}u$ is nef and irrational, which is precisely the case we would exclude by the above argument. Hence, we also obtain the following possible improvement of Theorem 4.22:

**Corollary 5.8.** *If Question 5.2 has an affirmative answer, then on threefolds of Calabi–Yau type of Type I the Weak Cone Conjecture for* $\mathrm{Nef}^+$ *holds with a fundamental domain which has at most one exceptional face.*

## Acknowledgement

I would like to express my thanks to my advisor Dr. Vladimir Lazić for providing me with a lot of insights, inspiration and motivation on the subject, for serving as a guide to the literature and for giving helpful recommendations on the presentation of this thesis. I also want to thank Christopher Miller for inspiring discussions on Diophantine Approximation problems and I am very grateful to Alexander Bertoloni Meli for referring me to the proof of Proposition 5.4.

Partly, this thesis was carried out during a stay at the Polish Algebraic Geometry mini-Semester. I would like to thank the Institute of Mathematics of the Polish Academy of Science for the hospitality and support.

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