How to square a linear space?

A story about invariant arrangements of linear spaces and their ideals

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Question

Given a linear space in projective space $L \subset \mathbb{P}^n_{\mathbb{C}}$, what is the set obtained by squaring each point of L coordinate-by-coordinate?

The coordinate-wise square

For a linear space $L \subset \mathbb{P}^n_{\mathbb{C}}$, denote by $L^{\circ 2} \subset \mathbb{P}^n_{\mathbb{C}}$ its **coordinate-wise square**, i.e. the image of L under

$$\varphi_2 \colon \mathbb{P}^n_{\mathbb{C}} \to \mathbb{P}^n_{\mathbb{C}}, \quad [x_0 : x_1 : \ldots : x_n] \mapsto [x_0^2 : x_1^2 : \ldots : x_n^2].$$

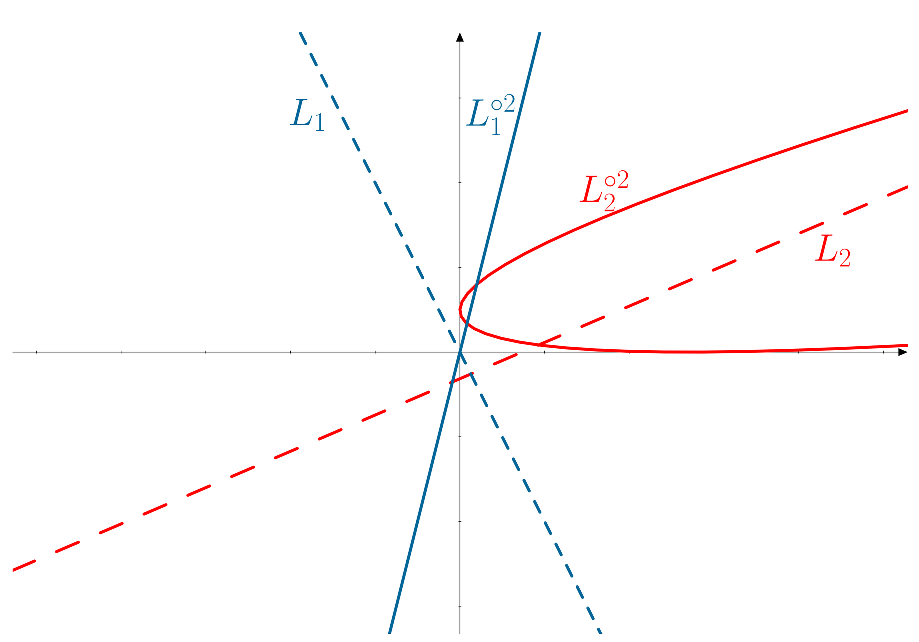


Figure 1: Coordinate-wise squares of two lines in $\mathbb{P}^2$

Motivation

Let $G \subset \operatorname{Aut}(\mathbb{P}^n)$ be the group generated by the coordinate hyperplane reflections. Consider the G-invariant arrangement of linear spaces $\mathcal{A} := \bigcup_{g \in G} gL \subset \mathbb{P}^n$. The polynomial equations vanishing on $\mathcal{A}$ form an ideal I in the invariant ring $\mathbb{C}[x_0, \ldots, x_n]^{\mathbb{Z}_2^{n+1}} = \mathbb{C}[y_0, \ldots, y_n]$, where $y_i = x_i^2$.

Describing $I \subset \mathbb{C}[\mathbf{y}]$ is the same as finding defining equations for the coordinate-wise square $L^{\circ 2} \subset \mathbb{P}^n$.

Squaring hyperplanes

The coordinate-wise square of a hyperplane $V(f) \subset \mathbb{P}^n$ is a hypersurface in $\mathbb{P}^n$ whose equation is computed as follows:

- Form the product over the orbit of $f \in \mathbb{P}(\mathbb{C}[\mathbf{x}]_1)$ under the action of $\mathbb{Z}_2^{n+1}$ on $\mathbb{P}(\mathbb{C}[\mathbf{x}]_1)$.
- The resulting polynomial lies in $\mathbb{C}[x_0^2, \ldots, x_n^2]$. Replace each occurrence of x_i^2 in it by x_i.

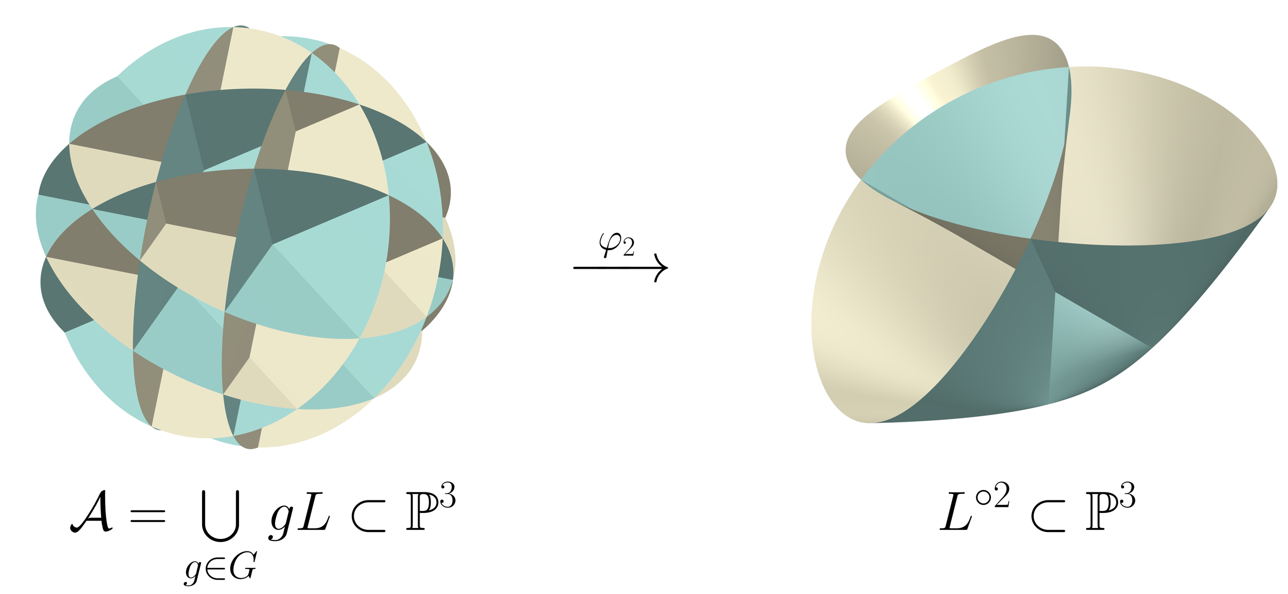


Figure 2: Factoring out the symmetry of the G-invariant arrangement $\mathcal{A} = \bigcup_{g \in G} gL \subset \mathbb{P}^n$ gives the coordinate-wise square $L^{\circ 2}$, illustrated here for $L = V(x_0 + x_1 + x_2 + x_3) \subset \mathbb{P}^3$.

The degree of $L^{\circ 2}$ as a matroid invariant

The combinatorial information how a linear space $L \subset \mathbb{P}^n$ lies relative to the coordinate hyperplanes is captured in its **linear matroid**:

$$\mathcal{M}_L := \{I \subset \{0, 1, \ldots, n\} \mid L \cap V(\{x_i \mid i \notin I\}) = \emptyset\}.$$

(collection of independent sets of the matroid). Some purely combinatorial definitions:

- An index $i \in \{0, 1, \ldots, n\}$ is a **coloop** of the matroid if $I \in \mathcal{M}_L \Leftrightarrow I \cup \{i\} \in \mathcal{M}_L$.
- A subset $E \subset \{0, 1, \ldots, n\}$ is a **component** of the matroid if there is no non-trivial partition $E = E_1 \sqcup E_2$ with $I \in \mathcal{M}_L \Leftrightarrow I \cap E_1, I \cap E_2 \in \mathcal{M}_L$, and E is maximal with this property.

Degree of the coordinate-wise square

Proposition. Let $L \subset \mathbb{P}^n$ be a linear space of dimension k. Then

$$\deg L^{\circ 2} = 2^{k+s-t+1},$$

where $s := \#\{\text{coloops of } \mathcal{M}_L\}$ and $t := \#\{\text{components of } \mathcal{M}_L\}$.

There exist planes $L_1, L_2 \subset \mathbb{P}^5$ with $\mathcal{M}_{L_1} = \mathcal{M}_{L_2}$ such that the ideal of defining equations of $L_1^{\circ 2}$ is minimally generated by 7 cubic forms, the ideal of $L_2^{\circ 2}$ by 6 quadratic forms.

The structure of the defining equations of $L^{\circ 2}$ depends on more than just the linear matroid of L.

Squaring via geometry of finite points

Let $L = \operatorname{im}(\mathbb{P}^k \xrightarrow{[\ell_0 : \ldots : \ell_n]} \mathbb{P}^n)$ with $\ell_i \in (\mathbb{C}^{k+1})^*$ and consider the finite set of points

$$Z := \{[\ell_i] \in (\mathbb{P}^k)^* \mid \ell_i \neq 0\} \subset (\mathbb{P}^k)^*.$$

Then $L^{\circ 2}$ only depends on the set of quadrics containing Z.

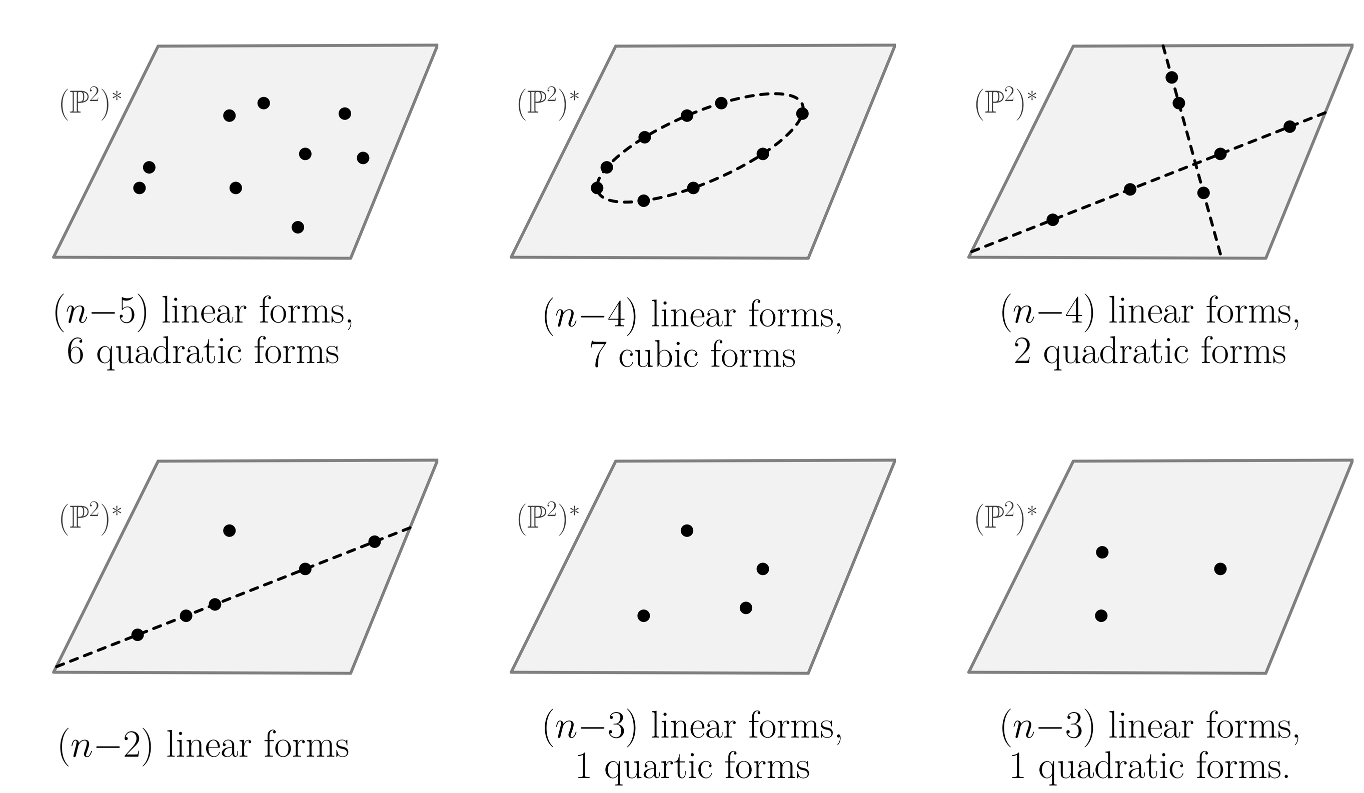
Theorem. If $Z \subset (\mathbb{P}^k)^*$ does not lie on any quadric or it lies on a unique quadric of rank $\neq 3$, then $L^{\circ 2} \subset \mathbb{P}^n$ is cut out by the vanishing of linear and quadratic forms.

Line arrangements

Proposition. The coordinate-wise square of a *line* $L \subset \mathbb{P}^n$ is an embedded plane conic if $|Z| > 2$, and a line otherwise.

Plane arrangements

Theorem. Depending on the geometry of $Z \subset (\mathbb{P}^2)^*$, the ideal of the coordinate-wise square of a *plane* $L \subset \mathbb{P}^n$ is minimally generated by:



P. Dey, P. Görlach, and N. Kaihnsa: *"Coordinate-wise Powers of Algebraic Varieties"*, arXiv:1807.03295