Modification of Coulomb's law in closed spaces

Pouria Pedram[a)]

*Plasma Physics Research Center, Science and Research Campus, Islamic Azad University,
Tehran 1477893855, Iran*

We obtain a modified version of Coulomb's law in two- and three-dimensional closed spaces. We demonstrate that in a closed space the total electric charge must be zero. We also discuss the relation between total charge neutrality of an isotropic and homogeneous universe to whether or not its spatial sector is closed. © *2010 American Association of Physics Teachers.*
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I. INTRODUCTION

One of the fundamental forces of nature is the electromagnetic force between charged particles. The interaction between charged particles in flat spaces is governed by Coulomb's law in two and three dimensions[1,2]

$$\mathbf{E} = \frac{\mathbf{F}}{q_0} = \frac{q}{2\pi\epsilon_0 r}\hat{\mathbf{r}} \quad \text{(two dimensions)}, \tag{1}$$

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} \quad \text{(three dimensions)}. \tag{2}$$

Here, $\mathbf{E}$ is the electric force per unit charge and $\hat{\mathbf{r}}$ is the unit vector directed from the first charge q to the second charge q_0 (see Fig. 1).

There are two basic assumptions that are valid in flat spaces for electrostatic forces. One is the superposition principle, which states that at any point in space, the total electric field of a group of charges equals the vector sum of the electric fields due to the individual charges. The other assumption in flat space is the absence of any restriction on the number of positive and negative charges; that is, we may have an arbitrary number of positive and negative charges with zero or nonzero total value. As we shall see, in curved spaces the correctness of the former is not clear and the latter assumption is not valid.[3]

The first attempt to solve electrostatic problems in curved space was done by Fermi in 1921.[4] In this paper, Fermi discussed the correction to the electric field of a single point charge held at rest within a gravitational field to first order in the gravitational acceleration. A few years later, Whittaker[5] solved this problem exactly both in the homogeneous gravitational field and in the Schwarzschild geometry cases. These efforts were further developed by Copson.[6]

In this paper, we obtain the modified form of Coulomb's law in two- and three-dimensional closed curved spaces. We show that in the vicinity of charges where the effect of curvature is negligible, we recover Coulomb's law. The geometrical interpretation of this result is that we can always find a tangent flat space for any point on a curved space. We also demonstrate the failure of the second assumption in these spaces. Finally, we discuss the charge neutrality of a closed isotropic and homogeneous universe.

II. MATHEMATICAL PRELIMINARIES

Because Coulomb's law is valid only in flat spaces, we outline some important properties of these spaces. In a flat space there always exists a (Cartesian) coordinate system where the distance between two infinitesimally close points can be written as

$$ds^2 = \sum_{i=1}^{d} (dx^i)^2, \tag{3}$$

where x^i are the coordinates, d is the dimension of the space, and ds^2 is the line element. We can define the metric of the space as a second-order covariant tensor g_{ij}, namely,

$$ds^2 = \sum_{i,j} g_{ij}dx^i dx^j. \tag{4}$$

Thus, in flat spaces, it is always possible to find a coordinate system in which the metric is diagonal with constant elements

$$g_{ij} = \pm\, \delta_{ij}, \tag{5}$$

where δ_{ij} is the Kronecker delta defined as

$$\delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j. \end{cases} \tag{6}$$

In particular, Euclidean space corresponds to $g_{11}=g_{22}=g_{33}=g_{44}=1$, and Lorentzian space corresponds to $g_{11}=g_{22}=g_{33}=1$, $g_{44}=-1$. A well-known example is the three-dimensional spherical coordinate system specified by r, θ, ϕ, and the line element

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{7}$$

or its equivalent metric tensor in the matrix form

$$[g_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{bmatrix}. \tag{8}$$

Using a suitable choice of the coordinate transformation

$$z = r\cos\theta,$$

$$x = r\sin\theta\cos\phi,$$

$$y = r\sin\theta\sin\phi, \tag{9}$$

we obtain the diagonal form of the line element

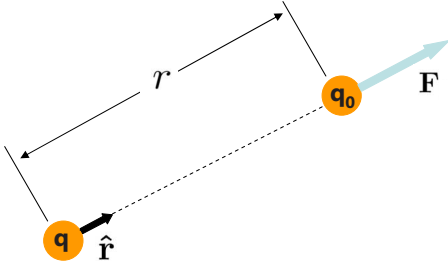$$ds^2 = dx^2 + dy^2 + dz^2, \tag{10}$$

and the metric tensor

Fig. 1. The electric force between two point charges separated by a distance r.

$$[g_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{11}$$

We now consider the surface of a 2-sphere as a two-dimensional closed curved space (see Fig. 2). The distance between two infinitesimally close points on this hypersurface is

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2 \theta d\phi^2, \tag{12}$$

where R is the radius of the sphere, which takes a constant value. For this case, it is not possible to find a proper coordinate transformation to diagonalize the metric tensor with constant elements. In other words, we cannot find a new set of variables $u(\theta, \phi)$ and $v(\theta, \phi)$ so that the line element takes the simple form $ds^2 = du^2 + dv^2$.

To be more precise, let us review the tensorial properties of the 2-sphere. We will show that because all components of the curvature tensor are nonzero, the metric tensor of a 2-sphere is not Euclidean.[13] The covariant metric tensor of the surface of a sphere can be written in the matrix form (12) as

$$[g_{ij}] = R^2 \begin{bmatrix} 1 & 0 \\ 0 & \sin^2 \theta \end{bmatrix}, \tag{13}$$

where i, j are elements of $\{\theta, \phi\}$. We have the following form for the contravariant metric tensor

$$[g^{ij}] = R^{-2} \begin{bmatrix} 1 & 0 \\ 0 & (\sin \theta)^{-2} \end{bmatrix}. \tag{14}$$

The Christoffel symbols of the first kind are defined as[13]

$$\Gamma_{i,jk} = \tfrac{1}{2}(\partial_k g_{ij} + \partial_j g_{ki} - \partial_i g_{jk}). \tag{15}$$

By substituting Eq. (13) into Eq. (15), we have $\Gamma_{\theta,\theta\theta}=0$, $\Gamma_{\theta,\theta\phi}=\Gamma_{\theta,\phi\theta}=0$, $\Gamma_{\theta,\phi\phi}=-R^2 \sin \theta \cos \theta$, $\Gamma_{\phi,\theta\theta}=0$, $\Gamma_{\phi,\theta\phi}=\Gamma_{\phi,\phi\theta}=+R^2 \sin \theta \cos \theta$, and $\Gamma_{\phi,\phi\phi}=0$. The Christoffel symbols of the second kind are also defined as

$$\Gamma^i_{jk} = g^{il}\Gamma_{l,jk}. \tag{16}$$

Thus, we have $\Gamma^\theta_{\theta\theta}=0$, $\Gamma^\theta_{\theta\phi}=\Gamma^\theta_{\phi\theta}=0$, $\Gamma^\theta_{\phi\phi}=-\sin \theta \cos \theta$, $\Gamma^\phi_{\theta\theta}=0$, $\Gamma^\phi_{\theta\phi}=\Gamma^\phi_{\phi\theta}=+(\sin \theta)^{-1}\cos \theta$, and $\Gamma^\phi_{\phi\phi}=0$.

We can use the Christoffel symbols to define the Riemann curvature tensor

$$\mathcal{R}^i_{jkl} = \partial_k\Gamma^i_{lj} - \partial_l\Gamma^i_{kj} + \Gamma^m_{lj}\Gamma^i_{km} - \Gamma^m_{jk}\Gamma^i_{lm}. \tag{17}$$

The Riemann curvature tensor in d-dimensional space has d^4 components. These components are not all independent, and the number of independent components is given by

$$n = \frac{d^2(d^2 - 1)}{12}, \tag{18}$$

which is equal to 1 for a 2-sphere ($d=2$). We can also define the Ricci tensor

$$\mathcal{R}_{ij} = \mathcal{R}^k_{ijk} = g^{kl}\mathcal{R}_{lijk} \tag{19}$$

and the Ricci scalar

$$\mathcal{R} = g^{ij}\mathcal{R}_{ij} = g^{ij}g^{lk}\mathcal{R}_{kijl}, \tag{20}$$

where the former is a symmetric tensor and the latter is proportional to the Gauss curvature. The only independent component of the Riemann curvature tensor for our case is $\mathcal{R}_{\theta\phi\theta\phi}$, which by Eq. (17) is

$$\mathcal{R}_{\theta\phi\theta\phi} = -\tfrac{1}{2}\partial_\theta\partial_\theta g_{\phi\phi} + g_{\phi\phi}(\Gamma^\phi_{\theta\phi})^2 = (R \sin \theta)^2. \tag{21}$$

Thus, we can obtain the Gauss curvature of the 2-sphere given by Eq. (20),

$$\frac{\mathcal{R}}{2} = g^{\theta\theta}g^{\phi\phi}\mathcal{R}_{\theta\phi\theta\phi} - g^{\theta\phi}g^{\theta\phi}\mathcal{R}_{\theta\phi\theta\phi}$$

$$= (g^{\theta\theta}g^{\phi\phi} - g^{\theta\phi}g^{\theta\phi})\mathcal{R}_{\theta\phi\theta\phi} \tag{22a}$$

$$= |g^{ij}|\mathcal{R}_{\theta\phi\theta\phi} = \frac{\mathcal{R}_{\theta\phi\theta\phi}}{g} = \frac{1}{R^2}, \tag{22b}$$

which is equal to the square inverse of its radius.

III. ELECTRIC FIELD ON A TWO-DIMENSIONAL SPHERICAL SPACE

We consider a 2-sphere as a simple two-dimensional closed curved space (see Fig. 2). The points on this space satisfy

$$x^2 + y^2 + z^2 = R^2. \tag{23}$$

Now, put a positive point charge q at its north pole. We assume that the electric fields exist only on the sphere's surface. In this situation, the field's lines come out from the north pole and after passing the equator meet each other at
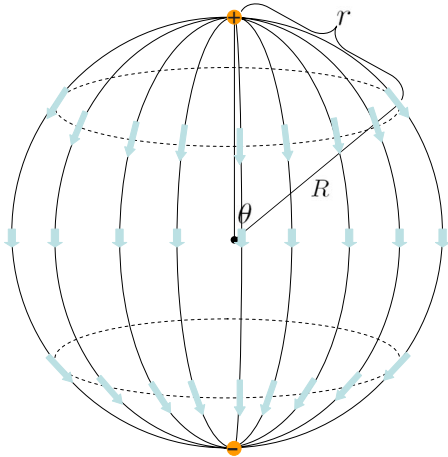

Fig. 2. The electric field on a 2-sphere in the presence of positive and negative point charges located at the north and south poles, respectively.

the south pole. The intersection point of the ingoing field lines corresponds to the presence of a negative charge. Thus, we will observe a negative point charge $-q$ at the south pole of the sphere. The presence of the negative charge shows that there is a one to one correspondence between positive and negative charges in this space. Therefore, the total charge on the sphere will be zero

$$\sum_i q_i = 0. \tag{24}$$

To obtain the electric field, we use Gauss's law in two dimensions

$$\oint \mathbf{E} \cdot d\mathbf{S} = \frac{q}{\epsilon_0}, \tag{25}$$

where the integration is over a one-dimensional closed curve. For instance, consider the upper dashed line in Fig. 2 as the integration contour. Because this path encloses both q and $-q$, it is not possible to relate the electric field to one charge only. So the resulting electric field can be decomposed into two components, which are related to each charge. For this case, the integration contour is a circle with circumference $2\pi R \sin \theta$ (see Fig. 2). The electric field on the sphere is

$$\mathbf{E} = \frac{q}{2\pi\epsilon_0 R \sin \theta} \hat{\boldsymbol{\theta}}, \tag{26}$$

where θ is the usual polar coordinate and R is the radius of the sphere. If we define r as the distance from the north pole on the sphere, we can rewrite the electric field in terms of r,

$$\mathbf{E} = \frac{q}{2\pi\epsilon_0 R \sin(r/R)} \hat{\mathbf{r}}. \tag{27}$$

For a large value of R or in the vicinity of the positive charge ($r \ll R$), Eq. (27) reduces to the flat two-dimensional form of Eq. (1),

$$\mathbf{E} \simeq \frac{q}{2\pi\epsilon_0 r} \hat{\mathbf{r}}. \tag{28}$$

Moreover, because the negative charge $-q$ is located at infinity in this limit, its effect is negligible in the vicinity of the positive charge.

Note that because the integration contour contains both positive and negative charges, it may seem that the superposition law fails in curved spaces. But, because Maxwell's equations are linear in any space-time, this conclusion is not true. The reason why the superposition principle seems to fail is related to the topology of the space: A general solution to Maxwell's equation will not be single-valued in a closed space, and additional conditions must be imposed to ensure that the electric field is globally well-defined. These additional conditions require the existence of an additional charge and seem to imply a failure of the superposition principle. However, the superposition principle is not violated because the correct solution is a sum $\mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3$, where $\mathbf{E}_1$ is the field of the positive charge, $\mathbf{E}_2$ is the field of the negative charge, and $\mathbf{E}_3$ is a solution to the homogeneous equation (Laplace's equation) that must be added to account for the topological conditions.

IV. ELECTRIC FIELD ON A THREE-DIMENSIONAL SPHERICAL SPACE

To have a more realistic model, we consider a 3-sphere, which is a set of points equidistant from a fixed central point in four-dimensional Euclidean space.[7,8] The points on this three-dimensional hypersurface satisfy the relation

$$x^2 + y^2 + z^2 + \tau^2 = R^2 \tag{29}$$

and can be expressed by $z = R \sin \chi \cos \theta$, $x = R \sin \chi \sin \theta \cos \phi$, $y = R \sin \chi \sin \theta \sin \phi$, R is the radius of the 3-sphere, χ is the extra polar coordinate, and $\tau = R \cos \chi$ is the fourth coordinate. Now, consider a positive charge located at the north pole of the 3-sphere. We require that the electric field is confined into the three-dimensional hypersurface. Thus, the electric field cannot propagate outside the 3-sphere. By using the constraint equation (29), we can rewrite the line element of this space as

$$ds^2 = dx^2 + dy^2 + dz^2 + \frac{(xdx + ydy + zdz)^2}{R^2 - (x^2 + y^2 + z^2)}. \tag{30}$$

If we use the corresponding spherical coordinates χ, θ, and ϕ instead of the coordinates x, y, and z, the line element takes the form

$$ds^2 = R^2[d\chi^2 + \sin^2 \chi(d\theta^2 + \sin^2 \theta d\phi^2)]. \tag{31}$$

To obtain the electric field, we need to use Gauss's law (25) over a closed two-dimensional surface with constant r (or χ) with the line element

$$ds^2 = R^2 \sin^2 \chi(d\theta^2 + \sin^2 \theta d\phi^2), \tag{32}$$

which is similar to the case of the 2-sphere ($R \to R \sin \chi$). The area of this hypersurface is equal to $\mathbf{S} = 4\pi R^2 \sin^2 \chi$. The use of Gauss's law results in the following form for the electric field

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0 R^2 \sin^2 \chi} \hat{\boldsymbol{\chi}}. \tag{33}$$

Equation (33) reduces to Coulomb's law (2) in the vicinity of the north pole ($r \ll R$). To show this result, we rewrite the line element (30) in terms of r', θ, and ϕ

$$ds^2 = \frac{dr'^2}{1 - r'^2/R^2} + r'^2(d\theta^2 + \sin^2 \theta d\phi^2) \tag{34}$$

$$= R^2\left[\frac{dr''^2}{1 - r''^2} + r''^2(d\theta^2 + \sin^2 \theta d\phi^2)\right], \tag{35}$$

where $r'^2 = x^2 + y^2 + z^2$ and $r'' = r'/R$. Moreover, we can obtain the relation between r' and the radius of a sphere on this hypersurface

$$r = \int_0^{r'} \frac{dr'}{\sqrt{1 - r'^2/R^2}} = R \sin^{-1} r'/R = R\chi, \tag{36}$$

which results in the electric field

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0 R^2 \sin^2\left(\dfrac{r}{R}\right)} \hat{\mathbf{r}} \tag{37}$$
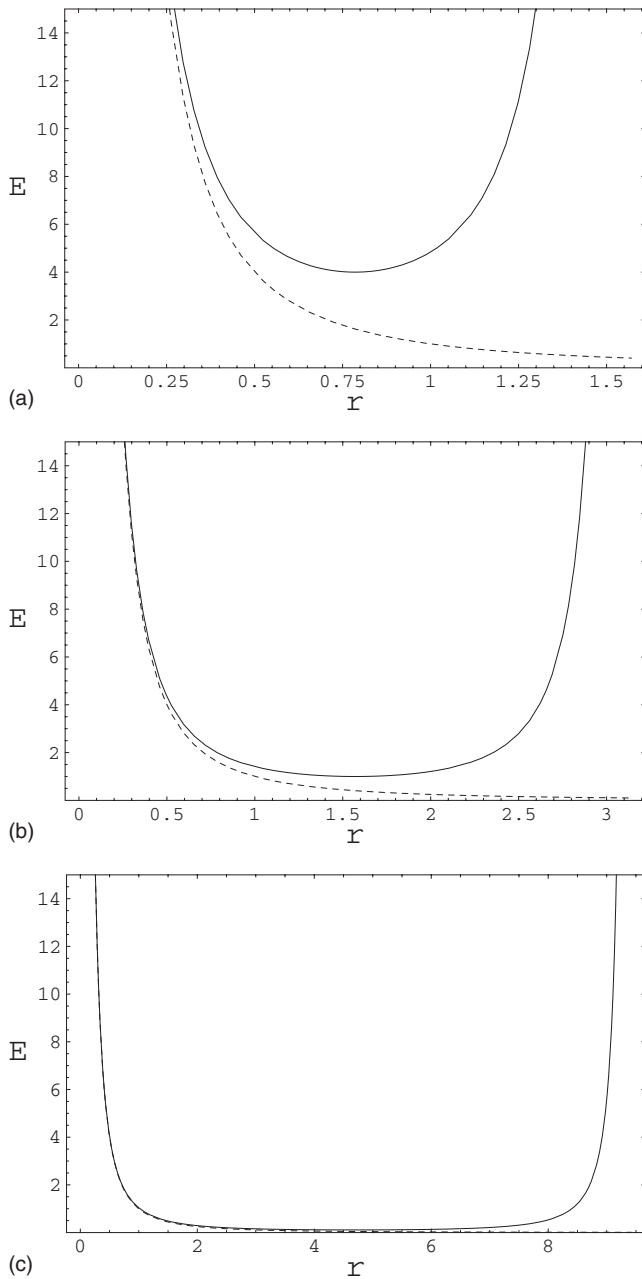
or

Fig. 3. The electric field on a 3-sphere [Eq. (33)] (solid line) and Coulomb's law [Eq. (2)] (dashed line) for (a) $R=1/2$, (b) $R=1$, and (c) $R=3$ with $q/(4\pi\epsilon_0)=1$.

$$\mathbf{E} \simeq \frac{q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} \qquad (38)$$

in the vicinity of the positive charge ($r \ll R$). In other words, we can recover the usual form of the electrostatic forces in the regions where the effect of the curvature is negligible. This result also shows that, similar to the two-dimensional case, the total charge of the 3-sphere should be zero. In Fig. 3 we have plotted Coulomb's law and its modified version on the 3-sphere for various values of R. As it can be seen, the correspondence between the two increases as R increases.

In reality, we live in a four-dimensional curved space-time with a matter distribution on it. Following the theory of gen-

eral relativity, the curvature of the universe comes from its mass and energy. Any massive object distorts its surrounding space-time. If we assume that the universe is homogeneous and isotropic, space-time can be expressed by the Friedmann–Robertson–Walker metric as[9–11]

$$ds^2 = -dt^2 + R^2(t)\left[\frac{dr''^2}{1 - kr''^2} + r''^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \qquad (39)$$

where $R(t)$ is the scale factor and $k=+1,0,-1$ corresponds to a closed, flat, or open universe, respectively. For a closed universe ($k=1$), the spatial section of the metric has the form Eq. (35), and thus can be considered as a 3-sphere imbedded in four-dimensional space-time, where the scale factor $R(t)$ plays the role of its radius.[7,9] Observations of our Universe suggest that the spatial section of the universe is flat at the present time ($k=0$). What would happen if we had a universe with positive curvature ($k=1$)? In early times, in the interval between a millisecond to a second after the Big Bang,[12] the radius of the universe was very small and the effect of the curvature was considerable. At the end of the radiation era, when the temperature of the universe decreased, charged particles could be created. For a closed universe, the curvature would allow the charged particles to be created in pairs. Consequently, the universe at large scales would be neutral. Thus, a closed universe results in the charge neutrality of the universe, but the inverse statement is not necessarily true. Hence, although observations show that our Universe is neutral at large scales, the flatness of our Universe indicates that the neutrality is only a coincidence from this point of view.

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[a]Electronic mail: pouria.pedram@gmail.com
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