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Applications of a constrained mechanics methodology in economics

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Abstract
This paper presents instructive interdisciplinary applications of constrained mechanics calculus in economics on a level appropriate for undergraduate physics education. The aim of the paper is (i) to meet the demand for illustrative examples suitable for presenting the background of the highly expanding research field of econophysics even at the undergraduate level and (ii) to enable the students to gain a deeper understanding of the principles and methods routinely used in mechanics by looking at the well-known methodology from the different perspective of economics. Two constrained dynamic economic problems are presented using the economic terminology in an intuitive way. First, the Phillips model of the business cycle is presented as a system of forced oscillations and the general problem of two interacting economies is solved by the nonholonomic dynamics approach. Second, the Cass–Koopmans–Ramsey model of economical growth is solved as a variational problem with a velocity-dependent constraint using the vakonomic approach. The specifics of the solution interpretation in economics compared to mechanics is discussed in detail, a discussion of the nonholonomic and vakonomic approaches to constrained problems in mechanics and economics is provided and an economic interpretation of the Lagrange multipliers (possibly surprising for the students of physics) is carefully explained. This paper can be used by the undergraduate students of physics interested in interdisciplinary physics applications to gain an understanding of the current scientific approach to economics based on a physical background, or by university teachers as an attractive supplement to classical mechanics lessons.

1. Introduction

Economic science has been influenced by physical concepts from its very beginning[1]. Although it is for contributions to physics and mathematics that Newton is celebrated, the Newtonian principles, formulated at the end of the 17th century, have undoubtedly powerfully influenced most branches of science–economics. Newton's mechanics introduced the doctrine of scientific determinism, the principle that all events are the inescapable results of preceding causes, under which (until the work of Planck and Einstein in the 20th century) scientists tend to think of nature as a mechanical device whose behaviour could be revealed by observation, experimentation, measurement and calculation. The idea of nature governed by natural laws dominated the new world order and many scholars have presumed human behaviour and economics to be governed by such laws as well (see [1]).

As economic thinking was developing through the centuries, the possibility that economic science could be inspired by physics was continuously debated. Nevertheless, the newly built physical theories and methodologies were repeatedly applied to economic problems by scholars trying to capture observed economic behaviour. As a well-developed branch of physics perfectly equipped with mathematical apparatus, mechanics has permanently served as an inspiration for theoretical constructions in economics (for a detailed discussion, see [2]).

Looking for recently arisen intersections of physics and economics, one arrives at *econophysics* which describes the phenomena of the development and dynamics of economic systems by using a strictly physically motivated methodology. The official birth of the term 'econophysics' dates back to a paper by Stanley in 1996 [3]. Currently, the applications of statistical physics and nonlinear dynamics are mainly considered to be the core of econophysics (see [4]), but in a broader context, econophysics can be considered an interdisciplinary research field applying theories and methods originally developed by physicists in order to solve problems in economics (for an approach employing solely the methods of classical mechanics, see [5]).

Econophysics is a modern, quickly expanding interdisciplinary branch of science, which has already been transferred from the area of purely scholarly interest into the real world including the establishment of graduate and postgraduate university study programmes. Although econophysics has become a part of university physics education, the teaching aids are still being developed. Since the core of econophysics lessons requires passing advanced physics courses, very little is available for undergraduate students to satisfy their curiosity about this 'econophysics fashion'. In this paper, we present instructive examples of how the methodology of constrained dynamic systems, commonly used in classical mechanics, can be used for solving economic problems and, in this way, we supply instruments to present and demonstrate this interdisciplinary topic at the undergraduate level of physics education. For recent educational contributions dealing with econophysics (but lacking concrete examples or inappropriate for undergraduates) see [6–9]. Being intended for the students of physics without preceding knowledge of economic theories, the examples given in the paper are based on intuitive economic terminology and supplemented by easy-to-read explanations and interpretations of the economic background.

Apart from the illustrative potential of the examples, the examination of the standard methodology procedures used in mechanics (such as the formulation of the equations of motion or handling the constrained systems) from a completely different point of view enables the reader to reach a better understanding of the physical background of these principles—a point which is often otherwise replaced by the calculation routine.

[1] Let us mention as an example Daniel Bernoulli, who was the originator of utility-based preferences, or one of the founders of neoclassical economic theory, Irving Fisher, who was originally trained in physics under J W Gibbs.

This paper can be used by undergraduate students of physics interested in solving interdisciplinary applications to get an idea about employing physical apparatus in economic problems. Moreover, the examples provided in this paper can be utilized by university teachers as an attractive supplement to traditional mechanics courses at the undergraduate level, or as motivation for the students to enroll in advanced interdisciplinary courses.

In section 2, the classical Phillips model of the business cycle is solved via the physical model of forced oscillations under friction: in section 2.1, the economic model is briefly developed in an intuitive way; in section 2.2, the problem of two interacting economies is stated and an appropriate, physically motivated model of a nonholonomic system is used for its solution, and in section 2.3, we demonstrate that the results obtained are in good qualitative correspondence with the observed behaviour of two economies. In section 3, the Cass–Koopmans–Ramsey model of economic growth is investigated by the calculus of variations. It is shown that the Lagrangian has the meaning of the overall (current value) utility in economy and the existing connection among the economic quantities is modelled by a velocity-dependent constraint. The vakonomic approach is used for solving the system, and the examples of a typical economic solution procedure and the interpretation of the solution are presented and discussed with respect to the standard procedures used in mechanics. A discussion of the difference between the nonholonomic and vakonomic approach together with the possibilities of their applications in economics and mechanics is provided in appendix A. A special paragraph is devoted to an interesting economic interpretation of the Lagrange multipliers and an additional example and comments on this topic are given in appendix B.

2. Physically motivated business cycle description

The business cycle (or economic cycle) refers to economy-wide fluctuations in economic activity over several months or years. These fluctuations occur around a long-term growth trend and typically involve periods of rapid economic growth (an expansion), and periods of relative stagnation or decline (a contraction or recession, see e.g. [10]). Business cycles exist in the economy of any country or region (e.g. we can detect business cycles in the Spanish economy or the business cycles of the European Union economy). Understanding how these cycles come into existence and what their determinants are is crucial for making good economical policies in a particular country/region since the level of economic activity is linked with the standard of living in a country. Therefore, mathematical models of business cycles form an important and intensively studied topic in the science of economics[2].

The business cycle is described via the time development of the gross domestic product (GDP) which refers to the market value of all final goods and services produced within a country in a given period. The term ‘gross’ refers to a particular methodology of enumerating the GDP and in economic models we speak simply about the product (denoting it Y). Hence, the product $Y = Y(t)$ provides us with the information on the amount of goods and services produced (and it is assumed that this amount is also sold) in the economy (of a country/region) in each time period. When trying to express $Y = Y(t)$ quantitatively, the economists face the problem of how to find the representative function. Unlike in mechanics where the time

[2] Note that currently there are a number of different mathematical models of business cycles and new models are still being developed. Generally, in economic science, for each economic phenomenon (such as a business cycle), there exist several competing approaches that use different assumptions and mathematical apparatus for building the model. Whatever the details of these approaches, all of them aim to follow the current and forecast the future behaviour of the economic quantities as precisely as possible. It is interesting from the physical point of view that there is no unique ‘correct’ model in almost any economic discipline. There are many models for every phenomenon, each having its advantages and disadvantages and each corresponding with reality to its own extent.

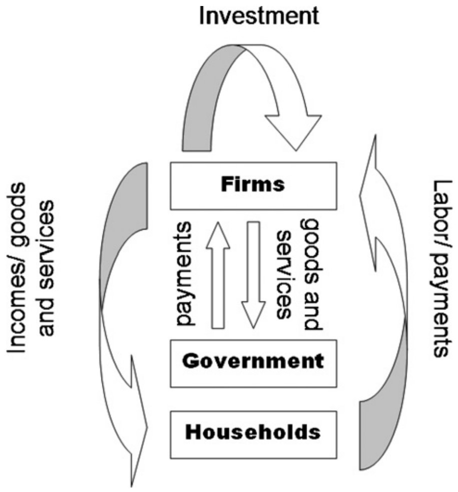

Figure 1. The illustration of money/goods and services flows in an economy.

development of the system is uniquely determined by forces and torques acting on the system, no such unequivocal approach exists in economics. Economists state that the characteristics playing a principal role in forming the business cycle are

- investments $I = I(t)$—the purchased amount of goods which are not consumed but are to be used for future production (e.g. purchases of new machines and buildings intended to be used in a production process),
- consumption of households $C = C(t)$—the amount of money spent by households for goods and services,
- total demand for goods and services $Z = Z(t)$—the amount of goods and services intended to be purchased in the economy. Let us explain more carefully the concept of total demand using the illustration of money/goods and services flows in an economy (see figure 1). There are three 'players' in an 'economy game': households that consume goods and services spending the money earned by working in firms, firms who are renting labour from workers to produce goods and services, and a government that purchases both goods and services. Hence, the total purchases of goods and services (i.e. demand Z) comprise the household consumption, firm investments and government purchases of goods and services.
- autonomous expenditures $A = A(t)$—expenditures independent of the total income in the economy (the total income is the amount of money earned by the individuals in the economy), i.e. whatever the economic performance of the economy is, this money will be spent (e.g. government expenditures to keep the roads functional). Note that product Y (which is expressed in monetary units) is identified by the total income in the economy[3].

[3] The product Y is the total amount of goods and services produced in an economy. But what was produced is to be sold and all the money is disbursed to the players in an economy game (e.g. the employees are paid wages, the material or semi-finished products must be bought from firms that must pay their employees, etc- and the surplus of corporations and entrepreneurs come as income to individuals in the economy). Hence, what was produced in an economy transforms into the total income in the economy.

2.1. The Phillips model of the business cycle

In the Phillips model of the business cycle (developed in 1954), which from the physicist's point of view is interesting due to the methodology used, the product function $Y(t)$ is deduced from the observed and presumed relationships between the above-mentioned characteristics of the economy (see e.g. [11]).

(i) The total demand in an economy $Z(t)$ is composed of the consumption of households C, investment purchases I and the autonomous expenditures A (we will consider it to be a given constant of government spending), i.e. $Z = C + I + A$.

The individuals divide their income between consumption (spending their money for goods and services) and saving (leaving the money in banks). So for the total income in an economy (which can also be denoted as Y since it has the same value as the product) we can write $Y = C + S$, where savings $S = sY$ and consumption $C = (1-s)Y$ are given by the income (product) Y and constant value s, $0 \leqslant s \leqslant 1$, which is a given characteristic of an economy. Then, the total demand Z is

$$Z = (1-s)Y + I + A. \tag{1}$$

(ii) The demand in the economy can be satisfied only via the product of the economy (since we do not take into account other economies). Hence, ideally the demand Z should be equal to supply (product) Y at any time. But it is assumed (and more realistic) that the product Y reacts to the demand with a delay. For example, if there is a demand for some fashionable consumer article and there is a lack of it in shops, the producers can react and increase its production, but this will take some time. Thus, the supply of the article reflects the past demand. In this way the total product Y 'persuades' the total demand Z in time and, particularly, in the Phillips model it is assumed that the dynamics of a product is given by

$$\frac{\mathrm{d}Y(t)}{\mathrm{d}t} = \lambda(Z - Y), \tag{2}$$

where λ is a positive constant.

(iii) It is assumed that the potential investment $\tilde{I}(t)$ depends on the product change in time $\tilde{I}(t) = v\frac{\mathrm{d}Y(t)}{\mathrm{d}t}$ (v being constant), i.e. the investments are needed only if we want to increase the production. For example, if we want to produce more products in a factory, we will need more inventories, machines, etc, which generates the spending falling within 'investments' in the economic language. But the true investment $I(t)$ is delayed from the potential one, i.e. in reality the investments do not react on the change in the product immediately (in our example, we can assume that we begin producing more products using the machines we already have more extensively and the new machines that will be bought later). The change of true investments in time is then supposed to be proportional to the difference $(\tilde{I}(t) - I(t))$, i.e.

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \kappa\left(v\frac{\mathrm{d}Y(t)}{\mathrm{d}t} - I\right), \tag{3}$$

where κ denotes a positive constant.

Now the above-mentioned equations describe the model of how the business cycle came into existence: the product is delayed from demand, and investment purchases are delayed from their immediate need. These two factors generate the oscillations in time development of the product Y[4] in the Phillips model. Combining equations (1)–(3) and eliminating Z and I, we obtain a second-order differential equation for the unknown variable Y which represents the business cycle:

$$\ddot{Y} + a\dot{Y} + bY = P, \tag{4}$$

where

$$a = \lambda s + \kappa(1 - \lambda v)$$

$$b = \kappa\lambda s$$

$$P = \kappa\lambda A$$

are constant values. Equation (4) reminds us of forced oscillations under friction where $b = \omega_0^2$ is the square of the free oscillation frequency and P is the external force acting on the system—generally it is a given function of time (this is also the case in the Phillips model once the autonomous expenditure A is not constant). In physics, the term $a\dot{Y}$, $a > 0$, represents the damping arising when the surrounding medium exerts a resistance. In an economic system, this term may cause both damped and explosive oscillations depending on the sign of a.

Solving the dynamical equation (4) for our economic problem, we can obtain harmonic oscillations, critical damping, damped or even explosive oscillation according to the given parameters of the economic system. Given all the other parameters, by changing the constant λ (which can be found in each of the constants a, b, P) we can obtain all mentioned types of time development of the system. The solution of (4) for several values of λ can be found in figure 2.[5] Note that we do not mention concrete units in the graphs, since the results are derived for artificial parameters and for the purpose of demonstrating the behaviour of the product Y. Visualizing 'nonspecified' graphs is a standard means of the qualitative presentation of economic behaviour. But if needed, one can scale the product Y in thousands of USD and the time axis in years.

The manner in which the product Y develops in time (harmonic, explosive, damped oscillations or the critical damping) depends on the particular choice of parameters κ, λ, s, v, A. Given the rest of the parameters, the intervals of λ-values corresponding to each possible solution in figure 2 can be derived from a general solution of the dynamical equation (4). Analysing these intervals, the economists conclude that oscillations will occur if the time lag of product reaction to the demand does not differ significantly from the time lag of the reaction of investment to product change. The damped oscillations will occur if the time lag of production to the demand is significantly higher than the time lag of the reaction of investments to product change, and the explosive oscillations will occur if the time lag is significantly lower. These explanations following from the mathematical model answer the question of how the business cycle came into existence only to some extent. As we have mentioned, there are many other approaches to business cycle modelling and there may always be a debate on the explanations and reasoning behind each particular model.

[4] The sources of the periodic oscillations observed in the time development of product Y are still a matter of investigation. A modern theory of *real business cycles* suggests that the fluctuations of the product can be to a large extent accounted for by real shocks (examples of such shocks include innovations, bad weather, quick oil price increase, stricter environmental regulations, etc) which appear more or less periodically. The general gist is that something occurs that directly changes the decisions of workers and firms about what they buy and produce and thus eventually affect the product Y. Note then that the Phillips model represents only one of the many possible reasonings for the product oscillations.

[5] The values of the parameters were chosen according to [11]: $\kappa = 1$, $A = 1$, $s = 0.25$, $v = 0.6$, and the initial conditions were set as $Y(0) = 10$, $\dot{Y}(0) = 4$.

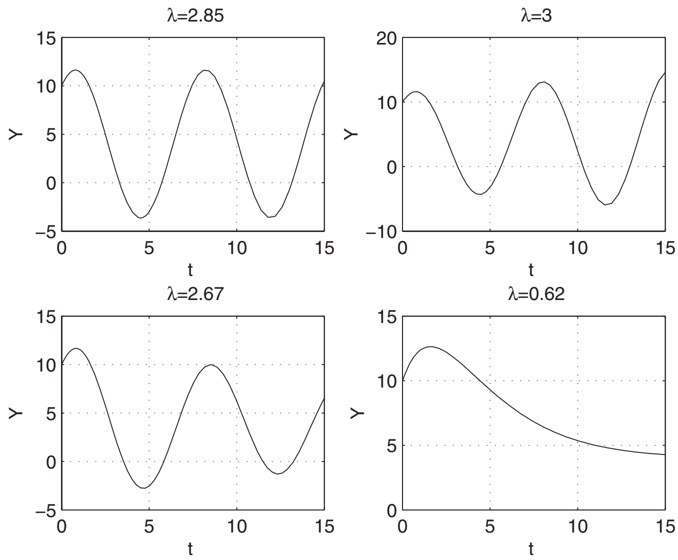

Figure 2. The solution of an unconstrained system; top-left: harmonic oscillations; top-right: explosive oscillations; bottom-left: damped oscillations; bottom-right: critical damping.

It is worth noting that the final 'equation of motion' (4) for the economic system (we will call it *a dynamical equation* when talking about economic systems) was obtained based on assumptions (1)–(3) about the dynamic relationships among the economic quantities. But there is no mention about the Lagrangian of the economic system[6] so as to generate the dynamic equation (4), nor are the terms in (4) interpreted as economic 'forces'. Thus, although the resulting dynamical equation is well known in physics, the method of obtaining it lacks the systematic spirit of physical methodology (recall the Lagrangian approach or the Newton equations in mechanics).

Hence, through certain assumptions the Phillips model arrived at the description of the oscillating behaviour of the product by the differential equation of second order. The ideal result of this model would be that once the constants of the economy $(\lambda, \nu, s, \kappa, A)$ are known, one could predict the future oscillations of Y. Practically, the political authorities would prefer only steady economical growth (i.e. growth of the product), which generates low unemployment, increasing wages and overall increasing the living standards of the people. Oscillations around this simple growth trend are disturbing and at least knowing how they come into existence or even better, what they will look like in the next period, would bring important support to the policy makers. But as we have mentioned before, such predictions and even the understanding of the oscillation phenomena are not complete and are the subject of ongoing research.

Let us conclude the description of the Phillips model of the business cycle by the note that using the well-known and developed physical model (harmonic oscillations) to describe observed economic behaviour (product time development) was a natural initial approach in economics. Although the current philosophy of quantitative economics diverges fundamentally from simply adopting the existing physical models for the description of economic systems, the physical methodology itself is still the focus of economists.

[6] Note that in the next paragraph we will see that the variational concept has also been adopted by quantitative economists in other models.

Accordingly, the harmonic-oscillation-based methodology is still a common approach used for the description of economic systems (for a recent application, see e.g. [12]). No matter how historical and simple the approach is, the Phillips model is able to produce at least qualitatively correct solutions (in the sense of correspondence with the observed behaviour) as we will see in section 2.2.

2.2. Two interacting economies

In the preceding model, we considered an isolated economy which had no contact with any other economy. In reality, economies interact through mutual purchases and money transfers. As is typical in economics, a number of models were developed for this more realistic assumption. The most common is the model where, when describing a single economy, we introduce another 'player' (apart from households, firms and government) representing external economies influencing the economy modelled. But let us continue to develop a different, possibly more general, model employing the physical description initialized in the preceding section.

Let us consider economies of two countries which are 'interacting' through purchases and money transfers, i.e. part of the goods and services in an economy are purchased in another. We can assume that the product in one economy will also be affected besides other things by the product of the other economy. (For example, if in Germany the total product decreases, i.e. there is less production of goods and services, then the demand of German firms for the sub-products and services imported from the Czech Republic decreases and this will naturally negatively affect the product of the Czech Republic. And, reciprocally, the smaller product in the Czech Republic means a lower income for households and lower consumption of all goods and services, including more expensive products e.g. those imported from Germany, so the lower demand in Czech Republic will affect—to some extent—the German product).

Assume that, if there was no such interaction between the economies, each of them could be described in terms of equation (4). Hence, we can describe two economies without interaction as a system with two degrees of freedom using the dynamical equations

$$\ddot{Y}_1 + a_1\dot{Y}_1 + b_1 Y_1 = P_1, \tag{5}$$

$$\ddot{Y}_2 + a_2\dot{Y}_2 + b_2 Y_2 = P_2, \tag{6}$$

with the constants

$$\begin{aligned} a_i &= \lambda_i s_i + \kappa_i(1 - \lambda_i v_i), \\ b_i &= \kappa_i \lambda_i s_i, \\ P_i &= \kappa_i \lambda_i A_i, \end{aligned}$$

where $i = 1, 2$ labels the economies. In mechanics, the system of equations (5)–(6) would be referred to as the equations of motion of the unconstrained system. This system of equations could represent two unconnected damped and driven oscillators. In that case, the variables Y_1 and Y_2 would represent the position of masses of the first and the second oscillator, respectively. Accordingly, concerning business cycles, the variables Y_1 and Y_2 in the system of equations (5)–(6) have the meaning of 'position of each economy' which, as we have discussed in the introduction to section 2, we identify with the current product of a particular economy.

Following the physically motivated design of the Phillips model, we will formulate the interaction between the economies as a constraint binding the products Y_1 and Y_2 in the first and second economy. In physics, the constraints are used if the real forces ensuring the observed behaviour are not known or are uneasy to quantify. In a constrained system, the constraint forces arise which ensure the prescribed behaviour and have the meaning of real physical forces acting on the system (recall, e.g., the rolling of the cylinder without slipping in mechanics where the constraint forces have the meaning of forces and torques stemming from the interaction of the cylinder with an underlay, see [13] or [14]). Although we have already mentioned that no ‘forces’ are defined in economics, the constraints seem to be an appropriate methodology since they make it possible to describe the behaviour observed in reality whose ‘dynamical generator’ is not known.

The relationship between the two economies in our problem can be stated using different types of constraints (holonomic, nonholonomic, rheonomic and scleronomic). In mechanics, we develop constraints according to known relationships among the variables on the basis of physical and geometrical rules. Hence, a particular form of the constraint results from the precise system knowledge. In economics, the main problem is the lack of knowledge about the dynamics of economic systems and the constraints representing the interaction between the economies can only be estimated. The estimation process begins with suggesting the general form of the constraint (the variables involved, linear or nonlinear relationship, etc). The suggestion for the constraint equation is based on the knowledge of the behaviour of the economic system and on the assumptions of the economic background of observed interaction. Once the form of constraint is suggested, the particular values of constants involved must be best fitted using the time series analysis which is the subject of econometrics. In our example, we focus only on the suggestion of the constraint in the general form, i.e. we are interested in the theoretical part of the model development, but remember that, in practice, the second—econometric—part of the process would have to be done consequently.

The form of constraint in our problem will be chosen according to observed correlations in the time development of the products of certain economies. From the economic background we can deduce that one economy will not only reflect the absolute level of the product in the other economy but will also react to the ‘product velocity’. For example, having a certain product in Germany Y_G, the expectations of people and firms in the Czech Republic, which play a key role in the consumption and production behaviour in the Czech economy, will differ according to whether the German economy is growing: $\dot{Y}_G > 0$, stagnating: $\dot{Y}_G = 0$, or declining: $\dot{Y}_G < 0$. The growing German product could make the firms optimistic concerning future orders of their products, they will be more willing to hire new employees, enlarge the contracts on sub-products, etc. Thus, the employees will be more sure to have jobs in the future and be willing to spend their money rather than to save it for harder times. These processes will have a positive effect on the Czech product.

Therefore, in our example of two interacting economies, we assume that the interaction between economies can be explained via the relationship between the products Y_1, Y_2 and their first time derivatives $\dot{Y}_1$, $\dot{Y}_2$. The corresponding constraint generally falls within the nonholonomic constraints and we will assume a linear relationship (which is still most common in economical modelling). Consequently, for our example, let us assume this constraint to be

$$\dot{Y}_2 = k\dot{Y}_1 + \alpha Y_1 + \beta Y_2, \tag{7}$$

where k, α, β are constants. Remember that the determination of constants involved is the subject of econometrics which fits the observed data to the prescribed equation (7).

Since the constraint is linear in velocities and the model itself is physically motivated, we will treat the constrained system (5)–(7) in a way common in mechanics—we will use the nonholonomic approach (see e.g. [15] or [16]). Thus, the dynamical equations for our constrained system take the form

$$\ddot{Y}_1 + a_1\dot{Y}_1 + b_1 Y_1 = P_1 - \mu k, \tag{8}$$

$$\ddot{Y}_2 + a_2\dot{Y}_2 + b_2 Y_2 = P_2 + \mu, \tag{9}$$

which, together with constraint (7), yields three equations for three unknown variables (Y_1, Y_2, μ), where μ is the Lagrange multiplier. Eliminating the Lagrange multiplier from (8)–(9) and substituting from the constraint (7), we obtain reduced equations of the constrained system:

$$(1 + k^2)\ddot{Y}_1 + A_1\dot{Y}_1 + B_{11}Y_1 + B_{12}Y_2 - P_1 - P_2 = 0, \tag{10}$$

$$\dot{Y}_2 - k\dot{Y}_1 + \alpha Y_1 + \beta Y_2 = 0, \tag{11}$$

where

$$\begin{aligned} A_1 &= a_1 + a_2k^2 + \alpha k + \beta k^2, \\ B_{11} &= b_1 + \alpha a_2 k + \alpha\beta k, \\ B_{12} &= \beta a_2 k + k b_2 + k\beta^2. \end{aligned} \tag{12}$$

The system of equations (10)–(11) provides two differential equations for two unknowns—the products Y_1 and Y_2. The parameters (12) involved in the system (10)–(11) can be derived using the constants of unconstrained systems representing economies without interaction (5)–(6) and the constants specifying constraint (7). As usual in economic models, these parameters need not have any special economic interpretation and can be observed only as parameters specifying the particular dynamic system. Note that the initial parameters s, λ, κ, ν used to derive the dynamical equation (4) for product Y had precise economic meanings (see e.g. [11]). However, processing these parameters first through the calculation leading to (4) and later (for the two-economy system) through the elimination of the Lagrange multiplier, we arrive at the reduced equations (10)–(11), where the arising parameters are just functions of the original parameters with an economic interpretation and the ones estimated for constraint (7). Remember that a somewhat similar situation takes place when treating mechanical systems with constraints: the parameters in the equations for the unconstrained system and the constraint equations have clear physical meaning, but the parameters remaining in the reduced equations (representing the constrained mechanical system) generally need not bear particular physical relevance.

2.3. Results and discussion

Solving the constrained system (5)–(7) for three different artificial sets of parameters k, α, β in (7):

Constraint A :	$k = 0.7$	$\alpha = 0$	$\beta = -0$
Constraint B :	$k = 0.7$	$\alpha = 0.4$	$\beta = 0.1$
Constraint C :	$k = 0.7$	$\alpha = -0.1$	$\beta = -0.2$,

we obtain the solutions in figure 3.[7] The sets of constraint parameters were chosen to correspond to the construction of the constraint mentioned in previous section, i.e. we chose $k > 0$ reflecting the assumption that economic growth in one economy induces growth in the other.

[7] The parameters of the unconstrained systems (isolated economies) and initial conditions used for our artificial example were adopted from [11] to correspond with the realistic behaviour of economies according to rules accepted in economics: $\lambda_1 = 3$, $\lambda_2 = 3$, $\kappa_1 = 1$, $\kappa_2 = 1$, $A_1 = 2$, $A_2 = 1$, $s_1 = 0.25$, $s_2 = 0.25$, $\upsilon_1 = 0.6$, $\upsilon_2 = 0.6$. Hence, the two economies are quite similar except for the autonomous expenditure A. The initial conditions for our problem were as follows: $Y_1(0) = 10$, $Y_2(0) = 5$, $\dot{Y}_1(0) = 8$.

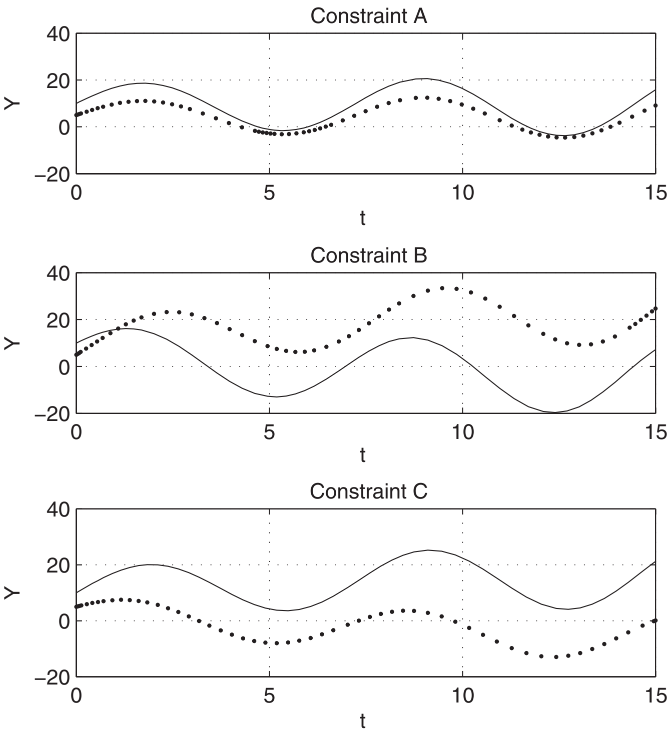

Figure 3. The solution of a constrained system for a particular setting of constraint parameters.

We set the value to $k = 0.7$ and then vary the other two parameters to cover several possible forms of constraint (7).

The solutions provide qualitative information about product dynamics for two economies which influence each other. The particular time behaviour of products Y_1 and Y_2 obviously depends on the particular form of the constraint. Note that for realistic modelling, the particular coefficients $k,\ \alpha,\ \beta$ should be estimated from observed data. Since the qualitative results of the test case are for guidance only, we can state that although being simple and based on an (old-fashioned) classical physically motivated model, the solution is in qualitative correspondence with the observed behaviour of the true interacting economies.

In figure 4, we can see the GDP time series of the Czech Republic and Germany in the period 2007–2010. As we have already mentioned, there is considerable trading between these two economies which possibly makes a connection between the time development of their products, as is qualitatively visible in figure 4. After a thorough econometric analysis, this connection could be qualitatively expressed by constraint (7) and the model of two interacting economies (10)–(11) could enable us to describe the business cycles of the two economies.

Thus, we have presented an example of how an economic model can be designed using the physical motivation. First, the classical model of product dynamics based on the mechanical model of an oscillator was employed to describe an isolated economy. Then, this model was extended to a model of two interacting economies using an oscillator-based approach, i.e. we treat each of the economies as a single component of the system with two degrees of freedom. The existing observed relationship between the economies was modelled by a constraint, and the qualitative results of the model when compared to the real data (see

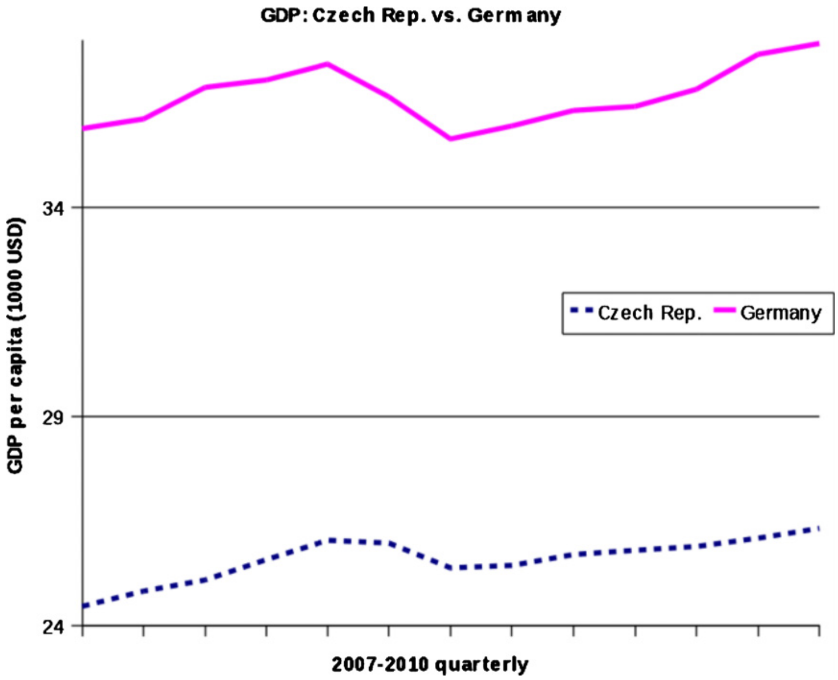

Figure 4. GDP per capita in Germany and Czech Republic: 2007–2010, quarterly (source: http://stats.oecd.org, expenditure approach, current prices, seasonally adjusted).

(This figure is in colour only in the electronic version)

figure 4) are promising. In economic science, such a model design could be successful after careful validation based on a concrete parameter estimation. As in other models, probably only some aspects of the observed behaviour could be captured by the newly developed model and, therefore, a debate about the relevance of the assumptions made and the methodology used might be sparked. Nevertheless, such debates take place for any economic model no matter whether it has a physical or other background. An important fact is that, so far, hardly any model has been validated in economics in the sense of physics, since there always exist significant discrepancies between the data predicted by the model and those observed in the economy in the world. Indeed, economic science is still trying to find the appropriate theories, methods and even the calculus to achieve the status of an unchallenged source of scholarly knowledge.

3. The economic growth model using calculus of variations

While the short run variation in product (measured by the GDP per capita) is usually termed a *business cycle*; the long term increase of per capita GDP is called *economic growth*. Economic growth is primarily driven by improvements in productivity, which involves producing more goods and services with the same inputs of labour, capital, energy and materials (therefore, there is a lasting demand for innovations and technological improvements in developed economies). The long-run path of economic growth is one of the central questions of economics. An increase in the GDP of a country greater than the population growth is generally taken as an increase in the standard of living of its inhabitants; hence, the aim of the policy makers is to achieve steady economic growth. To understand better the phenomena of economic growth and to obtain quantitative support for making appropriate decisions, a number of mathematical models of economic growth have been developed during the past 50 years. Again—as we have already mentioned when speaking about economic models in general—all models represent the economic reality to an individual extent and they differ in the assumptions made about how the economic characteristics mutually interact and influence the final growth.

In this section, the Cass–Koopmans–Ramsey model of economic growth based on an variational approach will be presented and the methodology and motivations behind an economic variational approach in comparison with traditional usage of variational calculus in mechanics will be investigated. Since the aim of this section is not to provide the reader with a thorough economic background of the model, we will describe the philosophy of the model intuitively (for more detailed notes about particular assumptions and the economic background, see e.g. [18]).

3.1. The Cass–Koopmans–Ramsey model

This time, there are two 'players' in the 'economy game': firms and households (so the flows are similar to the previous model illustrated in figure 1 except for the missing government purchases). The aim of the model is to answer the question of how much of the product of the economy at any point of time should be spent for immediate consumption to yield current utility, and how much of it should be saved (and invested) so as to enhance future production and consumption and, hence, yield future utility[8].

The criterion of optimality in the model is social welfare which is given by social utility U. There are two time-dependent variables through which the desired optimal development of the economy can be attained: consumption $c(t)$ and capital $k(t)$ (measured in units advantageous for the model purposes; for more details see [18]). Consumption c represents the purchases of goods and services by households while capital, k, is a factor of production, used to produce goods or services, that is not itself significantly consumed (though it may depreciate) in the production process (e.g. machinery, buildings, vehicles). Typically, in economics, we consider a relationship between the investment and the time derivative of the capital, k. In the Cass–Koopmans–Ramsey model, the particular relationship takes the form

$$\dot{k} = f(k) - c - (n+g)k, \tag{13}$$

where $f(k)$ denotes the product (more precisely it is a known production function prescribing how the total production in the economy is dependent on the capital available). The expression $f(k) - c$ has the meaning of investment[9] and $(n+g)k$ is the amount of investment that must be made just to keep capital, k, at its existing level (n and g being constants). Equation (13) states that the change in capital, k, is equal to the investments less the replacement purchases (i.e. if we buy new machinery only to replace the broken ones, then even if we made investments, we have not increased capital—we still have the same production factors for our production process as before the investment purchases).

The task in the Cass–Koopmans–Ramsey growth model is to maximize social utility $U(c)$ (which depends only on consumption) via time paths of the variables consumption, c,

[8] In economics, the product (which is considered to be equal to the income) can either be consumed or saved, but what is saved results in investment. For example, we can assume that the household saves some amount of money in a bank. But the bank uses the money to provide the firms with loans and the firms use the extra money for investment purchases.

[9] This expression is connected with the relationship $Y = C + S$ used in the previous example (see item (i)). Remember that it means that the total income (or product, which is equivalent as we have already discussed) in an economy is distributed among consumption and savings. Since what is saved is supposed to be invested, the savings define the investments in monetary units. Thus, investments≡ savings = product − consumption.

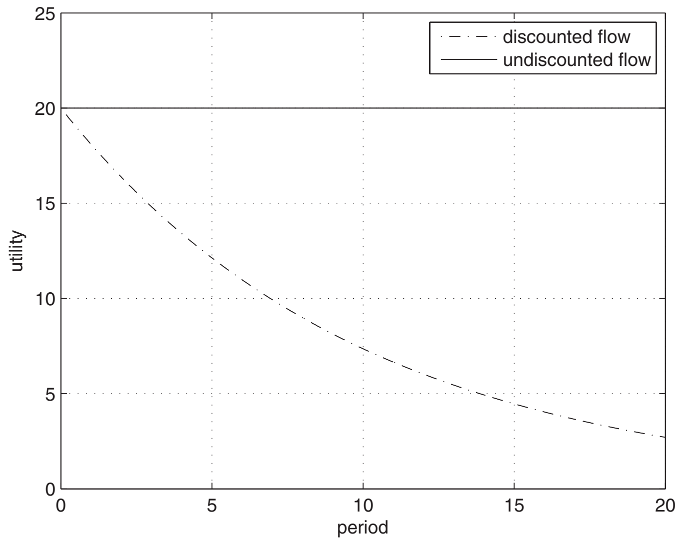

Figure 5. The model example of discounted and undiscounted social utility flow in time.

and capital, k, under constraint (13). Note that it is typically assumed in economics that the utility of the society is given only by the consumption of the people. This assumption can be questioned and actually several authors have done so. It is obvious that if the criterion of optimality in an economy is given simply by consumption, then many essential factors considering e.g. natural conditions can be omitted.

For the particular form of the utility function defined in the Ramsey model (see [17] or [18]), we can write the overall optimization problem as

$$\max \qquad U = \int_0^\infty B\,\mathrm{e}^{-\beta t}\,\frac{c^{1-\theta}}{1-\theta}\,\mathrm{d}t \tag{14}$$

$$\text{s.t.} \qquad f(k) - c - (n+g)k - \dot{k} = 0, \tag{15}$$

where $B,\ \beta,\ \theta,\ n,\ g$ are positive constants with $0 < \theta, \beta < 1$. The form of the utility function is a matter of intensive investigation and the particular form used in the Cass–Koopmans–Ramsey model represents just one approach. The term $\mathrm{e}^{-\beta t}$ ensures the *discounting* of economical quantities. It means that the future flows of a particular quantity (social utility in our problem) have a smaller weight than its current flow. To make this concept more clear, let us consider a model example (see figure 5). Let the consumption in society be constant ($c = 100$, measured in monetary units), i.e. in each period the amount of goods and services 'consumed' in the economy is the same. Now, setting the constants $B = 1$, $\theta = 1/2$ (without mentioning the units), the immediate utility function is (see (14))

$$u(t) = 20 \cdot \mathrm{e}^{-\beta t}.$$

If we did not discount ($\beta = 0$), the consumption in each period would bring to society the same utility $u(t) = \text{const} = 20$. When discounting is applied ($\beta = 0.1$ in our example), the further in the future the consumption, the less utility it provides. For example, the consumption of 100 after 15 periods from today earns only the utility of about 5 compared to the utility of 20 earned by today's consumption. Note that this approach was doubted by many authors because by this model the priority is given to (our) current consumption at the expense of the consumption of future generations.

From the mathematical point of view, this is a constrained variational problem with a velocity-dependent constraint (15). Solving (14)–(15), we obtain the optimal growth path; in other words, we realize how consumption, c, and capital, k, must behave over time to achieve the maximal lifetime utility from the consumption flow given relation (15) between the product, consumption and investment. Note that c in (14) could be substituted from (15) to obtain an unconstrained variational problem with the Lagrangian dependent on both variable k and its first derivative $\dot{k}$. Because the form (14)–(15) of the growth model is typically used in economics (thanks to its better economic interpretation and reasoning), we will not 'simplify' the initial problem (14)–(15) and it will be treated as a constrained variational problem for the solution of which the vakonomic approach will be employed. The Lagrangian for the constrained problems (14)–(15) takes the vakonomic form

$$L = B\,\mathrm{e}^{-\beta t}\frac{c^{1-\theta}}{1-\theta} + \lambda(f(k) - c - (n+g)k - \dot{k}), \tag{16}$$

where $\lambda = \lambda(t)$ is the Lagrange multiplier. Considering three variables $(c,\ k,\ \lambda)$ instead of the initial pair $(c,\ k)$, we obtain the variational dynamical equations

$$\mathrm{e}^{-\beta t}c^{-\theta} - \lambda = 0, \tag{17}$$

$$\lambda\frac{\mathrm{d}f}{\mathrm{d}k} - (n+g)\lambda + \dot{\lambda} = 0, \tag{18}$$

$$f(k) - c - (n+g)k - \dot{k} = 0. \tag{19}$$

Note that the vakonomic approach accounts for the main technique used to solve variational constraint dynamical problems in economics (see [17–19] for typical examples). For the description and some notes on the comparison of the vakonomic and the nonholonomic approach, see appendix A.

3.2. Results and discussion

Expressing λ from (18) and substituting it into equation (17), we arrive after some calculation at a system of two equations for the unknown variables consumption, c, and capital, k:

$$\frac{\dot{c}}{c} = \frac{\frac{\mathrm{d}f}{\mathrm{d}k} - \beta - n - g}{\theta} \tag{20}$$

$$\dot{k} = f(k) - c - (n+g)k. \tag{21}$$

Thus, it is not difficult to find a solution to our problem, i.e. to find optimal consumption and capital paths. Nevertheless, in economics, the functions are not normally given explicitly, but only assumed to have characteristic qualitative properties (such as that the utility is an increasing nonconvex function of consumption c). Thus, the problems have an air of 'theory' rather than 'computation' and reaching a particular solution is neither possible nor necessary. Yet, the content of the problems is meaningful and analysing the qualitative characteristics of the solutions often generates important insights into economic behaviour. In models of economic dynamic optimization, two variable diagrams are prevalently employed to obtain these qualitative analytical results.

The behaviour of an economy in the Cass–Koopmans–Ramsey problem is typically described in terms of the evolution of consumption, c, and capital, k, using a diagram in figure 6. This diagram reflects the dynamical equations (20)–(21) where the arrows show the directions of 'velocities' $\dot{c}$ and $\dot{k}$, e.g. in the upper-left 'quadrant' of the diagram, the arrows

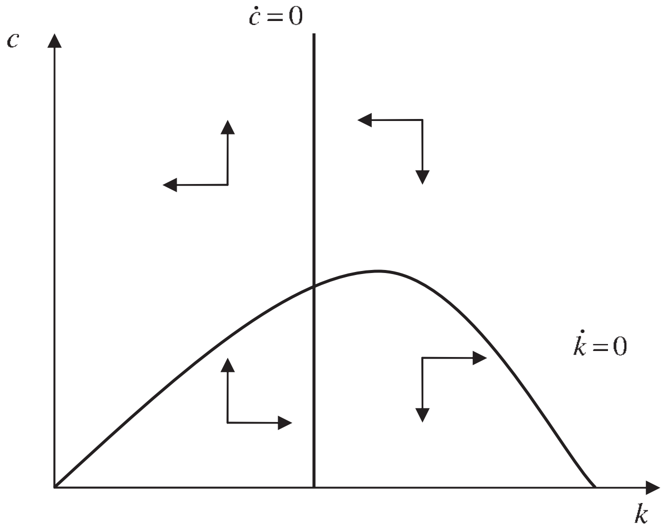

Figure 6. The dynamics of c and k in the Ramsey model: vertical arrows show the direction of $\dot{c}$, whereas the horizontal ones show the direction of $\dot{k}$.

indicate that for the combinations of consumption, c, and capital, k, in this area of the diagram, the consumption is increasing while capital is decreasing in time.

In mechanics, we can plot similar diagrams but this is mostly done for a particular solution. In such a case, the trajectory of the system in the configuration space is obtained. The diagram in figure 6 is used to identify the qualitative optimal behaviour of an economy given the initial conditions and additional requirements (such as $k \geqslant 0$). Thus, the result consists of a verbal description of the optimal behaviour rather than in an analytical solution. As an example, let us discuss the top-left 'quadrant' of the graph. The consumption, c, in the economy is high and rising and capital, k, eventually reaches zero. When c continues to rise k must become negative. But this cannot occur. Since the product is zero when capital k is zero, the consumption c must drop to zero. Therefore, such paths can be ruled out from the consideration of the realistic behaviour of an economy. In a similar way, other 'quadrants' in figure 6 can be analysed and we would arrive at the conclusion that the possible time path of the system is the one driving into the point where $\dot{k} = 0$, $\dot{c} = 0$. Then, the economy is said to 'move along the saddle path' to the equilibrium in $\dot{k} = 0$, $\dot{c} = 0$. The equilibrium is Pareto-efficient which means that it is impossible to make anyone better off without making someone else worse off. The explanation lies in the fact that all the households in the model have the same utility. The equilibrium produces the highest possible utility among allocations of capital and consumption that treat all households in the same way. Once the economy arrives at this equilibrium it cannot do better and therefore we do not change the level of capital and consumption anymore, i.e. $\dot{k} = 0$, $\dot{c} = 0$ (for more details, see [18]).

We can clearly see the difference between the application of variational calculus in mechanics where we are searching for a particular solution which prescribes the time evolution of the system precisely, and the economic application, where the behaviour of the system is only qualitatively analysed.

3.3. Lagrange multiplier interpretation

A first look at a nontrivial constrained optimization economic problem may be surprising for the student of physics. Leaving aside the unusual definition of a feasible region in the economic optimization problem (mostly restricted to nonnegative values of variables), the economic interpretation of the Lagrange multiplier is worth a discussion. Mathematical texts provide no interpretation of the Lagrange multiplier λ, leaving the student with the impression that λ has no significance beyond providing an extra variable which transforms a constrained problem into an unconstrained, higher-dimensional one. But in economic problems, the Lagrange multiplier can usually be interpreted as the rate of change of the optimal value of the criterion function relative to a parameter.

Constraint (15) describes the capital accumulation and thus λ should correspond to the value of having a little bit more capital (see appendix B for a simple explanatory example). Hence, we can find the meaning of the Lagrange multiplier by answering the question of how the unit change in the capital available in economics will affect the utility in optimum. Let us derive the influence of the change in capital onto a 'present value' instantaneous utility $\bar{u} = u(c)\,\mathrm{e}^{-\rho t}$ (see (14)):

$$\frac{\mathrm{d}\bar{u}}{\mathrm{d}k} = \frac{\partial \bar{u}}{\partial k} + \frac{\partial \bar{u}}{\partial c}\frac{\partial c}{\partial k}. \tag{22}$$

From the first variational dynamical equation (17), we obtain

$$\lambda = \frac{\partial \bar{u}}{\partial c}, \tag{23}$$

and differentiating constraint (19), we obtain

$$\frac{\partial c}{\partial k} = \frac{\partial f}{\partial k} - (n+g). \tag{24}$$

Substituting into relation (22) from (23)–(24) and having in mind that $\frac{\partial \bar{u}}{\partial k} = 0$, we obtain

$$\frac{\mathrm{d}\bar{u}}{\mathrm{d}k} = \lambda\left(\frac{\partial f}{\partial k} - (n+g)\right). \tag{25}$$

That is, the extra unit of capital will raise the flow of output by an amount $\partial f/\partial k$ each unit of which (without the unit replacement production $n-g$ needed for keeping the capital at the existing level and thus not intended for consumption) has a utility value of λ. The multiplier λ is referred to as *a shadow value of capital* (evaluated in the present value). New economic interpretation then arises from variational equations (17) and (23), respectively. It states that the marginal utility of consumption[10] is equal to the shadow value of capital λ. Thus, at the optimum, the consumer is indifferent between consuming an additional unit and investing it. Remember that the consumer must decide whether to consume or to save (which directly generates the investments) the additional unit (of income). If it is consumed it brings direct utility, and if it is invested (into capital) then it increases the product and future consumption. If the marginal utility of consumption is larger than the shadow value of capital,

$$\frac{\partial \bar{u}}{\partial c} > \lambda, \tag{26}$$

then capital would be too high and consuming more and saving less would increase the utility. Similarly, if the marginal utility of consumption is lower than the shadow value of capital,

$$\frac{\partial \bar{u}}{\partial c} < \lambda, \tag{27}$$

then capital is too low and the households should save more to increase utility through higher future consumption. This example of Lagrange multiplier interpretation together with the subsequent discussion of dynamic equations represents the typical approach of optimization problem analysis in economics.

[10] Utility gained (or lost) from an increase (or decrease) in the consumption.

In mechanics, no interpretation is used for multipliers in optimization problems, but remember that the multipliers arise in the constrained forces, which are important from the physical point of view. The concept of constrained forces, on the other hand, has no significance in economic problems although the multipliers do. Hence, although the computational routine remains similar, there are distinct interpretations of the tools used.

4. Conclusion

Classical mechanics has played a significant role in the development of economic thinking influencing both the principles and calculus. Currently, different attitudes to the adoption of physical methodology for economic purposes can be distinguished. A strong one supports the philosophy that applying the well-developed physical methods could quickly provide the economists with working and applicable models. This strong motion in economic science is reflected by the newly arisen term 'econophysics', which currently designates one of the possible fields where the undergraduates and graduates of physics could be involved.

The examples presented in this paper enable undergraduates to meet for the first time this highly modern and progressively expanding field, and to touch upon the different philosophy of building scholarly knowledge in economics. These examples may serve as a motivation for students to further study the economics–mechanics (or more generally physics) intersections. In addition, studying these problems makes the undergraduate students of physics face the different usage of a known methodology: the constraint variational calculus appears to be a flexible approach which in each discipline provides specific information and the nonholonomic dynamics of systems, based on constraint forces, could even be applied outside mechanics.

In physics education, the economic examples presented can not only serve as a demonstration of interdisciplinary applications of methods typically used in mechanics, but also provide the teachers with an aid for demonstrating what are general mathematical and what are specifically physical features of the mechanics methodology. In this way, the students can achieve a deeper insight into the physical background of what they have learned in classical mechanics courses.

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Appendix A

Consider a variational system with the Lagrangian $L = L(t, q, \dot{q})$, which is subject to the constraint $f(t, q, \dot{q}) = 0$. The *nonholonomic approach* consists of incorporating the constraint forces into the Lagrange equations:

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} = \left(\frac{\partial f}{\partial \dot{q}}\right)^T \cdot \mu, \tag{A.1}$$

$$f(t, q, \dot{q}) = 0, \tag{A.2}$$

where μ are the Lagrange multipliers. The nonholonomic approach is typically used for solving mechanical systems with nonholonomic constraints, although in mechanics the nonholonomic constraints are frequently affine in velocities. The form of constraint forces for classical ideal constraints is a consequence of the d'Alemberts principle. Note that the nonholonomic approach has also been extended to more general constraints by the work of Chetaev [20] and others (see e.g. [21]; for instructive nonholonomic mechanical problems, see e.g. [13, 14, 16]).

The mathematical concept of nonholonomic dynamics was essentially unchanged until 30 years ago when a new dynamics of velocity constrained mechanics system was introduced by Kozlov [22]. This new mechanics was called *vakonomic* being the 'variational axiomatic kind'. If we adopt a variational approach by requiring the motion to be a stationary curve of the action functional among all curves having the same end points and satisfying the nonholonomic constraints, then we get a vakonomic motion, i.e. we search for the solution of the (unconstrained) variational problem associated with the Lagrangian function

$$\bar{L}(t, q, \dot{q}, \lambda, \dot{\lambda}) = L(t, q, \dot{q}) + \lambda f(t, q, \dot{q}).$$

Namely, the vakonomic motions can be obtained by the Lagrange equations

$$\frac{\partial \bar{L}}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \bar{L}}{\partial \dot{q}} = 0, \tag{A.3}$$

$$\frac{\partial \bar{L}}{\partial \lambda} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \bar{L}}{\partial \dot{\lambda}} = 0, \tag{A.4}$$

which give

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \left(\frac{\partial f}{\partial \dot{q}}\right)^T \cdot \dot{\lambda} + \left(\frac{\partial f}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial \dot{q}}\right)^T \cdot \lambda = 0, \tag{A.5}$$

$$f(t, q, \dot{q}) = 0. \tag{A.6}$$

Comparing the nonholonomic (A.1) and vakonomic equations (A.5), we arrive at a conclusion that the arising systems of differential equations are not equivalent unless

$$\frac{\partial f}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial \dot{q}} = 0.$$

The author himself in [22] says: '...vakonomic dynamics, which is an internally consistent model that can be applied to description of the motion of any mechanical systems, is as "true" as traditional nonholonomic mechanics. The issue of the choice of model for each particular case is ultimately resolved by experiment'. While there are doubts that vakonomic dynamics is a satisfactory model for nonholonomic systems in mechanics (see e.g. [15]), in economics the vakonomic approach is the one typically used for the solution of dynamical optimization problems with velocity-dependent constraints. Intuitively, the nonholonomic concept based on the definition of constrained forces developed for mechanics could not be valid for the economic problems unless the background of the model is physically motivated. The vakonomic approach, if chosen for solving the constrained systems in economics, can be justified simply by the results it provides (for more examples on the vakonomic approach in economics, see e.g. [23, 24]).

Appendix B

In constrained optimization in economics, the value of the Lagrange multiplier at the optimal solution is referred to as a *shadow price*. Shadow price is the change in the objective value of the optimal solution obtained by relaxing the constraint by one unit.

Consider a simple (static) business optimization problem of profit maximization in a firm. Assume that there are only two products and the firm is deciding about the amount of these two products x_1, x_2 to be produced in the next period. The profit and number of working hours needed per unit of each product are known and denoted by a_1, a_2 and l_1, l_2, respectively. The operating time limit for the next period is B (for example, we have B working hours available on a certain machine for the next month). We get the linear programming problem

$$\max \quad \pi = a_1x_1 + a_2x_2 \tag{B.1}$$

$$\text{s.t.} \quad l_1x_1 + l_2x_2 = B. \tag{B.2}$$

We introduce a multiplier λ and form the Lagrangian

$$L = \pi + \lambda(B - l_1x_1 - l_2x_2).$$

Assuming that the firm maximizes profit (given constraint (B.2)), the optimal quantities $x_1^\star$, $x_2^\star$ and the multiplier $\lambda^\star$ necessarily satisfy the first-order conditions:

$$\frac{\partial L}{\partial x_1} = \frac{\partial \pi}{\partial x_1} - \lambda l_1 = 0, \tag{B.3}$$

$$\frac{\partial L}{\partial x_2} = \frac{\partial \pi}{\partial x_2} - \lambda l_2 = 0, \tag{B.4}$$

$$\frac{\partial L}{\partial \lambda} = B - l_1x_1 - l_2x_2 = 0, \tag{B.5}$$

and the differentiation of constraint (B.2) yields

$$l_1\frac{\partial x_1}{\partial B} + l_2\frac{\partial x_2}{\partial B} = 1. \tag{B.6}$$

Now, using the chain rule and equations (B.3), (B.4), (B.6), we obtain

$$\frac{\partial \pi}{\partial B} = \frac{\partial \pi}{\partial x_1}\frac{\partial x_1}{\partial B} + \frac{\partial \pi}{\partial x_2}\frac{\partial x_2}{\partial B} = \lambda l_1\frac{\partial x_1}{\partial B} + \lambda l_2\frac{\partial x_2}{\partial B} = \lambda. \tag{B.7}$$

Hence, the Lagrange multiplier, λ, measures how the total profit responds to the unit change in the total machine-operating time available. The shadow price here is the maximum price the manager would be willing to pay for operating the production line for an additional unit of time, based on the benefits he would get from this change. For example, if $\lambda = 1$ Euro, then running the machine and producing for an additional hour will gain the profit of 1 Euro and the manager should not pay more for this additional working hour than the value it produces (for more details, see e.g. [25]).

References

[1] Ray Canterbery E 2002 *A Brief History of Economics* (Singapore: World Scientific)

[2] Grattan-Guinness I 2010 How influential was mechanics in the development of neoclassical economics? A small example of a large question *J. Hist. Econ. Thought* **32** 531–81

[3] Stanley H *et al* 1996 Anomalous fluctuations in the dynamics of complex systems: from DNA and physiology to econophysics *Physica* A **224** 302–21

[4] Schulz M 2003 *Statistical Physics and Economics* (New York: Springer)

[5] Kitov I and Kitov O 2010 *Mechanomics: Economics as Classical Mechanics* (Saarbrucken: Lap Lambert Academic)

[6] Schinckus Ch 2010 Econophysics and economics: sister discipline? *Am. J. Phys.* **78** 325–7

[7] Walstad A 2010 Comment on Econophysics and economics: sister discipline? *Am. J. Phys.* **78** 789

[8] Stauffer D 2005 Teaching computational bio-socio-econo-physics *Eur. J. Phys.* **26** S79–84

[9] Mostardinha P, Durana E J and de Abereu F V 2006 The econophysics in the Euromillions lottery *Eur. J. Phys.* **27** 675–84

[10] O'Sullivan A and Sheffrin S M 2003 *Economics: Principles in Action* (Upper Saddle River, NJ: Pearson/Prentice-Hall)

[11] Allen R G D 1963 *Mathematical Economics* (New York: St Martin's)

[12] Ataullah A and Tippett M 2007 Equity prices as a simple harmonic oscillator with noise *Physica* A **382** 557–64

[13] Janová J, Musilová J and Bartoš J 2009 Coupled rolling motion: a student project in non-holonomic mechanics *Eur. J. Phys.* **30** 1257–69
[14] Janová J and Musilová J 2010 The streetboard rider: an appealing problem in non-holonomic mechanics *Eur. J. Phys.* **31** 333–45
[15] Zampieri G 2000 Nonholonomic versus vakonomic dynamics *J. Differ. Eqns* **163** 335–47
[16] Janová J and Musilová J 2011 Coupled rolling motion: considering rolling friction in non-holonomic mechanics *Eur. J. Phys.* **32** 245–57
[17] Chiang A C 2000 *Elements of Dynamic Optimization* (Long Grove, IL: Waveland)
[18] Romer D 2006 *Advanced Macroeconomics* (New York: McGraw-Hill)
[19] Aghion P and Howitt P 1998 *Endogenous Growth Theory* (Cambridge, MA: MIT Press)
[20] Chetaev N G 1932 On Gauss principle *Izv. Kazan. Fiz. Mat. Obs.* **6** 323–6 (in Russian)
[21] de Leon M and de Diego D M 1996 On the geometry of non-holonomic Lagrangian systems *J. Math. Phys.* **37** 1–26
[22] Kozlov V V 1982 Dynamics of systems with nonintegrable constraints: part I *Moscow Univ. Mech. Bull.* **37** 27–34
Kozlov V V 1982 Dynamics of systems with nonintegrable constraints: part II *Moscow Univ. Mech. Bull.* **37** 74–80
Kozlov V V 1983 Dynamics of systems with nonintegrable constraints: part III *Moscow Univ. Mech. Bull.* **38** 40–51
[23] Grubbstrom A 1990 Application of calculus of variations to financing alternatives *Omega* **19** 305–16
[24] Molinder A 1994 Application of calculus of variations to a continuous time aggregate production model *Int. J. Prod. Econ.* **41** 273–80
[25] Baxley J V 1984 Lagrange multiplier problems in economics *Am. Math. Mon.* **91** 404–12