Properties of lower bounds for the RCPSP

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1 Abstract

We show that the calculation of the well-known lower bound of Mingozzi for the RCPSP is an NP-hard problem and that the relative error of this lower bound can be equal to $O(\log n)$, where n is the number of jobs.

2 Introduction

Problem RCPSP may be formulated as follows. Given a set $N = \{1, \ldots, n\}$ of jobs. A constant amount of $Q_k > 0$ units of resource $k, k = 1, \ldots, K$, is available at any time. Job $j \in N$ has be processed for $p_j \geq 0$ time units without preemption. During this period, a constant amount of $q_{jk} \geq 0$ units of resource k is occupied. Furthermore, finish-start precedence relations $i \to j$ are defined between the jobs according to an acyclic directed graph G. The objective is to determine the starting times S_j for each job $j = 1, \ldots, n$, in such a way that: at each time t, the total resource demand is less than or equal to the resource availability for each resource type; the given precedence constraints are fulfilled; the makespan $C_{max} = \max_{j=1}^{n} C_j$, where $C_j = S_j + p_j$, is minimized.

Let C^*_{max} be the optimal value of the objective function for the problem when preemptions are not allowed and $C^*_{max}(pmtn)$ be the optimal value when preemptions are allowed.

3 Lower Bound of Mingozzi et al.

We consider a linear programming formulation that partially relaxes the precedence constraints and allows preemption. The columns of this LP correspond to the so-called non-dominated feasible subsets. A feasible set X is a set of jobs that may be processed simultaneously, i.e., there are no precedence relations between any pair $i, j \in X$ and all resource constraints are satisfied (i.e., $\sum_{i \in X} q_{ik} \leq Q_k$ for $k = 1, \ldots, K$). Such a set is called non-dominated if it is not a proper subset X of another feasible set Y. We consider all non-dominated feasible sets and additionally the one-element sets $\{i\}$ for all $i = 1, \ldots, n$.

We denote all these sets by $X_1, X_2, \ldots, X_f$, where f is the number of such sets, and associate each set X_j with an incidence vector $a^j \in \{0, 1\}^n$ defined by $a^j_i = 1$ if $i \in X_j$, and $a^j_i = 0$ otherwise, $j = 1, \ldots, f$.

Furthermore, let x_j be a variable denoting the number of time units over which all the jobs in X_j are processed simultaneously. Then the following linear programming problem provides a lower bound LB_M (Mingozzi A. *et. al.* 1998) for the $RCPSP$ by relaxing the precedence constraints and allowing preemption:

$$\sum_{j=1}^{f} x_j \longrightarrow \min \tag{1}$$

$$\begin{cases} \sum_{j=1}^{f} a_i^j x_j \geq p_i, \ i = 1, \ldots, n; \\ x_j \geq 0, \ j = 1, \ldots, f. \end{cases} \tag{2}$$

It is known that the calculation of LB_M is an NP-hard problem (by a reduction from the NP-hard Bin packing problem) (Lazarev A.A. and Gafarov E.R. 2008), and there are instances for which $\frac{C^*_{\max}}{LB_M} \approx 2$.

4 Relative errors of well-known lower bounds for the problem

In the paper (Lazarev A.A. and Gafarov E.R. 2008), there is a conjecture that $C^*_{\max} < 2 \cdot C^*_{\max}(pmtn)$. This conjecture is true for the special case of problem $Pm|prec|C_{\max}$ (Lawler E.L. *et. al.* 1998), for the special case of $RCPSP$ with a constant amount of $Q_1 > 0$ units of a single resource and without precedence constraints, and for the special case for which there are only one or two preempted job in an optimal schedule for RCPSP with preemptions. However, the conjecture is false for the general case.

Theorem 1. *There exists an instance of $RCPSP$ for which*

$$\frac{C^*_{\max}}{C^*_{\max}(pmtn)} = O(\log n).$$

Proof.
We consider an instance of the type given in Fig. 1 (a). For this instance, we have $m+1$ levels of jobs. At the highest level, we have a job j_1^1 with processing time Mp, at the second highest level, we have two jobs j_1^2 and j_2^2 with processing times $\frac{M}{2}p$, and so on. At the lowest level, we have a chain of short and very short jobs $e_1 \to j_1^{m+1} \to e_2 \to j_2^{m+1} \to \ldots \to e_M \to j_M^{m+1} \to e_{M+1}$, where $p_{e_i} = \varepsilon, \ q_{e_i} = m+1, \ i = 1, \ldots, M+1$, and $p_{j_i^{m+1}} = p, \ q_{j_i^{m+1}} = 1, \ i = 1, \ldots, M$. For each job j_i^k from level $k, \ k = 2, \ldots, m$, we have $p_{j_i^k} = \frac{M}{2(k-1)}p, \ q_{j_i^k} = 1$. At each level $k, \ k = 2, \ldots, m$, we have a chain of jobs $e_1 \to j_1^k \to e_{\frac{M}{2(k-1)}+1} \to j_2^k \to e_{2\frac{M}{2(k-1)}+1} \to \ldots \to j_{2^{k-1}}^k \to e_{M+1}$. Some of these precedence relations are illustrated in Fig. 1 (a). For this instance, we have $(M+1)\varepsilon \ll p$.

By dotted lines, we mark all jobs which will be processed in parallel in an optimal schedule for the non-preemptive problem.

In Fig. 1 (b), for an instance with only $m+1 = 6$ levels, we give the resulting optimal solution. For this instance, we obtain $C^*_{\max}(pmtn) = 32p + 33\varepsilon$ and $C^*_{\max} = 112p + 33\varepsilon$.

For the general case, we have $C^*_{\max}(pmtn) = Mp + (M+1)\varepsilon$ and $C^*_{\max} = Mp + \frac{m}{2}Mp + (M+1)\varepsilon$. Hence, we obtain

$$\frac{C^*_{\max}}{C^*_{\max}(pmtn)} \approx \frac{m+2}{2}.$$

Let us now express M by means of m. We have $2^m = M$. Then

$$n = 2M + 1 + 1 + 2 + \ldots + 2^{m-1} = 2M + 1 + 2^m - 1 = 2 \cdot 2^m + 2^m = 3 \cdot 2^m$$

from which we obtain

$$m = \log \frac{n}{3}.$$

Therefore, we get

$$\frac{C^*_{\max}}{C^*_{\max}(pmtn)} \approx \frac{m+2}{2} = \frac{\log n - \log 3 + 2}{2}.$$

□

As a consequence, there exists an instance of $RCPSP$ for which $\frac{C^*_{\max}}{LB_M} = O(\log n)$, and we obtain the following result:

Theorem 2. *There exists a type of instances of $RCPSP$ for which*

$$\frac{C^*_{\max}}{LB_M} = O(\log n),$$

and the calculation of LB_M is an NP-hard problem.

The idea of constructing such instances is not difficult. The instance contains two subsets of jobs N_1 and N_2. The jobs from the first subset correspond to the instance illustrated in Fig. 1, where $Q_1 = m+1$. In the set N_2, we have n independent jobs with unit processing times $p_j = 1$ and $\sum_{j \in N_2} q_j = 2m + 2$. Additionally, we have a dummy job o_1 such that $j \to o_1 \to l$ for all $j \in N_1,\ l \in N_2$. It is obvious that we can give a reduction from the partition problem to the problem of calculating LB_M for this type of instances.

Additionally, let us consider the relaxation of the problem in which we do not take into consideration non-preemptive jobs, or different processing times, or different values q_i, or we do not consider the precedence relations. Denote by C'_{max} the optimal value of the objective function for the relaxed instance. Then there exist instances for which

$$\frac{C^*_{max}}{C'_{max}} = O(\log n),$$

i.e., we have bad approximation ratio for the lower bound C'_{max}.

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References

Mingozzi A., Maniezzo V., Ricciardelli S. and Bianco L., 1998, “An exact algorithm for project scheduling with resource constraints based on new mathematical formulation", *Management Science*, Vol. 44, pp. 714–729.

Lawler E.L., Lenstra J.K., Rinnooy Kan A.H.G. and Shmoys D.B., 1998, “Sequencing and Scheduling: Algorithms and Complexity.", *Report BS-R8909, Centre for Mathematics and Computer Science, Amsterdam, The Netherlands*.

Lazarev A.A. and Gafarov E.R. , 2008, “On project scheduling problem.", *Automation and Remote Control*. Vol. 69, N 12, pp. 2070-2087

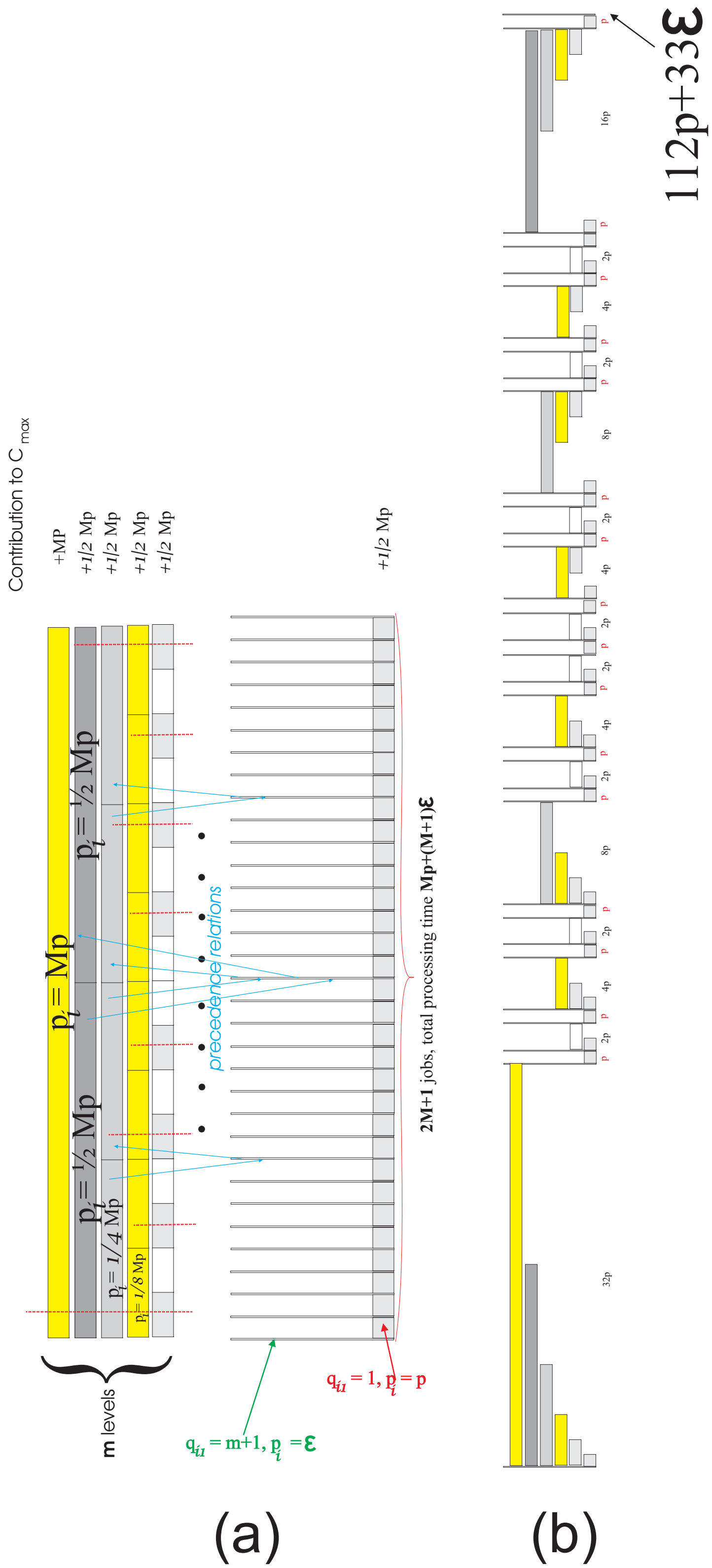

Fig. 1. An instance for illustrating Theorem 1