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On the quasi-convexity of a special job shop scheduling problem

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In this paper a structural neighbourhood graph is introduced for a polynomially solvable special case of the job shop problem. Then it is shown that the chosen objective $F = C_{\max}$ is quasi-convex on the considered graph.

1. Introduction

The structure of discrete optimization problems has been a topic of considerable research in the past few years. The aim of structural investigations consists in deducing suitable solution methods from our knowledge about the structure of the objective and the solution set, and separating subclasses of some problem types and developing special solution algorithms for these problems respectively. In connection with the investigation of planning and production processes sequencing and machine scheduling problems play an important role. Special permutation problems of the following form turn out to be a fundamental subclass of discrete optimization problems which appear in this sphere:

$$\min \{F(p) | p = (p_1, p_2, \ldots, p_m) \in P(M)\} \tag{1}$$

$$\text{or} \quad \min \left\{F(P) | P = (p^1, p^2, \ldots, p^n) \in X \subseteq \prod_{j=1}^{n} P(M^j)\right\}. \tag{2}$$

Thereby $P(M)$ denotes the set of permutations of the elements of the set $M = \{1,\ldots,m\}$, p represents a permutation with the assignments $i \to p_i (i = 1(1)m)$, $\prod_{j=1} P(M^j)$ is the cartesian product of the sets $P(M^1),\ldots, P(M^n)$, where $M^j \subseteq M$ $(j = 1(1)n)$, and the objective F (cost function) is a unique mapping of the solution set into the set R^1.

For instance one-machine scheduling problems, permutation flow shop problems (machine scheduling problems where each job is to be processed in the same order on the machines and we have to choose the same job order on each machine) as well as flow shop problems with at most 3 machines can be reduced to problem class (1). The job order on the machines is characterized by a permutation $p \in P(M)$ of the indices of the jobs. The so-called job shop problems belong to the type (2), where the machine order of the jobs is arbitrary but fixed.

The permutation $p^j \in P(M^j)$ of the job indices describes the order of those jobs which have to be processed on machine M_j.

In [10] the conception of the quasi-convexity of objectives has been adequately transfered from the field of non-discrete optimization to permutation problems of type (1) in order to obtain a base for a uniform solution methodology for special classes of such permutation problems. The considerations made in this paper could be extended to the case of more general discrete optimization problems.

In Section 2 the essential foundations on the quasi-convexity in discrete optimization are briefly summarized. Then in Section 3 the job shop problem is introduced in detail as a machine scheduling problem of type (2). In Section 4 we present a special permutation graph for the 2-machine job shop problem $[m/2/G, n_i \leq 2/C_{\max}]$ and verify the quasi-convexity of the chosen objective on this graph.

2. On the quasi-convexity in discrete optimization

Let $G_E(X) = (X, E)$ be a connected undirected graph, where the vertex set X is the set of feasible solutions of the discrete optimization problem $\min \{F(x) | x \in X,\ X$ finite$\}$ and the edge set E represents a special neighbourhood structure. A sequence of vertices which are connected by an edge of E in each case is denoted as simple chain if the same vertex does not occur twice in the sequence.

Definition 1: F is said to be quasi-convex on $G_E(X)$, if from $x^1, x^2 \in X$ with $F(x^2) \leq F(x^1)$ follows that there exists at least a simple chain K in $G_E(X)$ connecting x^1 with x^2 with the property $F(x) \leq F(x^1)$ for all $x \in K$.

Definition 2: A quasi-convex function F is said to be strictly quasi-convex on $G_E(X)$, if from $x^1, x^2 \in X$ with $F(x^2) < F(x^1)$ follows that there exists at least a simple chain K in $G_E(X)$ connecting x^1 with x^2 with the property $F(x) < F(x^1)$ for all $x \in K$ with $x \neq x^1$.

In connection with the quasi-convex non-discrete optimization the following theorems, the proofs of which are contained in [10], hold.

Theorem 1: If the objective F is strictly quasi-convex on $G_E(X)$, then each local optimum is also a global optimum.

Theorem 2: If the objective F is quasi-convex on $G_E(X)$, then the set G_0 of optimal points forms a connected subgraph of $G_E(X)$.

The aim consists in determining an edge set such that F is quasi-convex or even strictly quasi-convex on $G_E(X)$ for special problem classes and thereby each vertex $x \in X$ has a number of neighbour vertices in $G_E(X)$ as small as possible. A descent algorithm can be stated for such objectives which always yields a global optimum (cf. [2]).

In [2] different permutation graphs are introduced for the permutation problem (1). The vertex set consists of the set of feasible solutions $P(M)$ in each case. If we consider the undirected graph $G_V(m) = (P(M), E_V)$ of interchanges of neighbours, then two vertices are neighbours, if the corresponding permutations differ only in two neighbouring positions by the interchange of both elements, i. e.

$$E_V = \{(p^i, p^k) \in [P(M)]^2 \mid \exists j \in \{2,\ldots,m\} \text{ such that } p^k = z(j-1, j) \cdot p^i\},$$

where $z(j-1, j)$ is a standard transposition, which differs from the identical permutation only by the interchange of the positions $j-1$ and j. In [10] it is shown that F is

already quasi-convex on $G_V(m)$, if the *Smith* condition is satisfied (cf. [11]), i. e. if there exists a sequencing rule for the jobs.
In the following section a problem of the more general type (2) is described.

3. The $[m/n/G/C_{\max}]$ job shop problem

The job shop problem can be formulated as follows: m jobs $A_1, \ldots, A_m$ are to be processed on n machines $M_1, \ldots, M_n$. The job A_i $(i = 1(1)m)$ consists of n_i operations, where each operation corresponds to a processing of A_i on a machine. The processing order on the single machines is discribed for job A_i by its machine order $\mathrm{TRF}_i = (M_{i_1}, \ldots, M_{i_{n_i}})$. In the following we only consider the case that each job is to be processed at most once on each machine $(n_i \leq n)$. $t_{ij} > 0$ denotes the processing time of A_i on M_j, $t_{ij} = 0$ means that A_i is not to be processed on M_j. Each machine can handle only one job simultaneously, and it is not allowed to interrupt the processing of operations on a machine. We choose as objective exclusively the maximum completion time of the single jobs (i. e. $F = C_{\max}$). If a job shop problem with n machines is considered, a schedule P can be described by representing the job order on each machine by a permutation of the indices of those jobs, which have to be processed on the corresponding machine, i. e.

$$P = (p^1, p^2, \ldots, p^n) \in \prod_{i=1}^{n} P(M^j), M^j \subseteq M = \{1, \ldots, m\}.$$

M^j denotes the set of the indices of those jobs, which have to be processed on machine M_j.
A $[m/n/G/C_{\max}]$ problem can be described in a suitable manner by a disjunctive graph model (cf. for instance [7]). In order to represent a schedule we introudce in the following a slightly simplified directed graph $\boldsymbol{N}(P) = (V, \boldsymbol{U})$ with vertex valuations. The vertex set V consists of all operations (i, j) $(i \in M,\ j \in \{1, \ldots, n\})$, which represent a processing, as well as a fictitious initial operation $(0, 0)$ and a fictitious final operation $(*, *)$. The arc set $\boldsymbol{U}$ describes the machine order of the jobs as well as the job order on the single machines.
There exists an arc from a vertex $(i, j) \in V$ $(i \in M,\ j \in \{1, \ldots, n\})$ to the vertex of the succeeding operation (l, j) on the same machine as well as the vertex of the succeding operation (i, k) of the corresponding job, if these operations exist in each case. Moreover, we have an edge from vertex $(0, 0)$ to the sources as well as from all sinks to the vertex $(*, *)$. The vertex valuations are given by the processing times of the single operations, where the processing times of both fictitious operations $(0, 0)$ and $(*, *)$ are equal to zero. A schedule P is said to be feasible, if $\boldsymbol{N}(P)$ has no cycles.
Furthermore, we make the following definitions for a directed graph $\boldsymbol{N}(P) = (V, \boldsymbol{U})$ without cycles. A path w is a sequence of vertices, which are connected by an arc in each case, i. e. $w = (v_1, v_2, \ldots, v_r)$, $v_i \in V$ $(i = 1(1)r)$, $u_j = (v_j, v_{j+1}) \in \boldsymbol{U}$ $(j = 1(1)\ r - 1)$.
The vertices belonging to the path form the range $T(w)$ of the path w. The weigth $l(w)$ of the path w is given by the sum of the valuations of the vertices which comprise the path, i. e.

$$l(w) = \sum_{(i,j) \in T(w)} t_{ij}.$$

Furthermore, we denote the set of all paths from vertex $v_1 \in V$ to vertex $v_2 \in V$ in $\boldsymbol{N}(P)$ by $W^P(v_1, v_2)$.
$W(P) := W^P((0, 0), (*, *))$ is the set of all paths in $\boldsymbol{N}(P)$. $r_{ij}(P) = \max_{w \in W^P((0,0),(i,j))} \{l(w) - t_{ij}\}$ is called the head of operation (i, j) in $\boldsymbol{N}(P)$. Moreover, $z_{ij}(P) = \max_{w \in W^P((i,j),(*,*))} \{l(w) - t_{ij}\}$ is denoted as tail of operation (i, j) in $\boldsymbol{N}(P)$. The completion time $C_i(P)$ of job $A_i (i \in M)$ for a feasible schedule P is obtained by

$$C_i(P) = \max_{j \text{ with } i \in M^j} \{r_{ij}(P) + t_{ij}\}.$$

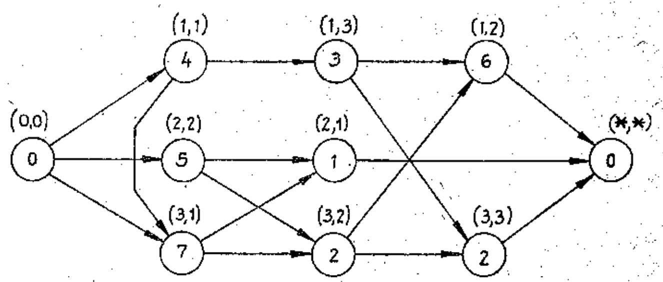
Then the cost of P is given by the maximum completion time of a job which corresponds to the weight of a critical path in $\boldsymbol{N}(P)$, i. e.

$$F(P) = C_{\max}(P) = \max_{i \in M} \{C_i(P)\} = z_{00}(P) = r_{**}(P).$$

Now we consider the following example for illustration. Let $m = 3$, $n = 3$, $\mathrm{TRF}_1 = (M_1, M_3, M_2)$, $\mathrm{TRF}_2 = (M_2, M_1)$, $\mathrm{TRF}_3 = (M_1, M_2, M_3)$ and

$$T = \begin{bmatrix} 4 & 6 & 3 \\ 1 & 5 & 0 \\ 7 & 2 & 2 \end{bmatrix}.$$

If we choose a schedule $P = (p^1, p^2, p^3)$ with $p^1 = (1, 3, 2)$, $p^2 = (2, 3, 1)$ and $p^3 = (1, 3)$, then $\boldsymbol{N}(P)$ looks as follows:



Obviously, $\boldsymbol{N}(P)$ has no cycles and, therefore, P is a feasible schedule with $C_{\max}(P) = 19$.
By using *Johnson's* algorithm for the $[m/2/F/C_{\max}]$ flow shop problem there was given an $0(m \log m)$ algorithm for solving the $[m/2/G, n_i \leq 2/C_{\max}]$ problem by *Jackson* (cf. [5]). Let A be the set of the indices of the jobs which have to be processed first on M_1 and then on M_2, B is the set of the indices of those jobs which have to be processed first on M_2 and then on M_1, and C and D, respectively, are the sets of all indices of the jobs which have to be processed only on M_1 and M_2, respectively, i. e. $M^1 = A \cup B \cup C$ and $M^2 = A \cup B \cup D$. We denote by $p^A \in P(A)$ the optimal permutation of the elements of the set A obtained by *Johnson's* algorithm and by $p^B \in P(B)$ the optimal permutation of the elements of the set B obtained in an analogous manner. $p^C \in P(C)$ and $p^D \in P(D)$ are arbitrary permutations of the elements of the sets C and D, respectively. Then $P = (p^1, p^2) \in P(M^1) \times P(M^2)$ is an optimal solution of the $[m/2/G, n_i \leq 2/C_{\max}]$ problem, where the mentioned partial permutations are joined in p^1 and p^2 as follows:

$$p^1 = (p^A, p^C, p^B) \in P(M^1) \quad \text{and} \quad p^2 = (p^B, p^D, p^A) \in P(M^2). \tag{3}$$

Minor extensions of the $[m/2/G, n_i \leq 2/C_{\max}]$ already lead to NP-hard problems. Indeed, the $[m/3/G, n_i \leq 2/C_{\max}]$ problem and, in the case that a job has to be processed on a machine more than once, also the $[m/2/G, n_i \leq 3/C_{\max}]$ problem belong to the class NP-hard.

4. On the quasi-convexity of the $[m/2/G, n_i \leq 2/C_{\max}]$ problem

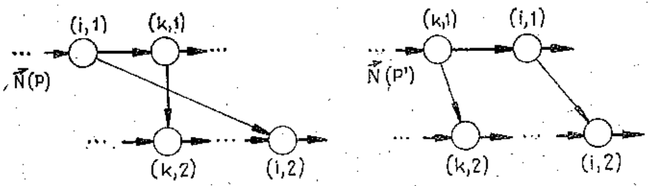
Let us consider the 2-machine job shop problem. Let A, B, C and D be the sets introduced in Section 3. In the following we present some types of interchanges of neighbouring elements in permutation p^1 of the feasible schedule $P = (p^1, p^2)$, which do not lead to a greater cost (independent of T). To this end we denote the feasible initial schedule always by P and that feasible schedule which is obtained if the mentioned interchange has been carried out by P'. Furthermore, $Po(p^j, i)$ denotes the position of the element $i \in M$ in p^j $(j = 1, 2)$. In the following proofs the vertex valuations are always omitted in the representation of the graphs $\vec{N}(P)$.

Theorem 3: The following interchanges of two elements i and k in p^1 of a feasible schedule $P = (p^1, p^2)$ with $Po(p^1, k) = Po(p^1, i) + 1$ do not lead to a greater cost:

1. interchange of i, k with $i \in A$, $k \in A$ and $Po(p^2, k) < Po(p^2, i)$;
2. interchange of i, k with $i \in B$, $k \in B$ and $Po(p^2, k) < Po(p^2, i)$;
3. interchange of i, k with $i \in B$, $k \in A$ and $Po(p^2, i) < Po(p^2, k)$;
4. interchange of i, k with $i \in C$, $k \in C$;
5. interchange of i, k with $i \in C$, $k \in A$;
6. interchange of i, k with $i \in B$, $k \in C$.

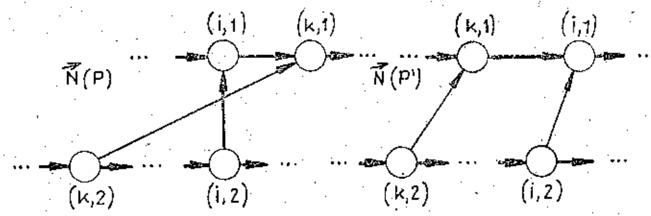
Proof: Clearly, by an interchange of type 1 up to 6 in P a feasible schedule P' is generated in each case. We consider $\vec{N}(P)$ and $\vec{N}(P')$ in each case. Let $M^* = M \setminus \{i, k\}$ for the further considerations.

1.



From $r_{i2}(P) \geq r_{i2}(P')$ (4) follows immediately $C_i(P) \geq C_i(P') = \max\{C_i(P'), C_k(P')\}$. By considering (4), $r_{k2}(P) \geq r_{k2}(P')$ and $r_{k1}(P) + t_{k1} = r_{i1}(P') + t_{i1}$ we obtain $r_{lj}(P) \geq r_{lj}(P')$ $(l \in M^*,\ j \in \{1, 2\})$ and, consequently, $C_l(P) \geq C_l(P')$. Therefore $C_{\max}(P) \geq C_{\max}(P')$ holds.

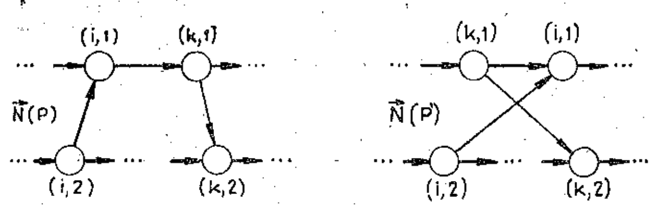
2.



We have $C_k(P) = r_{k1}(P) + t_{k1} \geq r_{i1}(P') + t_{i1} = C_i(P')$
$= \max\{C_i(P'), C_k(P')\}$. (5)

By using (5) $r_{lj}(P) \geq r_{lj}(P')$ $(l \in M^*,\ j \in \{1, 2\})$ is obtained, and, consequently, $C_l(P) \geq C_l(P')$. Therefore $C_{\max}(P) \geq C_{\max}(P')$ holds.

3.

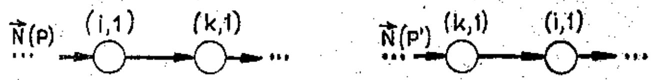


From $r_{k2}(P) \geq r_{k2}(P')$ follows immediately $C_k(P) \geq C_k(P')$. (6)

By $r_{k1}(P) + t_{k1} \geq r_{i1}(P') + t_{i1}$ (7)

we have $C_k(P) > r_{k1}(P) + t_{k1} \geq C_i(P')$. Thus $C_k(P) \geq \max\{C_i(P'), C_k(P')\}$.
By consideration of (6) and (7) we obtain $r_{lj}(P) \geq r_{lj}(P')$ $(l \in M^*,\ j \in \{1, 2\})$ and, consequently, $C_l(P) \geq C_l(P')$. Therefore $C_{\max}(P) \geq C_{\max}(P')$ holds.

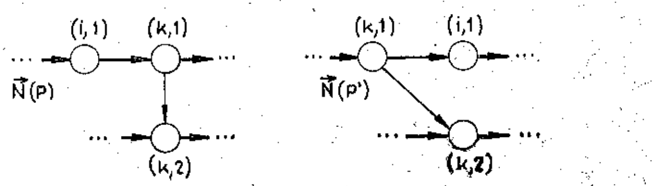
4.



We have $\max\{C_i(P), C_k(P)\} = C_k(P) = C_i(P') =$
$\max\{C_i(P'), C_k(P')\}$. (8)

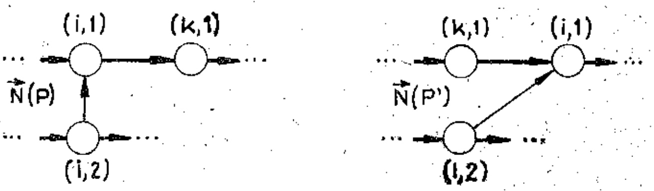
By considering (8) we obtain $r_{lj}(P) = r_{lj}(P')$ $(l \in M^*,\ j \in \{1, 2\})$ and, consequently, $C_l(P) = C_l(P')$. Therefore $C_{\max}(P) = C_{\max}(P')$ holds.

5.



We have $C_k(P) = r_{k2}(P) + t_{k2} \geq \max\{r_{i1}(P') + t_{i1}, r_{k2}(P') + t_{k2}\} = \max\{C_i(P'), C_k(P')\}$. By consideration of $r_{k1}(P) + t_{k1} = r_{i1}(P') + t_{i1}$ and $C_k(P) \geq C_k(P')$ $r_{lj}(P) \geq r_{lj}(P')$ $(l \in M^*, j \in \{1, 2\})$ is obtained, and, consequently, $C_l(P) \geq C_l(P')$. Therefore $C_{\max}(P) \geq C_{\max}(P')$ holds.

6.



We have $C_k(P) = r_{k1}(P) + t_{k1} \geq r_{i1}(P') + t_{i1} = C_i(P') =$
$\max\{C_i(P'), C_k(P')\}$. (9)

By considering of (9) we obtain $r_{lj}(P) \geq r_{lj}(P')$ $(l \in M^*,\ j \in \{1, 2\})$ and, consequently, $C_l(P) \geq C_l(P'_l)$. There fore $C_{\max}(P) \geq C_{\max}(P')$ holds.

In an analogous manner some types of interchanges of neighbouring elements in permutation p^2 of the feasible schedule $P = (p^1, p^2)$ can be presented, which do not lead to a greater cost.

Theorem 4: The following interchanges of two elements i and k in p^2 of a feasible schedule $P = (p^1, p^2)$ with $Po(p^2, k) = Po(p^2, i) + 1$ do not lead to a greater cost:

1. interchange of i, k with $i \in A$, $k \in A$ and $Po(p^1, k) < Po(p^1, i)$;
2. interchange of i, k with $i \in B$, $k \in B$ and $Po(p^1, k) < Po(p^1, i)$;
3. interchange of i, k with $i \in A$, $k \in B$ and $Po(p^1, i) < Po(p^1, k)$;
4. interchange of i, k with $i \in D$, $k \in D$;
5. interchange of i, k with $i \in D$, $k \in B$;
6. interchange of i, k with $i \in A$, $k \in D$.

The proof of this theorem is completely similar to that of Theorem 3 and, hence, is omitted.

In the following theorem we cousider two cases, where two elements are simultaneously interchanged in p^1 and p^2 which are neighbours in both permutations.

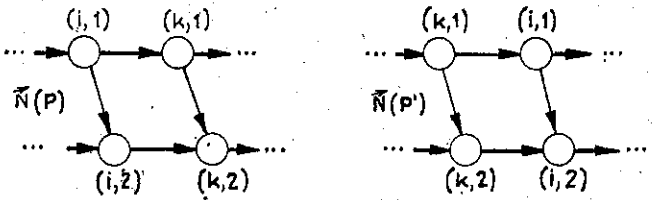
Theorem 5: The following interchanges of two elements i and k in p^1 and p^2 of a feasible schedule $P = (p^1, p^2)$ with $Po(p^j, k) = Po(p^j, i) + 1$ $(j = 1, 2)$ do not lead to a greater cost:

1. interchange of i, k with $i \in A$, $k \in A$ and $\min\{t_{k1}, t_{i2}\} \leq \min\{t_{k2}, t_{i1}\}$;
2. interchange of i, k with $i \in B$, $k \in B$ and $\min\{t_{k2}, t_{i1}\} \leq \min\{t_{k1}, t_{i2}\}$.

Proof: Clearly, by an interchange of type 1 or 2 in P a feasible schedule P' is generated in each case.

1. First of all, from $\min\{t_{k1}, t_{i2}\} \leq \min\{t_{k2}, t_{i1}\}$ follows immediately $\min\{-t_{i1} - t_{i2} - t_{k2}, -t_{i1} - t_{k1} - t_{k2}\} \leq \min\{-t_{k1} - t_{i1} - t_{i2}, -t_{k1} - t_{k2} - t_{i2}\}$ and, consequently, $\max\{t_{i1} + t_{i2} + t_{k2}, t_{i1} + t_{k1} + t_{k2}\} \geq \max\{t_{k1} + t_{i1} + t_{i2}, t_{k1} + t_{k2} + t_{i2}\}$. (10)

Let us consider $\vec{N}(P)$ and $\vec{N}(P')$.



From $r_{i1}(P) = r_{k1}(P')$ and (10) we obtain $r_{i1}(P) + \max\{t_{i1} + t_{i2} + t_{k2}, t_{i1} + t_{k1} + t_{k2}\} \geq r_{k1}(P') + \max\{t_{k1} + t_{i1} + t_{i2}, t_{k1} + t_{k2} + t_{i2}\}$ and, accordingly, $\max\{C_i(P), C_k(P)\} = C_k(P) = r_{k2}(P) + t_{k2} \geq r_{i2}(P') + t_{i2} = C_i(P') = \max\{C_i(P') C_k(P')\}$. (11)

By consideration of (11) and $r_{k1}(P) + t_{k1} = r_{i1}(P') + t_{i1}$ we have $r_{lj}(P) \geq r_{lj}(P')$ $(l \in M \setminus \{i, k\},\ j \in \{1, 2\})$ and, consequently, $C_l(P) \geq C_l(P')$. Therfore $C_{\max}(P) \geq C_{\max}(P')$ holds.

2. The proof is similar to that of part 1 and, hence, is omitted.

Now we introduce a graph $G_E(X) = (X, E)$ such that the objective $F = C_{\max}$ of a $[m/2/G, n_i \leq 2/C_{\max}]$ problem with given sets A, B, C and D (i. e. with given machine orders of the single jobs) is quasi-convex on $G_E(X)$ independent of T. Thereby X denotes the set of all feasible schedules $P = (p^1, p^2) \in P(M^1) \times P(M^2)$ (in order to guarantee that $\vec{N}(P)$ has no cycles, there may not exist an $i \in A$ and a $k \in B$ with $Po(p^1, i) > Po(p^1, k)$ and $Po(p^2, i) < Po(p^2, k)$). E is the set of all edges between vertices, where the corresponding schedules $\bar{P} = (\bar{p}^1, \bar{p}^2)$ and $P = (p^1, p^2)$ differ only by a special interchange of two neighbouring elements in a permutation or, in the case of a special structure of P, by the simultaneous interchange of two elements in both permutations, if the elements belong either to the set A or to the set B and if they are neighbours in both permutations of the schedule. Let $\hat{p}^A, p^A \in P(A)$, $p^B, \hat{p}^B \in P(B)$, $p^C \in P(C)$, $p^D \in P(D)$ and $Po(p^j, p^j_i) = i (j = 1, 2)$, then P and $\bar{P}$ are connected by an edge in $G_E(X)$ if

1. there exists an $i \in \{2, 3, \ldots, m\}$ such that $\bar{p}^1 = z(i - 1, i) \times p^1$, $\bar{p}^2 = p$, where a) $i = \min\{l \mid p^1_{l-1} \in M^1 \setminus A, p^1_l \in A\}$;
   or b) $\max_{k \in A}\{Po(p^1, k)\} = |A|^1$, $i = \max\{l \mid p^1_{l-1} \in B, p^1_l \in C\}$;
   or c) $p^1 = (\hat{p}^A, p^C, p^B)$, $p^2 = (\hat{p}^B, p^D, p^A)$, $i = \min\{l \mid p^1_{l-1}, p^1_l \in A, Po(p^2, p^1_{l-1}) > Po(p^2, p^1_l)\}$;
   or d) $p^1 = (p^A, p^C, p^B)$, $p^2 = (p^B, p^D, p^A)$, $p^1_{i-1}, p^1_i \in C$;
2. $p^1 = (p^A, p^C, p^B)$ and there exists an $i \in \{2, 3, \ldots, m\}$ such that $\bar{p}^2 = z(i - 1, i) \cdot p^2$, $\bar{p}^1 = p^1$,
   where a) $i = \min\{l \mid p^2_{l-1} \in M^2 \setminus B, p^2_l \in B\}$;
   or b) $\max_{k \in B}\{Po(p^2, k)\} = |B|$, $i = \max\{l \mid p^2_{l-1} \in A, p^2_l \in D\}$;
   or c) $p^2 = (\hat{p}^B, p^D, p^A)$, $i = \min\{l \mid p^2_{l-1}, p^2_l \in B, Po(p^1, p^2_{l-1}) > Po(p^1, p^2_l)\}$;
   or d) $p^2 = (p^B, p^D, p^A)$, $p^2_{i-1}, p^2_i \in D$;
3. $p^1 = (p^A, p^C, p^B)$, $p^2 = (p^B, p^D, p^A)$ and there exist $i, j \in \{2, 3, \ldots, m\}$ such that $\bar{p}^1 = z(i - 1, i) \cdot p^1$, $\bar{p}^2 = z(j - 1, j) \cdot p^2$,
   where a) $p^1_{i-1} = p^2_{j-1} \in A$, $p^1_i = p^2_j \in A$;
   or b) $p^1_{i-1} = p^2_{j-1} \in B$, $p^1_i = p^2_j \in B$.

Then the following assertion can be formulated:

Theorem 6: The objective $F = C_{\max}$ of the 2-machine job shop problem $[m/2/G, n_i \leq 2/C_{\max}]$ is quasi-convex on $G_E(X) = (X, E)$.

Proof: Let $\bar{P} = (\bar{p}^1, \bar{p}^2)$ be an arbitrary feasible schedule of the $[m/2/G, n_i \leq 2/C_{\max}]$ problem. Now we show that there exists a simple chain in $G_E(X)$ from $\bar{P}$ to the optimal solution P according to (3) on which the cost is monotonously nonincreasing. First of all, there are interchanged elements in $\bar{p}^1$. Thereby was arrange the elements of the set A on the first $|A|$ positions by successive interchanges of neighbouring elements in each case (the element of A which is most on the left in $\bar{p}^1$ is shifted to position 1 and so on; interchanges of type 3 and 5, respectively, of Theorem 3; edges of type 1a in $G_E(X)$). Then the elements of the set B are put on the last $|B|$ positions (the element of B which is most on the right is moved to the last position and so on; interchanges of type 6 of Theorem 3; edges of type 1b in $G_E(X)$). Similarly, starting from $\bar{p}^2$ we arrange the elements of the set B on the first $|B|$ positions (interchanges of type 3 and 5, respectively, of Theorem 4; edges of type 2a in $G_E(X)$) as well as the elements of the set A on the last $|A|$ positions (interchanges of type 6 of Theorem 4; edges of type 2b in $G_E(X)$). The result is a schedule $\tilde{P} = (\tilde{p}^1, \tilde{p}^2)$ with $\tilde{p}^1 = (\tilde{p}^A, \hat{p}^C, \hat{p}^B) \in P(M^1)$ and $\tilde{p}^2 = (\tilde{p}^B, \hat{p}^D, \hat{p}^A) \in P(M^2)$ (11), where $\hat{p}^A, \tilde{p}^A \in P(A)$, $\hat{p}^B, \tilde{p}^B \in P(B)$, $\hat{p} \in P(C)$ and $\hat{p}^D \in P(D)$.

If $\tilde{p}^A \neq \hat{p}^A$, let l be the number of pairs (i, k) where in $\hat{p}^A$ the elements $i, k \in A$ are in reversed order as in $\tilde{p}^A$. Then $\tilde{p}^A$ can be transformed into $\hat{p}^A$ by l successive interchanges of neighbouring elements in each case (interchanges of type 1 of Theorem 3; edges of type 1c in $G_E(X)$). Analogously, $\tilde{p}^B$ can be transformed into $\hat{p}^B$ by successive interchanges of neighbouring elements in each case (interchanges of type 2 of Theorem 4; edges of type 2c in $G_E(X)$). The resulting schedule $\hat{P}$ has the form $\hat{P} = (\hat{p}^1, \hat{p}^2)$ with $\hat{p}^1 = (\hat{p}^A, \hat{p}^C, \hat{p}^G, \hat{p}^B) \in P(M^1)$ and $\hat{p}^2 = (\hat{p}^B, \hat{p}^D, \hat{p}^A) \in P(M^2)$. (12)

Now we carry out simultaneous interchanges of two elements in the first and second permutations which are neighbours in both permutations in each case. If $\hat{p}^A \neq p^A$, where p^A is that partial permutation which is contained in the optimal solution according to (3), let l be the number of pairs (i, k) where in p^A the elements $i, k \in A$ are in reversed order as in $\hat{p}^A$. Then $\hat{p}^A$ can be transformed into p^A by l successive interchanges of neighbouring elements in each case (interchanges of type 1 of Theorem 5 since p^A was obtained by the algorithm of *Johnson*; edges of type 3a in $G_E(X)$). Analogously, $\hat{p}^B$ can be transformed into p^B by successive interchanges of neighbouring elements in each case (interchanges of type 2 of Theorem 5; edges of type 3b in $G_E(X)$).

Finally, the elements of the set C as well as D are put in the sequence according to (3) (interchanges of type 4 of Theorem 3 and 4, respectively; edges of type 1d and 2d, respectively, in $G_E(X)$). Now the optimal solution $P = (p^1, p^2)$ is obtained and, consequently, the simple chain required. Let P^1 and P^2 be two arbitrary feasible schedules of the $[m/2/G, n_i \leq 2/C_{\max}]$ problem with $F(P^2) \leq F(P^1)$. By the preceding considerations, there exist simple chains K_1 and K_2, respectively, from P^1 and P^2, respectively, to the optimal solution P in $G_E(X)$ with monotonously nonincreasing costs. Starting from P^1, let P' be the first vertex on K_1 which also belogs to K_2 (possibly only $P' = P$). Then the chain $K = [P^1, P', P^2]$, which passes from P^1 to P' on K_1 and from P' to P^2 on K_2, is a simple chain with $F(\bar{P}) \leq F(P^1)$ for all $\bar{P} \in K$. Therefore F is quasi-convex on $G_E(K)$.

In the definition of $G_E(X)$ we took into consideration that there always exists an optimal solution of type (12) for any processing time matrix. In order to guarantee a number of neighbour vertices as small as possible, E was chosen such that in the proof of Theorem 6 the partial chain to the first solution of type (12) is a unique one (cf. edges of type 1a—1c as well as 2a—2a). The edge set could also be defined such that another sequence is chosen for the interchanges of neighbours which have to be carried out (for instance if we begin with interchanges of neighbours in $\bar{p}^2$) or that the partial chain leads to another solution of type (12) (for instance starting from (11) $\hat{p}^A$ can be transformed into $\bar{p}^A$ by interchanges of type 1 of Theorem 4 and then $\hat{p}^B$ into $\bar{p}^B$ by interchanges of type 2 of Theorem 3). Nevertheless, such modifications do not lead to a structurally different neighbourhood definition.

5. Concluding remarks

By the transmission of the concept of a quasi-convex function from the field of non-discrete optimization to permutation problems a base is given for a classification of permutation functions with regard to their solution behaviour. In this paper the considerations on the quasi-convexity of objectives were extended from permutation problems, where the cost is determined by a permutation of the elements of the set $M = \{1,...,m\}$, to more general discrete problems.

For a special machine scheduling problem a structural graph was introduced with such a neighbourhood structure that the chosen objective is quasi-convex on the considered graph. The object of further investigations is the determination of such neighbourhood structures that the objective possesses the property of quasi-convexity on the corresponding structural graph for further special problem classes of the considered type. Moreover, such investigations are a good base for determining suitable neighbourhood structures in order to deduce iterative approximation methods for those problem classes of the considered type where the quasi-convexity is not obtained on the corresponding graph.

[1]) |A| denotes the cardinality of the set A.

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