EXERCISES CHAPTER 1

1. Which of the following terms are propositions? If they are propositions, what is their truth value?

   (a) $2 + 1 = 15$; (b) Please hold the line.

   (c) $\int x^2 dx$; (d) $5 < 10$;

   (e) 3 is an even number.

2. Given are the following propositions.

   A: Peter meets Ann.
   B: Peter meets Betty.
   C: Peter meets Cindy.

   Express the following propositions by use of conjunction, disjunction and (or) negation of the given propositions:

   D: Peter meets all three girls.
   E: Peter meets only Ann.
   F: Peter meets at least one of the three girls.
   G: Peter meets exactly one of the three girls.
   H: Peter meets no girl.
   I: Peter does not meet all three girls.

3. Verify by means of truth tables that

   (a) $\overline{A \wedge B} \Longleftrightarrow \overline{A} \vee \overline{B}$; (b) $(A \Rightarrow B) \Longleftrightarrow (\overline{B} \Rightarrow \overline{A})$

   are tautologies.

4. Find out in which cases A is sufficient for B or A is necessary for B or both.

   (a) $A : x \in \mathbb{N}$ is even;
   $B : x/2 \in \mathbb{N}$;

   (b) $A : x/3 \in \mathbb{N}$;
   $B : x$ is integer;

   (c) $A : x < 4$;
   $B : x^2 < 16$;

   (d) $A : x^2 = 16$;
   $B : (x = 4) \vee (x = -4)$;

   (e) $A : (x > 0) \wedge (y > 0)$;
   $B : xy > 0$.

5. Check whether the following propositions are true for $x \in \mathbb{R}$:

   (a)
   $$\bigwedge_x x^2 - 5x + 10 > 0;$$

   (b)
   $$\bigwedge_x x^2 - 2x > 0.$$

   Find the negations of the propositions and give their truth values.

6. Prove indirectly
$$x + \frac{1}{x} \geq 2$$
with the premise $x > 0$. Use both methods, proof by contradiction and proof of contrapositive.

7. Prove by induction:
   (a) $\sum\limits_{i=1}^{n}(2i-1) = n^2$;
   (b) $\sum\limits_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$;
   (c) Let $a > 0$ and $n = 1, 2, \ldots$. Prove the inequality $(1+a)^n \geq 1 + na$.

8. Which of the propositions
$$1 \in S; \qquad 0 \in S; \qquad 2 \notin S; \qquad -1 \notin S$$
are true and which are false in each of the following cases:
   (a) $S = \{1, 2\}$;
   (b) $S = \{x \mid x^2 + 2x - 3 = 0\}$;
   (c) $S = \{0, 1, 2\} \cup \{-1, 0\}$;
   (d) $S = \{0, 1, 2\} \cap \{-1, 0\}$.

9. Let $A = \{1, 3, 5, 7, 9, 11\}$ and $B = \{1, 2, 3, 7, 8, 9\}$. Find $A \cup B, A \cap B, |A|, |B|, |A \cup B|$, $|A \cap B|, A \setminus B$ and $|A \setminus B|$.

10. Given is $A = \{1, 2\}$. Enumerate all the subsets of the set A. What is the cardinality of set $P(P(A))$?

11. Illustrate by means of Venn diagrams:
    (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;
    (b) $(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$.

12. Prove that $(A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$.

13. Eight students of a group study mathematics, 13 students of the group do not live in the student hostel, and 17 students study mathematics or do not live in the student hostel. How many students of the group studying mathematics do live in the student hostel?

14. Among 1,100 students studying economics at a university, there are 550 students possessing a car and 400 students possessing a PC, and 260 students do neither have a car nor a PC. Determine the number of students having a car or a PC, the number of students having both a car and a PC, the number of students having a car but no PC and the number of students having a PC but no car.

15. Given are $A = \{1, 2\}, B = \{2, 3\}$ and $C = \{0\}$. Find $A \times A$, $A \times B$, $B \times A$, $A \times C$, $A \times C \times B$ and $A \times A \times B$.

16. Let the sets

$$M_1 = \{x \in \mathbb{R} \mid 1 \leq x \leq 4\}, \quad M_2 = \{y \in \mathbb{R} \mid -2 \leq y \leq 3\}, \quad M_3 = \{z \in \mathbb{R} \mid 0 \leq z \leq 5\}$$

be given. Find $M_1 \times M_2 \times M_3$. Illustrate this cartesian product which corresponds to a set of points in the three-dimensional space, in a rectangular coordinate system.

17. There are 12 books in a shelf. How many possibilities for arranging these books do exist? How does this number change if three books have to stand side by side?

18. A designer wants to arrange buttons in a row. Three buttons are blue and five are red. How many possibilities for arrangement do exist?

19. Given is the set of numbers $\{1, 2, 3, 4, 5\}$. How many three-figured numbers can you select? How does it change when all figures have to be different?

20. In a group of n people everyone says once 'hello' to each other. How many 'hello' can somebody from the group hear?

21. In the vicinity of a holiday resort 15 walks shall be marked with two colored parallel lines. How many colors are at least necessary if the same colors may be used and the order of the lines is unimportant?

22. A set of weights consists of weights of 1, 2, 5, 10, 50, 100 and 500 units.

    (a) How many combinations of these weights are possible (use combinatorics and consider the cases when first no weight is selected, then exactly one weight is selected and so on)?

    (b) Using your knowledge about sets explain that for a set of n weights, there are exactly 2^n combinations of weights.

    (c) Prove by induction:

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n.$$

23. Simplify the following fractions:

    (a) $\dfrac{4x+5y}{x^2+2xy} - \dfrac{3x-2y}{4y^2+2xy} + \dfrac{x^2-15y^2}{3x^2y+6xy^2}$;

    (b) $\dfrac{\dfrac{a}{a-b} - \dfrac{b}{a+b}}{1 + \dfrac{a^2+b^2}{a^2-b^2}}$.

24. Solve the following inequalities:

    (a) $5x + 3 \geq -2x - 4$;

    (b) $\dfrac{3}{2x-4} \leq 2$;

    (c) $\dfrac{x^2-1}{x+1} \leq \dfrac{2}{x}$;

    (d) $\dfrac{x+6}{x-2} > x + 1$.

25. Solve the following inequalities:

(a) $|x| < 2$; (b) $|x-3| < 2$; (c) $|2x-3| < 2$;

(d) $|x-4| < |x+4|$; (e) $\dfrac{|x-1|}{2x+2} \geq 1$.

26. Simplify the following terms:

(a) $\dfrac{4+5\sqrt{2}}{2-3\sqrt{2}} + \dfrac{11\sqrt{2}}{7}$; (b) $\dfrac{24a^3b^{-5}}{7c^3} : \dfrac{8b^{-4}c^{-4}}{21a^{-3}b^{-5}}$.

27. Solve the following equations for x:

(a) $\sqrt{3x-9} - \sqrt{2x-5} = 1$; (b) $9^{x^2-1} - 36 \cdot 3^{x^2-3} + 3 = 0$;

(c) $2\sqrt[3]{x+2} + 3 = \sqrt{15+3x}$.

28. Simplify the following terms:

(a) $a = -\log_3 (\log_3 \sqrt[3]{\sqrt[3]{3}})$; (b) $x = \dfrac{\log_a N}{\log_{ab} N} - \log_a b$.

29. Solve the following equations for x:

(a) $\frac{1}{4} \ln x^5 + 3 \ln \sqrt{x} - 3 \ln \sqrt[4]{x} = 2 (\ln 2 + \ln 3)$;

(b) $x^{2-(\lg x)^2 - \lg x^2} - \frac{1}{x} = 0$;

(c) $\log_3 x + \log_{\sqrt{x}} x - \log_{\frac{1}{3}} x = 6$.

30. Find the roots of the following equations and illustrate them in an Argand diagram:

(a) $x^2 - 3x + 9 = 0$; (b) $x^4 + 13x^2 + 36 = 0$.

31. Illustrate the set of all complex numbers z with $|z-i| < 4$ in an Argand diagram.

32. Find the sum, difference, product and quotient of the following complex numbers:

(a) $z_1 = 1 + 4i$; (b) $z_2 = -2 + i$.

33. Find the cartesian form of the following complex numbers:

(a) $z = \frac{1}{i}$;

(b) $z = \left(\frac{1+i}{1-i}\right)^2$;

(c) $z = re^{i\varphi}$ with $r = 2\sqrt{3}$ and $\varphi = -2\pi/3$.

34. Find the polar and exponential forms of the following complex numbers:

(a) $z = \frac{1}{i}$; (b) $z = \left(\frac{1+i}{1-i}\right)^2$;

(c) $z = \frac{3}{2} + 3\frac{\sqrt{3}}{2}i$; (d) $z = \dfrac{2-i}{3i + (i-1)^2}$.

35. Given are the complex numbers $z_1 = 2 - 2i$ and $z_2 = 1 + i$. Find z_1^4 and z_1/z_2 by using the polar form. Compare the results with those you get by means of the cartesian form of the complex numbers.

36. Show that i^i is a real number.

37. Find the solutions of the following equations:

(a) $z^4 = -8 + 8\sqrt{3}i$; (b) $z^3 + \frac{5}{8}i = \frac{15}{i}$.

38. Find the real numbers a_1 and a_2 so that $z = 4(\cos 40° + i \sin 40°)$ is a solution of the equation

$$z^3 - \frac{a_1(\sqrt{3} + a_2 i)}{5i + 2(1-i)^2} = 0.$$