EXERCISES CHAPTER 10

1. Given are the matrices

$$A = \begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix}; \qquad B = \begin{pmatrix} 2 & 1 \\ -2 & 4 \end{pmatrix};$$

$$C = \begin{pmatrix} -3 & -2 & 4 \\ -3 & 2 & 3 \\ -2 & -2 & 3 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & -2 \\ -1 & 0 & 2 \end{pmatrix}.$$

   Find the eigenvalues and the eigenvectors of each of these matrices.

2. Let x_t be the consumption value of a national economy in period t and y_t the capital investments of the national economy in this period. For the following period $t+1$, we have

$$x_{t+1} = 0.7x_t + 0.6y_t$$

   which describes the change in consumption from one period to the subsequent one depending on consumption and capital investment in the current period. Consumption increases by 30 per cent of consumption and by 40 per cent of the capital investments. The capital investments follow the same type of strategy:

$$y_{t+1} = 0.6x_t + 0.2y_t.$$

   Thus, we have the system

$$\mathbf{u}_{t+1} = A\mathbf{u}_t \qquad \text{with} \qquad \mathbf{u}_t = (x_t, y_t)^T, \qquad t = 1, 2, \ldots.$$

   (a) Find the greatest eigenvalue λ of the matrix A and the eigenvectors associated with this value.

   (b) Interpret the result above with λ as a factor of proportionate growth.

   (c) Let $10,000$ units be the sum of consumption value and capital investments in the first period. How does it have to be split for a proportionate growth? Assume you have the same growth rate λ for the following two periods, what are the values of consumption and capital investments?

3. Given are the matrices

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{pmatrix}.$$

   (a) Find the eigenvalues and the eigenvectors of each of these matrices.

   (b) Verify that the eigenvectors of matrix A form a basis of the space $\mathbb{R}^4$ and that the eigenvectors of matrix B are linearly independent and orthogonal.

4. Verify that the quadratic form $\mathbf{x}^T B\mathbf{x}$ with matrix B from Exercise 1 is positive definite.

5. Given are the matrices:

$$A = \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}; \qquad B = \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix};$$

$$C = \frac{1}{2}\begin{pmatrix} 5 & 1 & 0 \\ 1 & 5 & 0 \\ 0 & 0 & -8 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 5 \end{pmatrix}.$$

(a) Find the eigenvalues of each of these matrices.

(b) Determine by the given criterion (see Theorem 10.7 in the book resp. Theorem 5 in the lecture) which of the matrices A, B, C, D (and their associated quadratic forms, respectively) are positive definite and which are negative definite.

(c) Compare the results of (b) with (a).

6. Let $\mathbf{x} = (1,1,0)^T$ be an eigenvector associated with the eigenvalue $\lambda_1 = 3$ of the matrix

$$A = \begin{pmatrix} a_1 & 0 & 1 \\ 2 & a_2 & 0 \\ 1 & -1 & a_3 \end{pmatrix}.$$

(a) What can you conclude about the values of a_1, a_2 and a_3?

(b) Find another eigenvector associated with λ_1.

(c) Is it possible in addition to the answers concerning part (a) to find further conditions for a_1, a_2 and a_3 when A is positive definite?

(d) If your answer is affirmative for part (c), do you see a way to find a_1, a_2 and a_3 exactly when $\lambda_2 = -3$ is also an eigenvalue of the matrix A?