EXERCISES CHAPTER 11

1. (a) Given is a Cobb-Douglas function $f : \mathbb{R}^2_+ \to \mathbb{R}$ with

$$z = f(x, y) = A \cdot x_1^{\alpha_1} \cdot x_2^{\alpha_2},$$

where $A = 1, \alpha_1 = 1/2$ and $\alpha_2 = 1/2$. Graph isoquants for $z = 1$ and $z = 2$ and illustrate the surface in $\mathbb{R}^3$.

(b) Given are the following functions $f : D_f \to \mathbb{R},\ D_f \subseteq \mathbb{R}^2$, with $z = f(x, y)$:

$$(1)\ z = \sqrt{9 - x^2 - y^2}; \quad (2)\ z = \frac{xy}{x - y}; \quad (3)\ z = x^2 + 4x + 2y.$$

Graph the domain of the function and isoquants for $z = 1$ and $z = 2$.

2. Find the first-order partial derivatives for each of the following functions:

(a) $z = f(x, y) = x^2 \sin^2 y$; (b) $z = f(x, y) = x^{(y^2)}$;

(c) $z = f(x, y) = x^y + y^x$; (d) $z = f(x, y) = \ln(\sqrt{x}\sqrt{y})$;

(e) $z = f(x_1, x_2, x_3) = 2xe^{x_1^2 + x_2^2 + x_3^2}$; (f) $z = f(x_1, x_2, x_3) = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

3. The variable production cost C of two products P_1 and P_2 depends on the outputs x and y as follows:

$$C(x, y) = 120x + \frac{1,200,000}{x} + 800y + \frac{32,000,000}{y},$$

where $(x, y) \in \mathbb{R}^2$ with $x \in [20, 200]$ and $y \in [50, 400]$.

(a) Determine marginal production cost of products P_1 and P_2.

(b) Compare the marginal cost of P_1 for $x_1 = 80$ and $x_2 = 120$ and of P_2 for $y_1 = 160$ and $y_2 = 240$. Give an interpretation of the results.

4. Find all second-order partial derivatives for each of the following functions:

(a) $z = f(x_1, x_2, x_3) = x_1^3 + 3x_1x_2^2x_3^3 + 2x_2 + \ln(x_1x_3)$;

(b) $z = f(x, y) = \dfrac{1 + xy}{1 - xy}$; (c) $z = f(x, y) = \ln \dfrac{x + y}{x - y}$.

5. Determine the gradient of function $f : D_f \to \mathbb{R}, D_f \subseteq \mathbb{R}^2$, with $z = f(x, y)$ and specify it at the points $(x_0, y_0) = (1, 0)$ and $(x_1, y_1) = (1, 2)$:

(a) $z = ax + by$; (b) $z = x^2 + xy^2 + \sin y$; (c) $z = \sqrt{9 - x^2 - y^2}$.

6. Given is the surface

$$z = f(x, y) = x^2 \sin^2 y$$

and the domain $D_f = \mathbb{R}^2$, where the xy-plane is horizontal. Assume that a ball is located on the surface at point $(x, y, z) = (1, 1, z)$. If the ball begins to roll, what is the direction of its movement?

7. Determine the total differential for the following functions:

(a) $z = f(x, y) = \sin \frac{x}{y}$; (b) $z = f(x, y) = x^2 + xy^2 + \sin y$;

(c) $z = f(x, y) = e^{(x^2+y^2)}$; (d) $z = f(x, y) = \ln(xy)$.

8. Find the surface of a circular cylinder with radius $r = 2$ meters and height $h = 5$ meters. Assume that measurements of radius and height may change as follows: $r = 2 \pm 0.05$ and $h = 5 \pm 0.10$. Use the total differential for an approximation of the change of the surface in this case. Find the absolute and relative (percentage) error of the surface.

9. Let $f : D_f \to \mathbb{R}$, $D_f \subseteq \mathbb{R}^2$, be a function with

$$z = f(x_1, x_2) = x_1^2 e^{x_2},$$

where $x_1 = x_1(t)$ and $x_2 = x_2(t)$.

(a) Find the derivative dz/dt.

(b) Use the chain rule to find $z'(t)$ if

(1) $x_1 = t^2$; $x_2 = \ln t^2$;
(2) $x_1 = \ln t^2$; $x_2 = t^2$.

(c) Find $z'(t)$ by substituting the functions of (1) and (2) for x_1 and x_2, and then differentiate them.

10. Given is the function

$$z = f(x, y) = \sqrt{9 - x^2 - y^2}.$$

Find the directional derivatives in direction $\mathbf{r^1} = (1, 0)^T, \mathbf{r^2} = (1, 1)^T, \mathbf{r^3} = (-1, -2)^T$ at point (1,2).

11. Assume that

$$C(x_1, x_2, x_3) = 20 + 2x_1x_2 + 8x_3 + x_2 \ln x_3 + 4x_1$$

is the total cost function of three products, where x_1, x_2, x_3 are the outputs of these three products.

(a) Find the gradient and the directional derivative with the directional vector $\mathbf{r} = (1, 2, 3)^T$ of function C at point $(3, 2, 1)$. Compare the growth of the cost (marginal cost) in direction of fastest growth with the directional marginal cost in the direction $\mathbf{r}$. Find the percentage rate of cost reduction at point $(3, 2, 1)$.

(b) The owner of the firm wants to increase the output by six units altogether. The owner can do it in the ratio of 1:2:3 or of 3:2:1 for the products $x_1 : x_2 : x_3$. Further conditions are $x_1 \geq 1$, $x_3 \geq 1$, and the output x_2 must be at least four units. Which ratio leads to lower cost for the firm?

12. Success of sales z for a product depends on a promotion campaign in two media. Let x_1 and x_2 be the funds invested in the two media. Then the following function is to be used to reflect the relationship:

$$z = f(x_1, x_2) = 10\sqrt{x_1} + 20 \ln(x_2 + 1) + 50; \qquad x_1 \geq 0, x_2 \geq 0.$$

Find the partial rates of change and the partial elasticities of function f at point $(x_1, x_2) = (100, 150)$.

13. Determine whether the following function is homogeneous. If this is the case, what is the degree of homogenuity? Use Euler's theorem and interpret the result.

(a) $z = f(x_1, x_2) = \sqrt{x_1^3 + 2x_1x_2^2 + x_2^3}$; (b) $z = f(x_1, x_2) = x_1x_2^2 + x_1^2$.

14. Let $F(x, y) = 0$ be an implicitly defined function. Find dy/dx by the implicit-function rule.

(a) $F(x, y) = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0 \quad (y \geq 0)$; (b) $F(x, y) = xy - \sin 3x = 0$;

(c) $F(x, y) = x^y - \ln xy + x^2y = 0$.

15. We consider the representation of variables x and y by so-called polar coordinates:

$$x = r\cos\varphi, \qquad y = r\sin\varphi,$$

or equivalently, the implicitly given system

$$\begin{aligned} F_1(x, y; r, \varphi) &= r\cos\varphi - x = 0 \\ F_2(x, y; r, \varphi) &= r\sin\varphi - y = 0. \end{aligned}$$

Check by means of the Jacobian determinant whether this system can be put into its reduced form, i.e. whether variables r and φ can be expressed in terms of x and y.

16. Check whether the following function $f : D_f \to \mathbb{R}, D_f \subseteq \mathbb{R}^2$, with

$$z = f(x, y) = x^3y^2(1 - x - y)$$

has a local maximum at point $(x_1, y_1) = (1/2, 1/3)$ and a local minimum at point $(x_2, y_2) = (1/7, 1/7)$.

17. Find the local extrema of the following functions $f : D_f \to \mathbb{R},\ D_f \subseteq \mathbb{R}^2$:

(a) $z = f(x, y) = x^2y + y^2 - y$; (b) $z = f(x, y) = x^2y - 2xy + \frac{3}{4}e^y$.

18. The variable cost of two products P_1 and P_2 depends on the production outputs x and y as follows:

$$C(x, y) = 120x + \frac{1,200,000}{x} + 800y + \frac{32,000,000}{y},$$

where $D_C = \mathbb{R}^2_+$. Determine the outputs x^0 and y^0 which minimize the cost function and determine the minimum cost.

19. Given is the function $f : \mathbb{R}^3 \to \mathbb{R}$ with

$$f(x, y, z) = x^2 - 2x + y^2 - 2z^3y + 3z^2.$$

Find all stationary points and check whether they are local extreme points.

20. Find the local extrema of function $C : D_C \to \mathbb{R},\ D_C \subseteq \mathbb{R}^3$, with

$$C(\mathbf{x}) = C(x_1, x_2, x_3) = 20 + 2x_1x_2 + 8x_3 + x_2\ln x_3 + 4x_1$$

and $\mathbf{x} \in \mathbb{R}^3,\ x_3 > 0$.

21. The profit P of a firm depends on three positive input factors x_1, x_2, x_3 as follows:

$$P(x_1, x_2, x_3) = 90x_1x_2 - x_1^2x_2 - x_1x_2^2 + 60\ln x_3 - 4x_3.$$

Determine input factors which maximize the profit function and find the maximum profit.

22. Sales y of a firm depend on the expenses x of advertising. The following values x_i of expenses and corresponding sales y_i of the last 10 months are known:

x_i	20	20	24	25	26	28	30	30	33	34
y_i	180	160	200	250	220	250	250	310	330	280

(a) Find a linear function f by applying the criterion of minimizing the sum of the squared differences between y_i and $f(x_i)$.

(b) Which sales can be expected if $x = 18$, and if $x = 36$?

23. Given the implicitly defined functions

$$F(x, y) = \frac{(x-1)^2}{4} + \frac{(y-2)^2}{9} - 1 = 0.$$

Verify that F has local extrema at points $P_1 : (x = 1, y = 5)$ and $P_2 : (x = 1, y = -1)$. Decide whether they are a local maximum or a local minimum point and graph the function.

24. Find the constrained optima of the following functions:

(a) $z = f(x, y) = x^2 + xy - 2y^2$,
subject to

$$2x + y = 8;$$

(b) $z = f(x_1, x_2, x_3) = 3x_1^2 + 2x_2 - x_1x_2x_3$,
subject to

$$x_1 + x_2 = 3 \qquad \text{and} \qquad 2x_1x_3 = 5.$$

25. Check whether the function $f : D_f \to \mathbb{R}, D_f \subseteq \mathbb{R}^3$, with

$$z = f(x, y, z) = x^2 - xz + y^3 + y^2z - 2z,$$

subject to

$$x - y^2 - z - 2 = 0 \qquad \text{and} \qquad x + z - 4 = 0$$

has a local extremum at point $(11, -4, -7)$.

26. Find the dimension of a box for washing powder with a double bottom so that the surface is minimal and the volume amounts to 3,000 cm^3. How much card board is required if glued areas are not considered?

27. Find all the points (x, y) of the ellipse $4x^2 + y^2 = 4$ which have a minimal distance from point $P_0 = (2, 0)$ (use Lagrange multiplier method first, then substitute the constraint into the objective function).

28. The cost function $C : \mathbb{R}^3_+ \to \mathbb{R}$ of a firm producing the quantities x_1, x_2 and x_3 is given by

$$C(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 3x_1x_3 + 500.$$

The firm has to fulfil the constraint

$$2x_1 + 4x_2 + 3x_3 = 125.$$

Find the minimum cost (use Langrange multiplier method as well as optimization by substitution).