EXERCISES CHAPTER 4

1. Find the left-side limit and the right-side limit of function $f : D_f \to \mathbb{R}$ as x approaches x_0. Can we conclude from these answers that function f has a limit as x approaches x_0?

   (a) $f(x) = \begin{cases} a & \text{for} \quad x \neq x_0 \\ a+1 & \text{for} \quad x = x_0 \end{cases}$; (b) $f(x) = \begin{cases} \sqrt{x} & \text{for} \quad x \leq 1 \\ x^2 & \text{for} \quad x > 1 \end{cases}$, $x_0 = 1$;

   (c) $f(x) = |x|, \quad x_0 = 0$; (d) $f(x) = \begin{cases} x & \text{for} \quad x < 1 \\ x+1 & \text{for} \quad x \geq 1 \end{cases}$, $x_0 = 1$.

2. Find the following limits if they exist:

   (a) $\lim\limits_{x\to 2} \dfrac{x^3 - 3x^2 + 2x}{x-2}$; (b) $\lim\limits_{x\to 2} \dfrac{x^3 - 3x^2}{x-2}$; (c) $\lim\limits_{x\to 2} \dfrac{x^3 - 3x^2}{(x-2)^2}$.

   Which is the type of discontinuity at point $x_0 = 2$?

3. Check the continuity of function $f : D_f \to \mathbb{R}$ at point $x = x_0$ and define the type of discontinuity:

   (a) $f(x) = \dfrac{\sqrt{x+7} - 3}{x-2}$, $x_0 = 2$;

   (b) $f(x) = |x - 1|$, $x_0 = 1$;

   (c) $f(x) = \begin{cases} e^{x-1} & \text{for} \quad x < 1 \\ 2x & \text{for} \quad x \geq 1 \end{cases}$, $x_0 = 1$;

   (d) $f(x) = e^{1/(x-1)}$, $x_0 = 1$.

4. Are the following functions $f : D_f \to \mathbb{R}$ differentiable at points $x = x_0$ and $x = x_1$, respectively?

   (a) $f(x) = |x - 5| + 6x$, $x_0 = 5, \quad x_1 = 0$;

   (b) $f(x) = \begin{cases} \cos x & \text{for} \quad x < 0; \\ 1 + x^2 & \text{for} \quad 0 \leq x \leq 2 \\ 2x + 1 & \text{for} \quad x > 2. \end{cases}$, $x_0 = 0, \quad x_1 = 2$;

5. Find the derivative of each of the following functions $f : D_f \to \mathbb{R}$ with:

   (a) $y = 2x^3 - 5x - 3\sin x + \sin(\pi/8)$; (b) $y = (x^4 + 4x)\sin x$;

   (c) $y = \dfrac{x^2 - \cos x}{2 + \sin x}$; (d) $y = (2x^3 - 3x + \ln x)^4$;

   (e) $y = \cos(x^3 + 3x^2 - 8)^4$; (f) $y = \cos^4(x^3 + 3x^2 - 8)$;

   (g) $y = \sqrt{\sin(e^x)}$; (h) $y = \ln(x^2 + 1)$.

6. Find and simplify the derivative of each of the following functions $f : D_f \to \mathbb{R}$ with:

   (a) $f(x) = (\tan x - 1)\cos x$; (b) $f(x) = \ln\sqrt{\frac{1+\sin x}{1-\sin x}}$;

   (c) $f(x) = (1 + \sqrt[3]{x})^2$.

7. Find the derivatives of the following functions $f : D_f \to \mathbb{R}$ by using logarithmic differentiation:

   (a) $f(x) = (\tan x)^x$, $D_f = (0, \pi/2)$; (b) $f(x) = \sin x^{x-1}$, $D_f = (1, \infty)$;

   (c) $f(x) = \frac{(x+2)\sqrt{x-1}}{x^3(x-2)^2}$.

8. Find the third derivatives of the following functions $f : D_f \to \mathbb{R}$ with:

   (a) $f(x) = x^2 \sin x$; b) $f(x) = \ln(x^2)$;

   (c) $f(x) = \frac{2x^2}{(x-2)^3}$; d) $f(x) = (x+1)e^x$.

9. Given the total cost function $C : \mathbb{R}_+ \to \mathbb{R}$ with

$$C(x) = 4x^3 - 2x^2 + 4x + 100,$$

   where $C(x)$ denotes the total cost in dependence on the output x. Find the marginal cost at $x_0 = 2$. Compute the change in total cost resulting from an increase in the output from 2 to 2.5 units. Give an approximation of the exact value by using the differential.

10. Let $D : D_D \to \mathbb{R}_+$ be a demand function, where $D(p)$ is the quantity demand in dependence on the price p. Given $D(p) = 320/p$. Calculate approximately the change of quantity demand if p changes from 8 to 10 EUR.

11. Given the functions $f : D_f \to \mathbb{R}_+$ with

$$f(x) = 2e^{x/2}$$

   and $g : D_g \to \mathbb{R}_+$ with

$$g(x) = 3\sqrt{x}.$$

   Find the proportional rates of change $\rho_f(x)$, $\rho_g(x)$ and the elasticities $\varepsilon_f(x)$ and $\varepsilon_g(x)$ and specify them for $x_0 = 1$ and $x_1 = 100$. For which values $x \in D_f$ are the functions f and g elastic. Give the percentage rate of change of the function value when value x increases by one per cent.

12. Given the price-demand-function

$$D = D(p) = 1,000e^{-2(p-1)^2}$$

   with demand $D > 0$ and price $p > 0$, find the (point) elasticity of demand $\varepsilon_D(p)$. Check at which prices the demand is elastic.

13. Find all local extrema, the global maximum and the global minimum of the following functions $f : D_f \to \mathbb{R}$ with $D_f \subseteq [-5, 5]$, where:

(a) $f(x) = x^4 - 3x^3 + x^2 - 5$; (b) $f(x) = 4 - |x - 3|$;

(c) $f(x) = e^{-x^2/2}$; (d) $f(x) = \dfrac{x^2}{x-2}$;

(e) $f(x) = \dfrac{\sqrt{x}}{1+\sqrt{x}}$.

14. Assume that function $f : D_f \to \mathbb{R}$ with

$$f(x) = a \ln x + bx^2 + x$$

has local extrema at point $x_1 = 1$ and at point $x_2 = 2$. What can you conclude about the values of a and b? Check whether they are relative maxima or minima.

15. Determine the following limits by Bernoulli-l'Hospital's rule:

(a) $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$; (b) $\lim\limits_{x\to 1} \dfrac{e^{2(x-1)} - x^2}{(x^2-1)^2}$;

(c) $\lim\limits_{x\to\infty} (x^2)^{1/x}$; (d) $\lim\limits_{x\to 0+0} x^x$;

(e) $\lim\limits_{x\to 0} \left(\dfrac{1}{x \sin x} - \dfrac{1}{x^2}\right)$; (f) $\lim\limits_{x\to 0} \dfrac{1}{x^2} \ln \dfrac{\sin x}{x}$.

16. For the following functions $f : D_f \to \mathbb{R}$, determine and investigate domains, zeros, discontinuities, monotonicity, extreme points and extreme values, convexity and concavity, inflection points and limits as x tends to $\pm\infty$. Graph the functions f with $f(x)$ given as follows:

(a) $f(x) = \dfrac{x^2+1}{(x-2)^2}$; (b) $f(x) = \dfrac{3x^2-4x}{-2x^2+x}$;

(c) $f(x) = \dfrac{x^4+x^3}{x^3-2x^2+x}$; (d) $f(x) = e^{(x-1)^2/2}$;

(e) $f(x) = \ln \dfrac{x-2}{x^2}$; (f) $f(x) = \sqrt[3]{2x^2 - x^3}$.

17. Expand the following functions $f : D_f \to \mathbb{R}$ into Taylor polynomials with corresponding remainder:

(a) $f(x) = \sin \dfrac{\pi x}{4}$, $x_0 = 2$, $n = 5$;

(b) $f(x) = \ln(x+1)$, $x_0 = 0$, $n \in \mathbb{N}$;

(c) $f(x) = e^{-x} \sin 2x$, $x_0 = 0$, $n = 4$.

18. Calculate $1/\sqrt[5]{e}$ by using Taylor polynomials for function $f : D_f \to \mathbb{R}$ with $f(x) = e^x$. The error should be less than 10^{-6}.

19. Determine the zero $\overline{x}$ of function $f : D_f \to \mathbb{R}$ with

$$f(x) = x^3 - 6x + 2 \qquad \text{and} \qquad 0 \le \overline{x} \le 1$$

exactly to four decimal places. Use Newton's method and Regula falsi.

20. Find the zero $\overline{x}$ of function $f : D_f \to \mathbb{R}$ with

$$f(x) = x - \ln x - 3 \qquad \text{and} \qquad \overline{x} > 1.$$

Determine the value with an error less than 10^{-5} and use Newton's method.