EXERCISES CHAPTER 6

1. Given are the vectors
$$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ -4 \\ -2 \end{pmatrix} \qquad \text{and} \qquad \mathbf{c} = \begin{pmatrix} 2 \\ 2 \\ 6 \end{pmatrix}.$$
   (a) Find vectors $\mathbf{a}+\mathbf{b}-\mathbf{c}$, $\mathbf{a}+3\mathbf{b}$, $\mathbf{b}-4\mathbf{a}+2\mathbf{c}$, $\mathbf{a}+3(\mathbf{b}-2\mathbf{c})$.
   (b) For which of the vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ do the relations $>$ or $\geq$ hold?
   (c) Find the scalar products $\mathbf{a}^T \cdot \mathbf{b}$, $\mathbf{a}^T \cdot \mathbf{c}$, $\mathbf{b}^T \cdot \mathbf{c}$. Which of the vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are orthogonal? What is the angle between the vectors $\mathbf{b}$ and $\mathbf{c}$?
   (d) Compute vectors $(\mathbf{a}^T \cdot \mathbf{b}) \cdot \mathbf{c}$ and $\mathbf{a} \cdot (\mathbf{b}^T \cdot \mathbf{c})$.
   (e) Compare number $|\mathbf{b}+\mathbf{c}|$ with number $|\mathbf{b}|+|\mathbf{c}|$ and number $|\mathbf{b}^T \cdot \mathbf{c}|$ with number $|\mathbf{b}| \cdot |\mathbf{c}|$.

2. Find α and β so that vectors
$$\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ \alpha \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} \beta \\ 4 \\ -2 \end{pmatrix}$$
are orthogonal.

3. (a) What is the distance between the following points: $(1,2,3)$ and $(4,-1,2)$ in the 3-dimensional Euclidean space $\mathbb{R}^3$?
   (b) Illustrate the following sets of points in $\mathbb{R}^2$: $\mathbf{a} \geq \mathbf{b}$ and $|\mathbf{a}| \geq |\mathbf{b}|$.

4. Given are the vectors
$$\mathbf{a^1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{a^2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$
Find out which of the vectors
$$\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 \\ 0.5 \end{pmatrix}$$
are linear combinations of $\mathbf{a^1}$ and $\mathbf{a^2}$. Is one of the above vectors a convex combination of vectors $\mathbf{a^1}$ and $\mathbf{a^2}$? Graph all these vectors.

5. Given are the vectors
$$\mathbf{a^1} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \quad \mathbf{a^2} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}, \quad \mathbf{a^3} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{a^4} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$
Show that vector $\mathbf{a^4}$ can be expressed as a convex linear combination of vectors $\mathbf{a^1}$, $\mathbf{a^2}$ and $\mathbf{a^3}$. Find the convex combinations of vectors $\mathbf{a^1}$, $\mathbf{a^2}$ and $\mathbf{a^3}$ graphically.

6. Are the vectors

$$\mathbf{a^1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{a^2} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{a^3} = \begin{pmatrix} 5 \\ 4 \\ -2 \end{pmatrix}$$

linearly independent?

7. Do the two vectors

$$\mathbf{a^1} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{a^2} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$$

span the 2-dimensional space? Do they constitute a basis? Graph the vectors and illustrate their linear combinations.

8. Do the vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

constitute a basis in $\mathbb{R}^4$?

9. Let vectors

$$\mathbf{a^1} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \qquad \mathbf{a^2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{a^3} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

constitute a basis in $\mathbb{R}^3$.

(a) Express vector

$$\mathbf{a} = \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix}$$

as a linear combination of the three vectors $\mathbf{a^1}$, $\mathbf{a^2}$ and $\mathbf{a^3}$ above.

(b) Find all other bases for the 3-dimensional space which include vector $\mathbf{a}$ and vectors from the set $\{\mathbf{a^1}, \mathbf{a^2}, \mathbf{a^3}\}$.

(c) Express vector

$$\mathbf{b} = 2\mathbf{a^1} + 2\mathbf{a^2} + 3\mathbf{a^3} = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$$

by the basis vectors $\mathbf{a^1}, \mathbf{a^2}$ and $\mathbf{a}$.