EXERCISES CHAPTER 7

1. Given are the matrices

$$A = \begin{pmatrix} 3 & 4 & 1 \\ 1 & 2 & -2 \end{pmatrix}; \qquad B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ -1 & 0 \end{pmatrix};$$

$$C = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 3 & 1 & 0 \\ 4 & 2 & 2 \end{pmatrix}.$$

   (a) Find the transposes. Check whether some matrices are equal.
   (b) Calculate $A + D, A - D, A^T - B$ and $C - D$.
   (c) Find $A + 3(B^T - 2D)$.

2. Find a symmetric and an antisymmetric matrix so that their sum is equal to

$$A = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 1 & -1 \\ 3 & 0 & -2 \end{pmatrix}.$$

3. Calculate all defined products of matrices A and B:

   (a) $A = \begin{pmatrix} 4 & 2 \\ 3 & 2 \end{pmatrix}; \quad B = \begin{pmatrix} 3 & 5 & -1 \\ 7 & 6 & 2 \end{pmatrix};$

   (b) $A = \begin{pmatrix} 4 & 3 & 5 & 3 \\ 2 & 5 & 0 & 1 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & 2 & 6 \\ 4 & 2 & 3 \\ 4 & 5 & 2 \\ 3 & 4 & 5 \end{pmatrix};$

   (c) $A = \begin{pmatrix} 2 & 3 & 4 & 5 \end{pmatrix}; \quad B = \begin{pmatrix} -3 \\ 6 \\ -3 \\ 2 \end{pmatrix};$

   (d) $A = \begin{pmatrix} 2 & -1 & 3 \\ -4 & 2 & -6 \end{pmatrix}; \quad B = \begin{pmatrix} 2 & 3 \\ 1 & 0 \\ -1 & -2 \end{pmatrix};$

   (e) $A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 5 & 2 \end{pmatrix}; \quad B = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$

4. Use the matrices given in Exercise 7.3 (d) and verify the equalities

$$A^T B^T = (BA)^T \qquad \text{and} \qquad B^T A^T = (AB)^T.$$

5. Given are the matrices

$$A = \begin{pmatrix} 1 & 0 \\ 7 & -4 \\ 5 & 3 \end{pmatrix}; \qquad B = \begin{pmatrix} -1 \\ 2 \\ 0 \\ 7 \end{pmatrix} \qquad \text{and} \qquad C = \begin{pmatrix} -2 & 0 & 1 & 0 \\ 0 & 3 & 0 & 6 \end{pmatrix}.$$

(a) Find the dimension of a product of all three matrices if possible.

(b) Test the associative law of multiplication with the given matrices.

6. Calculate all powers of the following matrices:

$$\text{(a) } A = \begin{pmatrix} 0 & 2 & 8 & 1 \\ 0 & 0 & 7 & 3 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \qquad \text{(b) } B = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}.$$

7. A firm produces by means of two raw materials R_1 and R_2 three intermediate products S_1, S_2 and S_3, and with these intermediate products two final products F_1 and F_2. The numbers of units of R_1 and R_2, which are necessary for the production of one unit of S_j, $j \in \{1,2,3\}$, and the numbers of units of S_1, S_2 and S_3, which are necessary for one unit of F_1 and F_2, are given in the following tables:

	S_1	S_2	S_3
R_1	2	3	5
R_2	5	4	1

	F_1	F_2
S_1	6	0
S_2	1	4
S_3	3	2

Solve the problem by means of matrix operations.

(a) How many raw materials are required when 1,000 units of F_1 and 2,000 units of F_2 have to be produced?

(b) The costs for one unit of raw material are 3 EUR for R_1 and 5 EUR for R_2. Calculate the costs for intermediate and final products.

8. Given is the matrix

$$A = \begin{pmatrix} 2 & 3 & 0 \\ -1 & 2 & 4 \\ 0 & 5 & 1 \end{pmatrix}.$$

(a) Find the submatrices A_{12} and A_{22}.

(b) Calculate the minors $|A_{12}|$ and $|A_{22}|$.

(c) Calculate the cofactors of the elements a_{11}, a_{21}, a_{31} of matrix A.

(d) Evaluate the determinant of matrix A.

9. Evaluate the following determinants:

$$\text{(a) } \begin{vmatrix} 2 & 1 & 6 \\ -1 & 0 & 3 \\ 3 & 2 & 9 \end{vmatrix}; \qquad \text{(b) } \begin{vmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{vmatrix}; \qquad \text{(c) } \begin{vmatrix} 1 & 0 & 1 & 2 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 0 \end{vmatrix};$$

$$\text{(d) } \begin{vmatrix} 2 & 7 & 4 & 1 \\ 3 & 1 & 4 & 0 \\ 5 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{vmatrix}; \qquad \text{(e) } \begin{vmatrix} -1 & 2 & 4 & 3 \\ 2 & -4 & -8 & -6 \\ 7 & 1 & 5 & 0 \\ 1 & 5 & 0 & 1 \end{vmatrix}; \qquad \text{(f) } \begin{vmatrix} 3 & 3 & 3 & \cdots & 3 & 3 \\ 3 & 0 & 3 & \cdots & 3 & 3 \\ 3 & 3 & 0 & \cdots & 3 & 3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 3 & 3 & 3 & \cdots & 3 & 0 \end{vmatrix}_{(n,n)}.$$

10. Find the solutions x of the following equations:

(a) $\begin{vmatrix} -1 & x & x \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{vmatrix} = 27;$ (b) $\begin{vmatrix} x & 1 & 2 \\ 3 & x & -1 \\ 4 & x & -2 \end{vmatrix} = 2.$

11. Find the solution of the following systems of equations by Cramer's rule:

$$\begin{array}{rrrcr} 2x_1 & +4x_2 & +3x_3 & = & 1 \\ 3x_1 & -6x_2 & -2x_3 & = & -2 \\ -5x_1 & +8x_2 & +2x_3 & = & 4 \end{array}.$$

12. Let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear mapping described by matrix

$$A = \begin{pmatrix} 3 & 1 & 0 \\ -1 & 2 & 4 \\ 4 & 1 & 5 \end{pmatrix}.$$

Find the kernel of this mapping.

13. Given are three linear mappings described by the following systems of equations:

$$\begin{array}{rcrrr} u_1 & = & v_1 & -v_2 & +v_3 \\ u_2 & = & 2v_1 & -v_2 & -v_3 \\ u_3 & = & -v_1 & +v_2 & +2v_3, \end{array}$$

$$\begin{array}{rcrrr} v_1 & = & -w_1 & & +w_3 \\ v_2 & = & w_1 & +2w_2 & -w_3 \\ v_3 & = & & w_2 & -2w_3, \end{array}$$

$$\begin{array}{rcrrr} w_1 & = & x_1 & -x_2 & -x_3 \\ w_2 & = & -x_1 & -2x_2 & +3x_3 \\ w_3 & = & 2x_1 & & +x_3. \end{array}$$

Find the composite mapping $\mathbf{x} \in \mathbb{R}^3 \mapsto \mathbf{u} \in \mathbb{R}^3$.

14. Find the inverse of each of the following matrices:

(a) $A = \begin{pmatrix} 1 & 0 & 3 \\ 4 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix};$ (b) $B = \begin{pmatrix} 2 & -3 & 1 \\ 3 & 4 & -2 \\ 5 & 1 & -1 \end{pmatrix};$

(c) $C = \begin{pmatrix} 1 & 3 & -2 \\ 0 & 2 & 4 \\ 0 & 0 & -1 \end{pmatrix};$ (d) $D = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 2 & -1 & -2 & 3 \\ -1 & 2 & 2 & -4 \\ 0 & 1 & 2 & -5 \end{pmatrix}.$

15. Let

$$A = \begin{pmatrix} 1 & 2 & -1 & 0 & 4 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Find the inverse matrix by means of equality $A = I - C$.

16. Given are the matrices

$$A = \begin{pmatrix} -2 & 5 \\ 1 & -3 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 4 \\ -2 & 9 \end{pmatrix}.$$

Find $(AB)^{-1}$ and $B^{-1}A^{-1}$.

17. Given are the following matrix equations:

(a) $(XA)^T = B$; (b) $XA = B - 2X$; (c) $AXB = C$;
(d) $A(XB)^{-1} = C$; (e) $C^T XA + (X^T C)^T = I - 3C^T X$.

Find matrix X.

18. Given is an open Input-Output-Model (Leontief model) with

$$A = \begin{pmatrix} 0 & 0.2 & 0.1 & 0.3 \\ 0 & 0 & 0.2 & 0.5 \\ 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 \end{pmatrix}.$$

Let $\mathbf{x}$ be the total vector of goods produced and $\mathbf{y}$ the final demand vector.

(a) Explain the economic meaning of the elements of A.

(b) Find the linear mapping which maps all the vectors $\mathbf{x}$ into the set of all final demand vectors $\mathbf{y}$.

(c) Is a vector

$$\mathbf{x} = \begin{pmatrix} 100 \\ 200 \\ 200 \\ 400 \end{pmatrix}$$

of goods produced possible for some final demand vector $\mathbf{y}$?

(d) Find the inverse mapping of (b) and interpret it economically.

19. A firm produces by means of three factors R_1, R_2 and R_3 five products $P_1, P_2, \ldots, P_5$, where some of these products are also used as intermediate products. The relationships are given in the graph presented in Figure 1. The numbers besides the arrows describe how many units of R_i respectively P_i are necessary for one unit of P_j. Let p_i denote the produced units (output) of P_i and q_i denote the final demand for the output of P_i, $i \in \{1, 2, \ldots, 5\}$.

(a) Find a linear mapping $\mathbf{p} \in \mathbb{R}^5_+ \mapsto \mathbf{q} \in \mathbb{R}^5_+$.

(b) Find the inverse mapping.

(c) Let $\mathbf{r}^T = (r_1, r_2, r_3)$ be the vector which contains the required units of the factors R_1, R_2, R_3. Find a linear mapping $\mathbf{q} \mapsto \mathbf{r}$. Calculate $\mathbf{r}$ when $\mathbf{q} = (50, 40, 30, 20, 10)^T$.

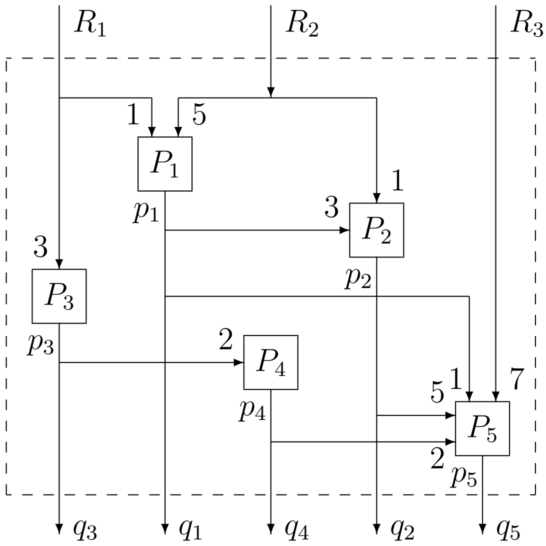

Figure 1. Relationships between raw materials and products in Exercise 19