EXERCISES CHAPTER 8

1. Decide whether the following systems of linear equations are consistent and find the solutions. Train both methods of Gaussian elimination and pivoting (use the rank criterion).

(a)
$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & 3x_3 & = & 5 \\ 2x_1 & + & 3x_2 & + & x_3 & = & 8 \\ 3x_1 & + & x_2 & + & 2x_3 & = & 5 \end{array}$$

(b)
$$\begin{array}{rcrcrcl} 3x_1 & + & x_2 & + & x_3 & = & 3 \\ 4x_1 & - & x_2 & + & 2x_3 & = & 4 \\ x_1 & - & x_2 & + & x_3 & = & 1 \\ 4x_1 & - & x_2 & + & 2x_3 & = & 5 \end{array}$$

(c)
$$\begin{array}{rcrcrcrcl} 3x_1 & + & 4x_2 & + & x_3 & + & 6x_4 & = & 8 \\ 3x_1 & + & 8x_2 & + & 6x_3 & + & 5x_4 & = & 7 \\ 8x_1 & + & 5x_2 & + & 6x_3 & + & 7x_4 & = & 6 \\ 6x_1 & + & 2x_2 & + & 5x_3 & + & 3x_4 & = & 5 \end{array}$$

(d)
$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & 3x_3 & = & 16 \\ 8x_1 & + & 7x_2 & + & 6x_3 & = & 74 \\ 4x_1 & + & 5x_2 & + & 9x_3 & = & 49 \\ 5x_1 & + & 4x_2 & + & 2x_3 & = & 43 \\ x_1 & + & 4x_2 & + & x_3 & = & 22. \end{array}$$

2. (a) Solve the following homogeneous systems of linear equations $A\mathbf{x} = \mathbf{0}$ with

(i) $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 \\ 1 & 1 & 2 & 4 \\ 2 & 3 & 5 & 0 \end{pmatrix}$; (ii) $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 6 & 3 \\ 1 & 1 & 2 & 4 \\ 2 & 3 & 5 & 0 \end{pmatrix}$

(b) Find the solutions of the nonhomogeneous system $A\mathbf{x} = \mathbf{b}$ with matrix A from (i) resp. (ii) and $\mathbf{b} = (0, 0, -2, 2)^T$.

3. Find the general solutions of the following systems and specify two different basic solutions for each system:

(a)
$$\begin{array}{rcrcrcrcr} 4x_1 & - & x_2 & - & 3x_3 & + & 5x_4 & = & -2 \\ -x_1 & + & x_2 & + & x_3 & - & x_4 & = & 4 \\ 2x_1 & - & x_2 & - & x_3 & - & 2x_4 & = & 1 \end{array}$$

(b)
$$\begin{array}{rcrcrcrcrcr} x_1 & + & 2x_2 & - & x_3 & & & + & 4x_5 & = & 2 \\ x_1 & + & 4x_2 & - & 5x_3 & + & x_4 & + & 3x_5 & = & 1 \\ 2x_1 & - & 2x_2 & + & 10x_3 & + & x_4 & - & x_5 & = & 11 \\ 3x_1 & + & 2x_2 & + & 5x_3 & + & 2x_4 & + & 2x_5 & = & 12 \end{array}$$

4. What restriction on the parameter a does ensure the consistence of the following system? Find the solution depending on a.

$$\begin{array}{rcrcrcl} 3x & + & 4y & + & 4z & = & 2 \\ 3x & + & 2y & + & 3z & = & 3 \\ 4x & + & 5y & + & az & = & 4 \end{array}$$

5. Check the consistence of the following system as a function of the parameter λ:

$$\begin{array}{rcrcrcl} 3x & + & 2y & + & z & = & 0 \\ 6x & + & 4y & + & \lambda z & = & 0 \end{array}$$

Do the following cases exist:

(a) there is no solution;
(b) there is a unique solution.

Find the solution if possible.

6. Given is the system

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & a & 0 \\ 0 & 1 & 0 & b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix}.$$

   (a) Decide with respect to the parameters a and b in which cases the system is consistent or inconsistent. When does a unique solution exist?
   (b) Find the general solution if possible.
   (c) Calculate the solution for $a = 1$ and $b = 0$.

7. Decide whether the following system of linear equations is consistent and find the solution in dependence on parameters a and b:

$$\begin{array}{rcrcrcl} a\,x & + & (a+b)\,y & + & b\,z & = & 3a+5b \\ b\,x & + & ab\,y & + & a\,z & = & a(2b+3)+b \\ a\,x & + & b\,y & + & b\,z & = & a+5b \end{array}$$

8. Find the kernel of the linear mapping described by matrix

$$A = \begin{pmatrix} 2 & -2 & 2 \\ 5 & -1 & 7 \\ 3 & -1 & 4 \end{pmatrix}.$$

9. Given are the two systems of linear equations

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & 3x_3 & = & 5 \\ 3x_1 & + & 1x_2 & + & 2x_3 & = & 5 \\ 2x_1 & + & 3x_2 & + & 1x_3 & = & 8 \end{array} \qquad \text{and} \qquad \begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & 3x_3 & = & 14 \\ 2x_1 & + & 3x_2 & + & x_3 & = & 13 \\ 3x_1 & + & x_2 & + & 2x_3 & = & 21. \end{array}$$

   Solve them in only one tableau.

10. Find the inverse matrices by pivoting:

(a) $A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 5 & 3 \\ -3 & -8 & -4 \end{pmatrix}$; (b) $B = \begin{pmatrix} 1 & 4 & 6 \\ 3 & 2 & 1 \\ 7 & 8 & 8 \end{pmatrix}$;

(c) $C = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 2 & -1 & -2 & 3 \\ -1 & 2 & 2 & -4 \\ 0 & 1 & 2 & -5 \end{pmatrix}$.

11. Find the solutions of the following matrix equations:

    (a) $XA = B$ with

$$A = \begin{pmatrix} -1 & 3 & 2 \\ 2 & 5 & 3 \\ -3 & -8 & -4 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & -4 & 2 \\ 3 & 3 & -2 \end{pmatrix};$$

    (b) $AXB = C$ with

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \qquad \text{and} \qquad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

    (c) $XA = -2(X + B)$ with

$$A = \begin{pmatrix} -3 & 1 & 1 \\ -1 & 0 & 1 \\ 2 & 1 & -2 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 2 & -4 & 0 \\ 1 & 3 & -2 \end{pmatrix}.$$

12. Assume we have a 4-industry economy. Each industry is to produce an output just sufficient to meet the input requirements of itself and of the other three industries as well as the final demand of the open sector. This is, the output level x_i must satisfy the following equation:

$$x_i = x_{i1} + x_{i2} + x_{i3} + x_{i4} + y_i, \quad i \in \{1, 2, 3, 4\},$$

where y_i is the final demand for industry i and x_{ij} is the amount of x_i needed as input for the industry $j \in \{1, 2, 3, 4\}$. Let

$$X = (x_{ij}) = \begin{pmatrix} 5 & 15 & 15 & 10 \\ 25 & 25 & 10 & 30 \\ 10 & 20 & 20 & 20 \\ 10 & 30 & 15 & 25 \end{pmatrix} \qquad \text{and} \qquad \mathbf{y} = \begin{pmatrix} 5 \\ 10 \\ 30 \\ 20 \end{pmatrix}.$$

    (a) Find the input-coefficient matrix A which satisfies the equation

$$\mathbf{x} = A\mathbf{x} + \mathbf{y}$$

    with $\mathbf{x} = (x_1, x_2, x_3, x_4)^T$ and $x_{ij} = a_{ij} x_j$.

    (b) How does the matrix X change if the final demand changes to $\mathbf{y} = (10, 20, 20, 15)^T$ and the input-coefficients are constant?

13. Given is the bounded convex set with the extreme points

$$\mathbf{x^1} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{x^2} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{x^3} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{x^4} = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}.$$

Do the points

$$\mathbf{a} = \frac{1}{2}\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

belong to the convex set given above?

14. Additionally to its main production a firm produces two products A and B with two machines I and II. The products are manufactured on machine I and packaged on machine II. Machine I has a free capacity of 40 hours and machine II can be used 20 hours. Production of one ton of A takes four minutes on machine I and six minutes on machine II. For the production of one ton of B ten minutes on machine I and three minutes on machine II are required. What output combinations are possible?

    (a) Model the problem by means of a system of linear inequalities.

    (b) Solve the problem graphically.

    (c) Find the general solution by calculation.

15. Solve the following problem graphically:

$$\begin{array}{rcrcl} -x_1 & + & x_2 & \leq & 4 \\ -2x_1 & + & x_2 & \leq & 3 \\ x_1 & - & 2x_2 & \leq & 1 \\ x_1 & , & x_2 & \geq & 0 \end{array}$$

16. Find the general solutions of the following systems by calculation:

$$\text{(a)}\quad \begin{array}{rcrcrcl} 2x_1 & + & 5x_2 & - & 2x_3 & \leq & 5 \\ 4x_1 & + & x_2 & + & x_3 & \leq & 3 \\ & & & & x_1, x_2, x_3 & \geq & 0 \end{array} \qquad \text{(b)}\quad \begin{array}{rcrcrcl} x_1 & + & x_2 & + & x_3 & \leq & 3 \\ x_1 & & & + & 5x_3 & \leq & 3 \\ & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$