

Graphs with Maximal Induced Matchings of the Same Size ^{*}

Philippe Baptiste¹ Mikhail Y. Kovalyov² Yury L. Orlovich³
Frank Werner⁴ Igor E. Zverovich⁵

¹*Ecole Polytechnique, Laboratoire d'Informatique (LIX), Palaiseau,
France (e-mail: philippe.baptiste@polytechnique.fr)*

²*United Institute of Informatics Problems, National Academy of
Sciences of Belarus, Minsk, Belarus (e-mail:
kovalyov_my@newman.bas-net.by)*

³*Faculty of Applied Mathematics and Computer Science, Belarusian
State University, Minsk, Belarus (e-mail: orlovich@bsu.by)*

⁴*Institute of Mathematical Optimization, Otto-von-Guericke-University
of Magdeburg, Magdeburg, Germany (e-mail:
frank.werner@mathematik.uni-magdeburg.de)*

⁵*Faculty of Mechanics and Mathematics, Belarusian State University,
Minsk, Belarus (e-mail: zverovich@bsu.by)*

Abstract: A graph is *well-indumatched* if all its maximal induced matchings are of the same size. We first prove that recognizing the class \mathcal{WIM} of well-indumatched graphs is a co-NP-complete problem even for $(2P_5, K_{1,5})$ -free graphs. We then show that the well-known decision problems such as INDEPENDENT DOMINATING SET, INDEPENDENT SET, and DOMINATING SET are NP-complete for well-indumatched graphs. We also show that \mathcal{WIM} is a co-indumatching hereditary class and characterize well-indumatched graphs in terms of forbidden co-indumatching subgraphs. However, we prove that recognizing co-indumatching subgraphs is an NP-complete problem. A graph G is *perfectly well-indumatched* if every induced subgraph of G is well-indumatched. We characterize the class of perfectly well-indumatched graphs in terms of forbidden induced subgraphs. Finally, we show that both INDEPENDENT DOMINATING SET and INDEPENDENT SET can be solved in polynomial time for perfectly well-indumatched graphs, even in their weighted versions, but DOMINATING SET is still NP-complete.

Keywords: Combinatorial mathematics, Graph theory, Maximal induced matching, Well-covered graph, Optimization problems, Computational complexity.

1. INTRODUCTION

In this paper, we study the recognition problem of graphs, where all maximal (with respect to set inclusion) induced matchings have the same number of edges, and examine the computational complexity of certain fundamental graph problems such as maximum independent set, minimum dominating set, and minimum independent dominating set within these graphs.

An *induced matching* M of a graph G is a set of pairwise non-adjacent edges such that no two edges of M are joined by an edge in G . Induced matchings have applications in the areas of communication network testing (Stockmeyer and Vazirani, 1982), concurrent transmission of messages in wireless ad hoc networks (Balakrishnan et al., 2004), secure communication channels in broadcast networks (Golombic and Lewenstein, 2000), and many others.

A graph G is called *well-indumatched* if all maximal induced matchings in G have the same size. For example, the

^{*} This work has been partially supported from Belarusian BRFFR grants (Projects F10FP-001 and F11K-095), French CNRS grant (Project PICS No. 5379), and by DAAD.

graph obtained from a star $K_{1,n}$ by subdividing each edge of $K_{1,n}$ by two vertices is a well-indumatched graph. We denote by \mathcal{WIM} the class of well-indumatched graphs. An interesting property of a well-indumatched graph is that its maximal induced matching is a maximum induced matching. Therefore, the class \mathcal{WIM} forms the set of greedy instances with respect to the maximum induced matching problem, because it is solved for well-indumatched graphs by any greedy type algorithm. Greedy instances of other combinatorial problems can be defined in a similar way (see Caro et al., 1996; Tankus and Tarsi, 2007).

Caro et al. (1996) investigated the complexity of recognizing greedy instances of several combinatorial problems, one of which is the maximum independent set problem. The class of well-covered graphs is a class of greedy instances of the maximum independent set problem and has applications, in particular, in distributed computing systems (Yamashita and Kameda, 1999).

The concept of a *well-covered graph* (with all its maximal independent sets of the same size) was first introduced by Plummer (1970) and has been investigated extensively in the literature. Ravindra (1977) characterized the well-

covered trees and bipartite graphs. Finbow et al. (1993) characterized the well-covered graphs with a girth of at least 5. Then, Campbell et al. (1993) characterized the well-covered cubic graphs. In a series of papers, Finbow et al. (2004, 2009, 2010) characterized the well-covered plane triangulations. The problem of recognizing the well-covered graphs was shown to be co-NP-complete for general graphs, independently by Sankaranarayanan and Stewart (1992), and Chvátal and Slater (1993). It is co-NP-complete even for $K_{1,4}$ -free graphs (Caro et al., 1996), and it is solvable in polynomial time for $K_{1,3}$ -free graphs.

A *matching* of a graph G is a set of edges in G with no common end-vertices. The problem of graphs recognition, in which all maximal matchings have the same size, was first considered by Lesk et al. (1984). Graphs which satisfy this property are known in the literature as *equimatchable*. Lesk et al. (1984) showed that there exists a polynomial-time algorithm which decides whether a given input graph is equimatchable (see also Lovász and Plummer (1986) for a detailed description of equimatchable graphs).

The class WIM of well-indumatched graphs is a natural analogue of well-covered graphs. We show that recognizing the class WIM is a co-NP-complete problem even for $(2P_5, K_{1,5})$ -free graphs. Thus, it is unlikely that there exists a characterization of well-indumatched graphs which provides its polynomial recognition.

Let $IMatch(G)$ be the set of all maximal induced matchings of graph G . Define the *minimum maximal induced matching number* as $\sigma(G) = \min\{|M| : M \in IMatch(G)\}$ and the *maximum induced matching number* as $\Sigma(G) = \max\{|M| : M \in IMatch(G)\}$. In a greedy way we can find both $\sigma(G)$ and $\Sigma(G)$ in any well-indumatched graph G . It is well known that the decision analogue of the problem of computing $\Sigma(G)$ is NP-complete (Stockmeyer and Vazirani, 1982; Cameron, 1989). We prove NP-completeness for the problem with $\sigma(G)$ even if graphs have maximal induced matchings of at most two sizes.

Ko and Shepherd (2003) investigated relations between $\Sigma(G)$ and $\gamma(G)$, the domination number of G . They mentioned that they do not know any class of graphs for which exactly one of γ, Σ is polynomial-time computable. We show that INDEPENDENT DOMINATING SET, INDEPENDENT SET, and DOMINATING SET are NP-complete problems for well-indumatched graphs. Thus, for the class WIM , γ is hard to find, while Σ is easily computable. Our construction for the INDEPENDENT SET problem implies that the well-known problems PARTITION INTO TRIANGLES and CHORDAL GRAPH COMPLETION are NP-complete for well-indumatched graphs. Furthermore, PARTITION INTO SUBGRAPHS ISOMORPHIC TO P_3 is an NP-complete problem for well-indumatched graphs. This implies that computing Σ is NP-hard even if the input is restricted to Hamiltonian line graphs of well-indumatched graphs, which generalizes results of Kobler and Rotics (2003). Also, GRAPH k -COLORABILITY and CLIQUE are NP-complete problems for well-indumatched graphs.

We show that WIM is a co-indumatching hereditary class, i.e., it is closed under deleting the end-vertices of an induced matching along with their neighborhoods. We characterize well-indumatched graphs in terms of forbid-

den co-indumatching subgraphs. It means that we specify the minimal set of graphs \mathcal{X} such that G is well-indumatched if and only if G does not contain any graph in \mathcal{X} as a co-indumatching subgraph. Unfortunately, recognizing co-indumatching subgraphs is NP-complete.

Finally, we consider *perfectly well-indumatched* graphs, i.e., graphs in which every induced subgraph is well-indumatched. We characterize the class of perfectly well-indumatched graphs in terms of forbidden induced subgraphs, thus obtaining a new polynomial-time recognizable class, where both parameters σ and Σ are easy to compute. We show that both INDEPENDENT SET and INDEPENDENT DOMINATING SET can be solved in polynomial time for perfectly well-indumatched graphs, even in their weighted versions, but DOMINATING SET is NP-complete.

In this paper, we use standard graph-theoretic terminology of Bondy and Murty (1976). Let G be a graph with the vertex set $V = V(G)$ and the edge set $E = E(G)$. The subgraph of G induced by a set $X \subseteq V$ is denoted by $G(X)$, and $G - X = G(V \setminus X)$. $N_G(x)$ is the neighborhood and $N_G[x] = \{x\} \cup N_G(x)$ is the closed neighborhood of a vertex $x \in V(G)$. Also, for a set $X \subseteq V(G)$, $N_G(X) = \cup_{x \in X} N_G(x)$, $N_G[X] = X \cup N_G(X)$, and $EN_G(X)$ is the set of all edges in G that have at least one end-vertex in X . The complete graph, the chordless path and the chordless cycle on n vertices are denoted by K_n , P_n and C_n , respectively. Let K_0 be the *null graph*, i.e., K_0 has no vertices and edges. $K_4 - e$ is a graph obtained from the complete graph K_4 by deleting an edge. The *star* $K_{1,n}$ consists of a dominating vertex and n pendant vertices. The star $K_{1,3}$ is also known as the *claw*. We use the notation nG for the disjoint union of n copies of G . We denote by G^2 the *square* of graph G , i.e., the graph on $V(G)$ in which two vertices are adjacent if and only if they have a distance of at most 2 in G . Finally, $L(G)$ is the *line graph* of a graph G , i.e., $V(L(G)) = E(G)$, and two vertices e and e' are adjacent in $L(G)$ if and only if the edges e and e' are adjacent in G .

2. COMPLEXITY OF RECOGNIZING WELL-INDUMATCHED GRAPHS

We consider the following decision problem.

NON-WELL-INDUMATCHED GRAPHS

Instance: A graph G .

Question: Are there two maximal induced matchings M and N in G with $|M| \neq |N|$?

We first prove that NON-WELL-INDUMATCHED GRAPHS is an NP-complete problem. Then we extend this result by showing that the recognition of graphs having maximal induced matchings of at most t sizes is co-NP-complete for any value of $t \geq 1$.

For the proof of NP-completeness, we will use a polynomial-time reduction from the following well-known NP-complete problem 3-SATISFIABILITY, abbreviated as 3-SAT (see, e.g., Garey and Johnson, 1979).

3-SAT

Instance: A collection $C = \{c_1, c_2, \dots, c_m\}$ of clauses over a set $X = \{x_1, x_2, \dots, x_n\}$ of 0-1 variables such that $|c_j| = 3$ for $j = 1, 2, \dots, m$.

Question: Is there a truth assignment for X that satisfies all the clauses in C ?

Theorem 1. NON-WELL-INDUMATCHED GRAPHS is NP-complete.

Proof. Obviously, the problem is in NP. To show that it is NP-hard, we construct a polynomial-time reduction from 3-SAT. Let $C = \{c_1, c_2, \dots, c_m\}$ and $X = \{x_1, x_2, \dots, x_n\}$ be an instance of 3-SAT. Without loss of generality, we may assume that no clause in C contains a variable x_i and its negation \bar{x}_i , since such a clause is satisfied by any truth assignment and therefore can be eliminated. Also, we may assume that $m = |C| \geq 2$.

We construct a graph $G = G_{(C,X)}$ with the vertex set $C' \cup X'$, where $C' = \{c_1, c_2, \dots, c_m\}$ and $X' = \{x_i, a_i, b_i, \bar{x}_i : i = 1, 2, \dots, n\}$ are disjoint sets, in the following way:

- The set C' induces a clique.
- The set X' induces n four-paths $P^i = (x_i, a_i, b_i, \bar{x}_i)$ (with the edges $x_i a_i$, $a_i b_i$ and $b_i \bar{x}_i$).
- For each clause $c_j = (l_j^1 \vee l_j^2 \vee l_j^3)$, introduce the three edges $c_j l_j^1$, $c_j l_j^2$, $c_j l_j^3$ between C' and X' in G .

The graph G associated with the instance (C, X) of 3-SAT, where $X = \{x_1, x_2, x_3, x_4, x_5\}$ and $C = \{c_1 = (x_1 \vee x_2 \vee x_3), c_2 = (\bar{x}_2 \vee \bar{x}_3 \vee x_4), c_3 = (\bar{x}_1 \vee x_3 \vee x_5), c_4 = (\bar{x}_3 \vee \bar{x}_4 \vee x_5)\}$, is shown in Fig. 1.

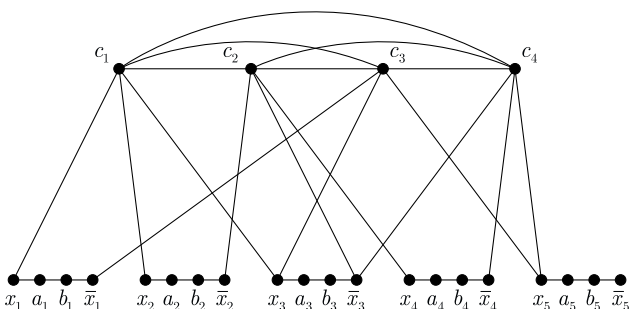


Fig. 1. An illustration of the construction

It is clear that the graph $G = G_{(C,X)}$ can be constructed in time polynomial in $m = |C|$ and $n = |X|$. We denote $X_i = \{x_i, a_i, b_i, \bar{x}_i\}$, $i = 1, 2, \dots, n$.

Claim 1. For an arbitrary maximal induced matching M in G , the following statements hold:

- $|EN_G(C') \cap M| \leq 1$,
- $EN_G(X_i) \cap M \neq \emptyset$ for each $i = 1, 2, \dots, n$, and
- $|M| \in \{n, n + 1\}$.

Proof. (i) Since C' induces a clique, each induced matching in $EN_G(C')$ has at most one edge, i.e., $|EN_G(C') \cap M| \leq 1$.

(ii) If M and $EN_G(X_i)$ are disjoint for some i ($1 \leq i \leq n$), then we can add the edge $a_i b_i$ to M , contradicting the maximality of M .

(iii) By (i), $|EN_G(C') \cap M| \leq 1$. Also, M contains at most n edges of $G(X')$. Thus, $|M| \leq n + 1$. It follows from (ii) that $|M| \geq n$. \square

Note that G has a maximum induced matching of size $n + 1$, for example $\{c_1 c_2\} \cup \{a_i b_i : 1, 2, \dots, n\}$. Recall that $m = |C| \geq 2$.

We show that C is satisfiable if and only if the graph G has a maximal induced matching of size n . First, suppose that there exists a truth assignment ϕ satisfying C . We construct an induced matching $M \subseteq \{x_i a_i, b_i \bar{x}_i : i = 1, 2, \dots, n\}$ choosing the n edges that correspond to true literals under ϕ . That is, if $\phi(x_i) = 1$, we include the edge $x_i a_i$ into M , otherwise the edge $b_i \bar{x}_i$ is included into M . Since ϕ satisfies C , each vertex of C' is adjacent to an end-vertex of an edge in M . It implies that we cannot extend M by adding an edge $e \notin E(G(X'))$. Indeed, such an edge is always incident to a vertex of C' . Thus, M is a maximal induced matching of size n .

Conversely, let M be a maximal induced matching in G of size n .

Claim 2. Relation $M \subseteq E(G(X'))$ holds.

Proof. Claim 1 (ii) and $|M| = n$ imply that M contains exactly one edge from each set $EN_G(X_i)$. Moreover, M cannot have an edge connecting a vertex of X' with a vertex of C' . Indeed, suppose that an edge $x_i c_j$ is in M . By Claim 1 (i), $M \setminus \{x_i c_j\} \subseteq E(G(X'))$. The vertex \bar{x}_i is non-adjacent to both x_i and c_j , since the clause c_j cannot contain both x_i and \bar{x}_i . Also, b_i is non-adjacent to both x_i and c_j . It follows that $M \cup \{b_i \bar{x}_i\}$ is an induced matching, a contradiction to the maximality of M . \square

We say that a vertex $c_j \in C'$ is *dominated* by M if c_j is adjacent to an end-vertex of an edge in M .

Claim 3. At most one vertex in C' is not dominated by M .

Proof. Suppose that c_1 and c_2 are not dominated by M . Then $M \cup \{c_1 c_2\}$ is an induced matching, contradicting to the maximality of M . \square

Claim 4. There exists a maximal induced matching $M' \subseteq E(G(X'))$ that dominates all vertices in C' .

Proof. If M dominates all vertices in C' , then we may set $M' = M$, since $M \subseteq E(G(X'))$ according to Claim 2. Otherwise, by Claim 3, we may assume that c_1 is the only vertex in C' that is not dominated by M . Without loss of generality, let $c_1 = x_1 \vee x_2 \vee x_3$. By the maximality of M , we cannot add the edge $b_i \bar{x}_i$ to M for each $i = 1, 2, 3$. Since $M \subseteq E(G(X'))$ and $x_i a_i \notin M$, the edges $a_i b_i$ are in M , $i = 1, 2, 3$. Now we define $M' = (M \setminus \{a_1 b_1\}) \cup \{x_1 a_1\}$. Since both a_1 and b_1 do not dominate any vertices in C' , the induced matching M' dominates all vertices in C' . In particular, M' is maximal. \square

The matching M' of Claim 4 covers at most one of the vertices x_i, \bar{x}_i for each $i = 1, 2, \dots, n$. Hence, we can define a partial truth assignment ϕ' satisfying C letting a literal be true if the corresponding vertex of C' is covered by M' . It remains to extend ϕ' to a full assignment. This completes the proof of Theorem 1. \square

A graph G is said to be *bi-size indumatched* if there exists an integer $k \geq 1$ such that $|M| \in \{k, k + 1\}$ for every maximal induced matching M in G . Theorem 1 implies the following interesting corollaries.

Corollary 1. NON-WELL-INDUMATCHED GRAPHS is an NP-complete problem even for bi-size indumatched graphs.

Corollary 2. NON-WELL-INDUMATCHED GRAPHS is NP-complete for $(2P_5, K_{1,5})$ -free graphs.

Corollary 3. The decision problem corresponding to the problem of computing $\sigma(G)$ is NP-complete within bi-size indumatched graphs.

Let $WIM(t)$ be the class of graphs having maximal induced matchings of at most t sizes. Note that, if $t = 1$, then $WIM(1)$ is the class of well-indumatched graphs. By means of Theorem 1, one can obtain the following result.

Theorem 2. For any positive integer t , the problem of recognizing the class $WIM(t)$ is co-NP-complete even for $(2P_5, K_{1,5})$ -free graphs.

3. NP-COMPLETENESS RESULTS FOR WELL-INDUMATCHED GRAPHS

A set $I \subseteq V(G)$ is called *independent* or *stable* if no two vertices in I are adjacent. The *independence number* of a graph G , denoted $\alpha(G)$, is the maximum cardinality of an independent set in G . A set $D \subseteq V(G)$ is a *dominating* set if each vertex in $V(G) \setminus D$ is adjacent to a vertex of D . The minimum cardinality of a dominating set in G is the *domination number* of G , denoted by $\gamma(G)$. A set $I \subseteq V(G)$ is called an *independent dominating* set if I is an independent set and I is a dominating set. The minimum cardinality of an independent dominating set of G is the *independent domination number*, and it is denoted by $i(G)$.

The following three decision problems are known to be NP-complete (see, e.g., Garey and Johnson, 1979).

INDEPENDENT SET

Instance: A graph G and an integer k .

Question: Is $\alpha(G) \geq k$?

DOMINATING SET

Instance: A graph G and an integer k .

Question: Is $\gamma(G) \leq k$?

INDEPENDENT DOMINATING SET

Instance: A graph G and an integer k .

Question: Is $i(G) \leq k$?

Here we prove that INDEPENDENT SET and INDEPENDENT DOMINATING SET are NP-complete problems for well-indumatched graphs. Below we show that the DOMINATING SET problem is NP-complete even in a hereditary subclass of well-indumatched graphs.

Theorem 3. INDEPENDENT DOMINATING SET is an NP-complete problem for well-indumatched graphs.

Proof. Clearly, the problem belongs to NP. To show that it is NP-hard, we establish a polynomial-time reduction from 3-SAT. For any instance (C, X) of 3-SAT with the clauses $C = \{c_1, c_2, \dots, c_m\}$ and variables $X = \{x_1, x_2, \dots, x_n\}$, we construct a graph G as follows.

- For each variable x_i , we introduce a complete graph $F_i = K_4$ with two special vertices x_i and \bar{x}_i called the *literal vertices*.
- For each clause c_j , we introduce a graph $H_j = P_7 = (a_j, b_j, d_j, c_j, d'_j, b'_j, a'_j)$, where c_j is called the *clause vertex*.
- Edges connecting $V(F_i)$ and $V(H_j)$ are defined as follows: a clause vertex c_j is connected to the three literal vertices corresponding to the literals in the clause c_j .

It is easy to see that the graph G can be constructed in time polynomial in $m = |C|$ and $n = |X|$.

Claim 5. Each maximal induced matching M in G has exactly $2m + n$ edges.

Proof. Clearly, M contains exactly one edge from each set $EN_G(V(F_i))$, $i = 1, 2, \dots, n$. Also, M contains exactly two edges from each path H_j , $j = 1, 2, \dots, m$. \square

Claim 6. There exists a satisfying truth assignment for C if and only if G has an independent dominating set of size $k = 2m + n$.

Proof. First, suppose that C has a satisfying truth assignment. We construct an independent dominating set I in G as follows. If x_i is assigned value 1, then include the literal vertex x_i into I ; otherwise, \bar{x}_i is included into I . Finally, the set $\{b_j, b'_j : j = 1, 2, \dots, m\}$ is included into I . It is straightforward to verify that I is an independent dominating set in G of cardinality $2m + n$, as required.

Conversely, suppose that I is an independent dominating set with $|I| = 2m + n$. Clearly, I contains exactly one vertex from each F_i . Also, I must contain at least two vertices from each set $S_j = \{a_j, b_j, a'_j, b'_j\}$ to dominate a_j and a'_j . Since $|I| = 2m + n$, I has exactly two vertices from each S_j . To dominate d_j and d'_j , we must have $I \cap S_j = \{b_j, b'_j\}$. However, the set S_j does not dominate c_j , so the vertex c_j must be dominated by some variable vertex. Thus, we can define a truth assignment $\phi : X \rightarrow \{0, 1\}$ by $\phi(x_i) = 1$ if $x_i \in I$, and $\phi(x_i) = 0$ otherwise. \square

Claims 5 and 6 imply that INDEPENDENT SET is NP-complete for well-indumatched graphs. \square

We now prove that INDEPENDENT SET is NP-complete for well-indumatched graphs.

Theorem 4. INDEPENDENT SET is an NP-complete problem for well-indumatched graphs.

Proof. We give a polynomial-time reduction from the INDEPENDENT SET problem for arbitrary graphs. Given a graph G with the edge set $E(G) = \{e_1, e_2, \dots, e_m\}$, $m \geq 1$, construct a new graph G^* as follows. First, for each edge $e_i = u_i v_i$, $i = 1, 2, \dots, m$, add a triangle (x_i, y_i, z_i) , and join x_i to u_i and y_i to v_i , respectively. Thus, $V(G^*) = V(G) \cup \{x_i, y_i, z_i : i = 1, 2, \dots, m\}$ and $E(G^*) = \{e_i, x_i u_i, y_i v_i, x_i y_i, y_i z_i, x_i z_i : i = 1, 2, \dots, m\}$.

Claim 7. G^* is a well-indumatched graph.

Proof. Let $G_i^* = G^*(\{u_i, v_i, x_i, y_i, z_i\})$, $i = 1, 2, \dots, m$. An arbitrary maximal induced matching M contains at most one edge of each G_i^* , therefore $|M| \leq m$. Suppose that M and $E(G_i^*)$ are disjoint for some $i \in \{1, 2, \dots, m\}$. By symmetry, we may assume that M does not contain any edge that is incident to u_i . It implies that $M \cup \{x_i z_i\}$ is an induced matching, a contradiction to the maximality of M . Thus, $|M| = m$. \square

Claim 8. $\alpha(G^*) = \alpha(G) + m$, where $m = |E(G)|$.

Proof. Let $Z = \{z_1, z_2, \dots, z_m\}$. If I is a maximum independent set in G , then $I^* = I \cup Z$ is an independent set in G^* . Hence, $\alpha(G^*) \geq |I^*| = \alpha(G) + m$. Conversely, let I^* be a maximum independent set in G^* . Without loss of generality, we may assume that $Z \subseteq I^*$. The set $I = I^* \setminus Z$ is independent in G , therefore $\alpha(G) \geq |I| = \alpha(G^*) - m$. \square

Claims 7 and 8 imply that the described mapping of an instance (G, k) to the instance $(G^*, k + |E(G)|)$ gives the required reduction, and the result follows. \square

The transformation used in the proof of Theorem 4 yields the following result for well-indumatched graphs.

Corollary 4. PARTITION INTO TRIANGLES and CHORDAL GRAPH COMPLETION are NP-complete problems for well-indumatched graphs.

Let H be a given graph. Consider the following decision problem.

PARTITION INTO SUBGRAPHS ISOMORPHIC TO H

Instance: A graph G with $|V(G)| = q|V(H)|$ for some integer q .

Question: Is there a partition $V_1 \cup V_2 \cup \dots \cup V_q = V(G)$ such that $G(V_i)$ contains a subgraph isomorphic to H for all $i = 1, 2, \dots, q$?

It is well known that this problem is NP-complete for each H that contains a connected component of three or more vertices (see, e.g., Garey and Johnson, 1979).

Theorem 5. PARTITION INTO SUBGRAPHS ISOMORPHIC TO P_3 is NP-complete for well-indumatched graphs.

Proof. We give a polynomial-time reduction from the problem with $H = P_3$ for arbitrary graphs. Given a graph G with $|V(G)| = 3q$ and $E(G) = \{e_1, e_2, \dots, e_m\}$, $m \geq 1$, construct a new graph G^{**} as follows. First, for each edge $e_i = u_i v_i$, $i = 1, 2, \dots, m$, add a new edge $x_i y_i$ and join x_i to u_i and y_i to v_i , respectively. Then attach two pendant vertices z' and z'' to each $z \in \{x_i, y_i\}$. More precisely,

$$V(G^{**}) = V(G) \cup \{x_i, x'_i, x''_i, y_i, y'_i, y''_i : i = 1, 2, \dots, m\}$$

and $E(G^{**})$ consists of

- e_i for all $i = 1, 2, \dots, m$,
- $x_i u_i, y_i v_i$, for all $e_i = u_i v_i$, $i = 1, 2, \dots, m$, and
- $x_i y_i, x_i x'_i, x_i x''_i, y_i y'_i, y_i y''_i$ for all $i = 1, 2, \dots, m$.

It is easy to see that each maximal induced matching contains exactly one edge from each subgraph induced by $\{u_i, v_i, x_i, y_i, x'_i, x''_i, y'_i, y''_i\}$. Thus, G^{**} is a well-indumatched graph. Also, G has a partition into subgraphs P_3 if and only if G^{**} has a partition into subgraphs P_3 . Indeed, all 3-paths (x'_i, x_i, x''_i) and (y'_i, y_i, y''_i) are always in a partition of G^{**} into 3-paths. \square

Kobler and Rotics (2003) proved that computing Σ is NP-hard for line graphs and therefore, for claw-free graphs. One interesting special case of the problem is when the input line graph $L(G)$ is obtained from a graph G with polynomial-time computable Σ . Corollary 5 shows that this special case is also NP-complete even if restricted to Hamiltonian line graphs of well-indumatched graphs.

Corollary 5. Computing Σ for Hamiltonian line graphs $L(G)$ is NP-hard even if G is a well-indumatched graph.

Kobler and Rotics (2003) also proved that computing Σ is NP-hard for Hamiltonian graphs. This proof is not related to their proof for claw-free graphs. Corollary 6 implies a unified result.

Corollary 6. Computing Σ is NP-hard for Hamiltonian claw-free graphs.

The reduction used in the proof of Theorem 5 also gives the following result.

Corollary 7. GRAPH k -COLORABILITY and CLIQUE are NP-complete problems for well-indumatched graphs.

4. CO-MATCHING SUBGRAPHS

The class WIM of well-indumatched graphs is not hereditary. For example, the path P_7 is a well-indumatched graph, while P_5 is not. However, P_7 contains P_5 as an induced subgraph. We introduce a new hereditary system that is similar to that for well-covered graphs (Caro et al., 1998; Zverovich, 2000).

The *neighborhood* of an edge $e = uv$ is defined as $N(e) = N(u) \cup N(v)$. Clearly, $u, v \in N(e)$. Also, $N(S) = \cup_{e \in S} N(e)$ for a set S of edges. Let M be an induced matching in G , including $M = \emptyset$. A subgraph $G - N(M)$ of G is called *co-indumatching*. We denote by $CIMSub(G)$ the set of co-indumatching subgraphs in G . For example, the cycle C_8 has the following co-indumatching subgraphs: C_8, P_4, K_1, K_0 and thus, $CIMSub(C_8) = \{C_8, P_4, K_1, K_0\}$. A class of graphs \mathcal{M} is *co-indumatching hereditary* if $CIMSub(G) \subseteq \mathcal{M}$ for each graph $G \in \mathcal{M}$. A *minimal forbidden co-indumatching subgraph* for a co-indumatching hereditary class \mathcal{M} is a graph F such that $CIMSub(F) \setminus \mathcal{M} = \{F\}$.

Proposition 1. WIM is a co-indumatching hereditary class.

Proposition 2. Each co-indumatching hereditary class can be characterized in terms of forbidden co-indumatching subgraphs.

Proposition 2 means that for each co-indumatching hereditary class \mathcal{M} , there is a set of graphs \mathcal{X} such that $G \in \mathcal{M}$ if and only if G does not contain each graph in \mathcal{X} as a co-indumatching subgraph. Such a set \mathcal{X} must contain all minimal forbidden co-indumatching subgraphs for \mathcal{M} . Conversely, every set \mathcal{X} that contains all minimal forbidden co-indumatching subgraphs for \mathcal{M} characterizes \mathcal{M} .

For example, P_5 is a forbidden co-indumatching subgraph for WIM . Indeed, it is not well-indumatched, but $P_5 - N(e)$ is well-indumatched for each edge e of P_5 . We shall characterize well-indumatched graphs in terms of forbidden co-indumatching subgraphs.

Another important problem concerns the complexity of recognizing co-indumatching subgraphs.

CO-INDUMATCHING SUBGRAPH

Instance: A graph G and a set $U \subseteq V(G)$ that induces a subgraph H .

Question: Is H a co-indumatching subgraph of G ?

We prove that the problem is hard.

Theorem 6. CO-INDUMATCHING SUBGRAPH is an NP-complete problem.

Let G_1, G_2, \dots, G_k be pairwise vertex-disjoint graphs. The *join* of graphs G_1, G_2, \dots, G_k is the graph $\sum_{i=1}^k G_i = \overline{G_1} \cup \overline{G_2} \cup \dots \cup \overline{G_k}$, where $\overline{G_i}$ is the complement of G_i . We denote by Z_W the set of all graphs of the form $\sum_{i=1}^k G_i$, where each graph G_i is well-covered and $\alpha(G_i) \neq \alpha(G_j)$ for $1 \leq i \neq j \leq k$. It is easy to see that Z_W consists of non-

well-covered graphs. Let Z_{WIM} be the class of all graphs F such that $(L(F))^2 \in Z_W$. Here is our characterization.

Theorem 7. Z_{WIM} is the set of all minimal forbidden co-indumatching subgraphs for the class WIM .

5. PERFECTLY WELL-INDUMATCHED GRAPHS

In this section, we consider a hereditary subclass of the class WIM . A graph G is *perfectly well-indumatched* if every induced subgraph of G is well-indumatched. We characterize perfectly well-indumatched graphs in terms of forbidden induced subgraphs.

Kite is the graph consisting of five vertices u, v, w, x, y and edges uv, uw, vw, wx, xy . *Butterfly* is the graph obtained from kite by adding the edge wy .

Theorem 8. For a graph G , the following statements are equivalent:

- (i) G is a perfectly well-indumatched graph.
- (ii) G is a $(P_5, \textit{kite}, \textit{butterfly})$ -free graph.
- (iii) $(L(G))^2$ is a $(K_4 - e)$ -free graph.

Theorem 8 implies that the class of perfectly well-indumatched graphs contains all $2K_2$ -free graphs and therefore, all split graphs. Recall that Foldes and Hammer (1977) characterized split graphs as $(2K_2, C_4, C_5)$ -free graphs. It is well known that DOMINATING SET is an NP-complete problem for split graphs (Bertossi, 1984).

Corollary 8. DOMINATING SET is an NP-complete problem for perfectly well-indumatched graphs.

As we have shown, INDEPENDENT SET and INDEPENDENT DOMINATING SET are NP-complete problems for well-indumatched graphs. However, they can be solved in polynomial time for perfectly well-indumatched graphs.

Theorem 9. The INDEPENDENT SET problem and the INDEPENDENT DOMINATING SET problem can be solved in polynomial time for perfectly well-indumatched graphs, even in their weighted versions.

REFERENCES

- Balakrishnan, H., Barrett, C.L., Anil Kumar, V.S., Marathe, M.V., and Thite S. (2004). The distance-2 matching problem and its relationship to the MAC-layer capacity of ad hoc wireless networks. *IEEE Journal on Selected Areas in Communications*, volume 22, 1069–1079.
- Bertossi, A. (1984). Dominating sets for split and bipartite graphs. *Information Processing Letters*, volume 19, 37–40.
- Bondy, J.A. and Murty, U.S.R. (1976). *Graph Theory with Applications*. American Elsevier Publishing Company, New York.
- Cameron, K. (1989). Induced matchings. *Discrete Applied Mathematics*, volume 24, 97–102.
- Campbell, S.R., Ellingham, M.N., and Royle, G.F. (1993). A characterization of well-covered cubic graphs. *Journal of Combinatorial Mathematics and Combinatorial Computing*, volume 13, 193–212.
- Caro, Y., Sebö, A., and Tarsi, M. (1996). Recognizing greedy structures. *Journal of Algorithms*, volume 20, 137–156.
- Caro, Y., Ellingham, M.N., and Ramey, J.E. (1998). Local structure when all maximal independent sets have equal weight. *SIAM Journal on Discrete Mathematics*, volume 11, 644–654.
- Chvátal, V., and Slater, P.J. (1993). A note on well-covered graphs. *Annals of Discrete Mathematics*, volume 55, 179–182.
- Finbow, A., Hartnell, B., and Nowakowski, R. (1993). A characterization of well-covered graphs of girth 5 or greater. *Journal of Combinatorial Theory, Series B*, volume 57, 44–68.
- Finbow, A., Hartnell, B., Nowakowski, R., and Plummer, M. (2004). On well-covered triangulations: Part I. *Discrete Applied Mathematics*, volume 132, 97–108.
- Finbow, A., Hartnell, B., Nowakowski, R., and Plummer, M. (2009). On well-covered triangulations: Part II. *Discrete Applied Mathematics*, volume 157, 2799–2817.
- Finbow, A., Hartnell, B., Nowakowski, R., and Plummer, M. (2010). On well-covered triangulations: Part III. *Discrete Applied Mathematics*, volume 158, 894–912.
- Foldes, S., and Hammer, P.L. (1977). Split graphs. *Congressus Numerantium*, volume XIX, 311–315.
- Garey, M.R. and Johnson, D.S. (1979). *Computers and intractability. A guide to the theory of NP-completeness*. W.H. Freeman and Co., San Francisco.
- Golumbic, M.C. and Lewenstein, M. (2000). New results on induced matchings. *Discrete Applied Mathematics*, volume 101, 157–165.
- Ko, C.W. and Shepherd, F.B. (2003). Bipartite domination and simultaneous matroid covers. *SIAM Journal on Discrete Mathematics*, volume 16, 517–523.
- Kobler, D. and Rotics, U. (2003). Finding maximum induced matchings in subclasses of claw-free and P_5 -free graphs, and in graphs with matching and induced matching of equal maximum size. *Algorithmica*, volume 37, 327–346.
- Lesk, M., Plummer, M.D., and Pulleyblank, W.R. (1984). Equimatchable graphs. In B. Bollobás (ed.), *Graphs Theory and Combinatorics*, 239–254. Academic press, London.
- Lovász, L. and Plummer, M.D. (1986). *Matching theory*. North-Holland Publishing Co., Amsterdam.
- Plummer, M.D. (1970). Some covering concepts in graphs. *Journal of Combinatorial Theory, Series B*, volume 8, 91–98.
- Ravindra, G. (1977). Well-covered graphs. *Journal of Combinatorics, Information & System Sciences*, volume 2, 20–21.
- Sankaranarayana, R.S., and Stewart, L.K. (1992). Complexity results for well-covered graphs. *Networks*, volume 22, 247–262.
- Stockmeyer, L.J. and Vazirani, V.V. (1982). NP-completeness of some generalizations of the maximum matching problem. *Information Processing Letters*, volume 15, 14–19.
- Tankus, D. and Tarsi, M. (2007). Greedily constructing Hamiltonian paths, Hamiltonian cycles and maximum linear forests. *Discrete Mathematics*, volume 307, 1833–1843.
- Yamashita, M. and Kameda, T. (1999). Modeling k -coterie by well-covered graphs. *Networks*, volume 34, 221–228.
- Zverovich, I.E. (2000). A characterization of well-covered graphs in terms of forbidden costable subgraphs. *Mathematical Notes*, volume 67, 41–44.