Fakultät für Mathematik
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Examination in Mathematics II

(8 February 2012)

Working time: 120 minutes

The derivation of the results must be given clearly. The statement of the result only is not sufficient.

Tools:

- pocket calculator
- printed collection of formulas
- **either** two individually prepared one-sided sheets of paper (write '2' on cover sheet) **or** textbook 'Mathematics of Economics and Business (write 'B' on cover sheet)

It is not allowed to use mobile phones.

Distribution of points obtainable for the problems:

problem	1	2	3	4	5	6	sum
points	7	9	11	8	10	5	50

Problems:

1. Given are the vectors

$$\mathbf{a} = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 1 \\ -4 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{d} = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}.$$

   (a) Confirm that vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are linearly independent and express vector $\mathbf{d}$ as a linear combination of $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$.

   (b) Give all basises of $\mathbb{R}^3$ using a subset of the above vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$.

2. A company produces by means of three raw materials R_1, R_2, R_3 three intermediate products I_1, I_2, I_3, and with these intermediate products three final products P_1, P_2, P_3. Let

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 2 & 1 & 2 \end{pmatrix},$$

   where a_{ij} denotes the number of units of R_i necessary for one unit of I_j and b_{jk} denotes the number of units of I_j necessary for the production of one unit of P_k, $i, j, k = 1, 2, 3$. Let x_i be the number of produced units of P_i and y_j the required number of units of R_j.

   (a) Give a matrix equation describing the relationship between the vectors $\mathbf{y} = (y_1, y_2, y_3)^T$ and $\mathbf{x} = (x_1, x_2, x_3)^T$.

   (b) Let the production vector be $\mathbf{x} = (20, 10, 30)^T$. Give the required vector $\mathbf{y}$ of raw materials.

   (c) Let $\mathbf{y} = (320, 160, 120)^T$. Give the resulting production vector $\mathbf{x}$.

3. Given is the following system of linear equations:

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & 4x_3 & = & 1 \\ x_1 & + & \lambda x_2 & + & 2x_3 & = & 0 \\ 2x_1 & + & 4x_2 & + & (\lambda^2 + 4)x_3 & = & \mu \end{array}$$

(a) For which values $\lambda, \mu \in \mathbb{R}$ is the given system inconsistent?

(b) For which values $\lambda, \mu \in \mathbb{R}$ is the solution uniquely determined?

(c) For which values $\lambda, \mu \in \mathbb{R}$ do infinitely many solutions exist?

(d) Determine the general case when λ is positive in case (c).

(e) Determine the particular solution with

$$\sum_{i=1}^{3} x_i = \frac{3}{4}$$

when λ is negative in case (c).

4. A self-employed teacher has a monthly working time (for teaching classes and giving private lessons) of at most 120 hours. He wants to divide it in such a way that the time for teaching classes is at least the same as the time for private lessons plus 20 hours because he gets 30 EUR per hour for teaching classes and only 20 EUR per hour for giving private extra lessons. Unfortunately the orders for teaching classes per month amount to at most 80 hours, and he has to give private lessons for at least 20 hours because of contracts he made.

   (a) Let x_1 be the number of hours for teaching classes and x_2 be the number of hours for private lessons per month. Formulate a linear programming problem assuming that he wishes to maximize his income.

   (b) Setup the *first* tableau for applying the simplex algorithm. Which variables would exchange their places in the set of basic and nonbasic variables after the first pivoting step (new tableau is *not* required).

   (c) Determine graphically an optimal solution.

5. The cost function of a company is given by

$$C(x_1, x_2, x_3) = 2x_1^2 + 3x_2^2 + x_3^2 + 2x_1x_3 + x_2x_3 + 10,$$

where x_i denotes the produced quantity of product $i, i = 1, 2, 3$. The prices p_i of the products i are $p_1 = 40, p_2 = 28$ and $p_3 = 33$.

Determine the quantities x_1, x_2, x_3 maximizing the profit of the company if all produced products are sold.

6. Given is the differential equation

$$\frac{dy}{dx} = -3x^2y^2 - e^{x-2}y^2$$

(a) Determine the general solution.

(b) Determine the particular solution satisfying $y(2) = 1$.