Fakultät für Mathematik
Institut für Mathematische Optimierung
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Examination in
'Mathematics II'
(3 August 2013)

Working time: 120 minutes

The derivation of the results must be given clearly. The statement of the result only is not sufficient.

Tools:

- pocket calculator
- **either** two individually prepared one-sided sheets of paper (write '2' on cover sheet) **or** textbook 'Mathematics of Economics and Business (write 'B' on cover sheet)

It is not allowed to use mobile phones.

Distribution of points obtainable for the problems:

problem	1	2	3	4	5	6	sum
points	9	10	10	4	11	6	50

Problems:

1. Given are the vectors

$$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}, \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

   (a) Determine all vectors $\mathbf{x}$ which are orthogonal to vector $\mathbf{b}$ and which satisfy $\mathbf{x}^T \cdot \mathbf{a} = 3$.

   (b) Among all vectors $\mathbf{x}$ found in (a) determine those vectors $\mathbf{x}^*$ with $|\mathbf{x}^*| = \sqrt{18}$.

   (c) Determine the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$.

2. Given is the following system of linear equations:

$$\begin{array}{rcrcrcrcl} x_1 & + & x_2 & + & x_3 & + & x_4 & = & 1 \\ 2x_1 & + & x_2 & & & + & ax_4 & = & b \\ 3x_1 & + & 2x_2 & + & x_3 & & & = & 0 \end{array}$$

   (a) By means of rank investigations check for which values of $a, b \in \mathbb{R}$ the given system is consistent/inconsistent and characterize the solution in dependence on a and b (give also the degrees of freedom).

   (b) Consider the case $a = 3$ and $b = 7$. Determine the general solution and the particular solution satisfying

$$x_1 + x_2 = 1 \; .$$

3. Given is the system of linear inequalities

$$\begin{array}{rcrcr} x_1 & - & x_2 & \leq & 2 \\ -4x_1 & - & 3x_2 & \leq & -12 \\ 6x_1 & + & 8x_2 & \leq & 48 \end{array}$$

$$x_1, x_2 \geq 0$$

   (a) Graph die feasible region, determine the extreme points of the given system of linear inequalities exactly and give the general solution of this system of linear inequalities.

(b) Consider in addition to the above system the objective function

$$f(x_1, x_2) = 3x_1 + 2x_2 \rightarrow \max!$$

Determine the **starting** tableau for applying the simplex algorithm, and determine graphically the optimal extreme point.

4. Given is the matrix B as follows:

$$B = \begin{pmatrix} 3 & 1 & 3 \\ 1 & 0 & 0 \\ 3 & 0 & 2 \end{pmatrix}.$$

Verify that $\lambda = -1$ is an eigenvalue of matrix B and determine all other eigenvalues of matrix B.

5. Consider the constrained optimization problem

$$F(x_1, x_2, x_3) = x_1 x_2 x_3 + w^2 \rightarrow \max!$$

s.t.

$$x_1 + x_2 + x_3 = \sqrt{w},$$

where $w \in \mathbb{R}$ is a given parameter.

(a) Determine all solutions $x_1(w), x_2(w), x_3(w)$ satisfying the necessary conditions for a local maximum point, for which all variables x_1, x_2, x_3 are positive.

(b) Check the sufficient condition for the solution(s) obtained (with positive values x_1, x_2, x_3).

6. Determine the general solution of the separable differential equation

$$(x^3 + 8)\,\frac{dy}{dx} = (3x^3 + x^2 + 24) \cdot y$$

and the particular solution satisfying the initial condition

$$(x_0, y_0) = (0, 2).$$