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<u>Examination in Mathematical Economics</u>

(06.02.2007)

Working time: 120 minutes

The derivation of the results must be given clearly. The statement of the result only is not sufficient.

Tools:

- pocket calculator
- printed collection of formulas
- two individually prepared double-sided sheets of paper with arbitrary material
- textbook 'Mathematics of Economics and Business'

It is not allowed to use mobile phones.

Distribution of points obtainable for the problems:

problem	1	2	3	4	5	6	sum
points	9	11	5	9	9	7	50

Problems:

1. Given is the nonlinear programming problem:

$$F(x,y,z) = x^2 - 27x + 2y^2 - 45y + 10z + 15 \rightarrow \text{min!}$$

s.t.

$$\begin{array}{rcrcrcr} x & + & y & - & z & \leq & 0 \\ & & & & z & \leq & 17.25 \end{array}$$

   (a) Determine a solution of the Karush-Kuhn-Tucker conditions.

   (b) Can one conclude that a solution of the KKT conditions is globally optimal?

2. Consider the nonlinear programming problem:

$$F(x,y) = (x-3)^2 + (y-3)^2 \rightarrow \text{min!}$$

s.t.

$$\begin{array}{rcrcr} x^2 & + & y^2 & \leq & 4 \\ -x & + & y & \geq & -1 \end{array}$$

$$x \geq 0, \quad y \geq 0$$

   (a) Setup the Karush-Kuhn-Tucker conditions.

   (b) Check whether point $(x^*, y^*) = (0, 2)$ satisfies the KKT conditions.

   (c) Solve the problem graphically.

   (d) Consider **in addition** to the above constraints the following constraint:

$$y \geq \frac{1}{x} - 1.$$

   Does the set of feasible solutions satisfy the Slater condition?

3. Given is the function $f : \mathbb{R} \rightarrow \mathbb{R}$ with

$$f(x) = -\frac{x^2}{1+x^2}.$$

(a) Sketch the graph of function f.

(b) Check whether function f satisfies the necessary condition for quasi-concavity and the sufficient condition for strict quasi-concavity.

4. Two products are in competition on a market. Let $x_i(t)$ denote the sales of product $i, i \in \{1, 2\}$. The changes in the sales are described by the following system of differential equations:

$$\begin{aligned} \dot{x_1}(t) &= -0.2x_1(t) + 0.5x_2(t) \\ \dot{x_2}(t) &= 0.2x_1(t) - 0.5x_2(t) \end{aligned}$$

   (a) Find the general solution by the eigenvalue method.

   (b) Determine the particular solution when the initial sales are given by $x_1(0) = 60$ and $x_2(0) = 80$.

   (c) What is the equlibirium state? Is it globally asymptotically stable?

5. Given is the economic model

$$\begin{aligned} \dot{K} &= 3\sqrt{K} - C \\ \dot{C} &= \frac{1}{3\sqrt{K}}\,C - \frac{1}{30}\,C \end{aligned}$$

   where $K(t)$ denotes the capital stock and $C(t)$ the consumption at time t.

   (a) Draw the nullclines into a phase diagram.

   (b) For the nullclines and each sector resulting from them, draw the directions of motion into the phase diagram.

   (c) Give the equilibrium state (K^*, C^*) with $K^* > 0$ and $C^* > 0$ and check whether it is a local saddle point.

6. Consider the following control theory problem:

$$\max \int_0^{20} (-2u^2 - x^2)dt, \quad \dot{x} = 4u, \quad x(0) = 1,\ x(20) \text{ free },\ u \in \mathbb{R}$$

Determine an optimal pair $(x^*(t), u^*(t))$ and the costate variable $p(t)$ (all functions may still include constants). Which conditions on function p must be used to find the constants in the term for the costate variable (do **not** compute the constants in the solution).