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Examination in Mathematical Economics

(19.07.2007)

Working time: 120 minutes

The derivation of the results must be given clearly. The statement of the result only is not sufficient.

Tools:

- pocket calculator
- printed collection of formulas
- two individually prepared double-sided sheets of paper with arbitrary material
- textbook 'Mathematics of Economics and Business'

It is not allowed to use mobile phones.

Distribution of points obtainable for the problems:

problem	1	2	3	4	5	sum
points	10	14	8	11	7	50

Problems:

1. (a) Determine the quadratic approximation of function $F : D_f \to \mathbb{R}$ with
$$F(x, y) = 3e^{xe^{2y}} + x + 5$$
at the point $(x^0, y^0) = (0, 0)$.

   (b) Let S and T be convex sets. Moreover, let
$$U = \{ax + by \mid x \in S, y \in T;\ a, b \in \mathbb{R}\}.$$
Prove that U is convex.

2. Consider the nonlinear programming problem:
$$F(x, y) = e^{-x} + e^{-y} + e^{-z} - 100 \to \min!$$
s.t.
$$\begin{array}{rcl} x + y + z & \leq & a \\ x & \leq & b, \end{array}$$
where $a, b \in \mathbb{R}$.

   (a) Setup the Karush-Kuhn-Tucker (KKT) conditions.

   (b) Check whether point $(x^*, y^*, z^*) = (\frac{1}{3}\,a, \frac{1}{3}\,a, \frac{1}{3}\,a)$ satisfies the KKT conditions for $a < 3b$.

   (c) Determine a solution of the KKT conditions for the case when $a \geq 3b$.

   (d) Let $f^*(a, b)$ be the minimal value function for the case when $a \geq 3b$ considered in (c). Compute the partial derivatives of f^* with respect to parameters a and b, and relate them to the Lagrangean multipliers.

3. Given is the problem
$$F(x, y) = -\ln(x + 1) - y \to \min!$$
s.t.
$$px + y \leq m$$
$$x \geq 0,\ y \geq 0$$

(a) Determine a solution of the Karush-Kuhn-Tucker (KKT) conditions for $p \in (0, 1]$ and $m > 1$.

(b) Is the solution obtained in (a) globally optimal?

4. Consider the following system of differential equations:

$$\begin{aligned} \dot{x} &= x - 4y \\ \dot{y} &= 2x - 5y \end{aligned}$$

(a) Draw the nullclines into a phase diagram.

(b) For each sector resulting from the nullclines, draw the directions of motion into the phase diagram.

(c) Find the general solution by the eigenvalue method.

(d) Transform the given system into a differential equation of order two with constant coefficients.

5. Find the solution to the following control problem:

$$\max \int_0^T \left(1 - tx(t) - u^2(t)\right) dt,$$

$$\dot{x}(t) = u(t), \quad x(0) = x_0, \quad x(T) = \text{ free}, \quad u \in \mathbb{R},$$

where x_0 and T are positive constants.