Fakultät für Mathematik
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<u>Examination in 'Methods for Economists'</u>

(29 January 2013)

Working time: 120 minutes

The derivation of the results must be given clearly. The statement of the result only is not sufficient.

Tools:

- pocket calculator
- two individually prepared double-sided sheets of paper with arbitrary lecture material
- textbook 'Mathematics of Economics and Business'

It is not allowed to use mobile phones.

Distribution of points obtainable for the problems:

problem	1	2	3	4	5	sum
points	9	9	14	8	10	50

Problems:

1. Given is the function

$$f(x, y, z) = e^{x+y} - 2xy + z^3$$

   (a) Give the requirements on the variables x, y, z such that the Hessian matrix H_f is positive definite (and thus, function F is strictly convex).

   (b) Determine the directional derivative of f at the point

$$(x^0, y^0, z^0) = (0, 1, 1)$$

   in the direction given by the vector $(2, 1, 2)^T$.

2. Consider the utility maximization problem

$$U(x, y) = \frac{1}{2}\ln(x + 1) + \frac{1}{4}\ln(y + 1)$$

   s.t.

$$2x + 3y = t,$$

   where $t \in \mathbb{R}$.

   (a) Determine a solution $x(t), y(t)$ and the multiplier $\lambda(t)$ of the necessary optimality conditions in dependence on the parameter t (do **not** check the sufficient optimality conditions!).

   (b) Find an explicit expression for the (maximum) value function $U^*(t)$ and verify that

$$\frac{dU^*}{dt} = -\lambda(t)$$

3. Consider the optimization problem

$$F(x, y, z) = x - 3\ln(1 + x) + 2y + z \to \min!$$

   s.t.

$$x^2 - y - z \le 0$$

$$x \ge 0, \quad y \ge 0, \quad z \ge 0.$$

(a) Determine all solutions of the Karush-Kuhn-Tucker (KKT) conditions.

(b) Is the Slater condition satisfied?

4. Consider the following system of differential equations:

$$\begin{aligned} \dot{x} &= x + y \\ \dot{y} &= 3x - y \end{aligned}$$

(a) Determine the equilibrium point.

(b) Draw the nullclines into a phase diagram and for each sector resulting from the nullclines, draw the directions of motion into the phase diagram.

(c) Find the general solution by the eigenvalue method.

5. Consider the following control problem:

$$\int_0^{10} \Big(-x^2 - \frac{1}{2}u^2 \Big) \cdot e^{-3t}\, dt \to \max!$$

$$\dot{x} = x + u, \quad x(0) = 3, \quad x(10) \text{ free}, \quad u \in \mathbb{R}.$$

(a) Formulate the necessary optimality conditions obtained from the maximum principle (use $\lambda_0 = 1$) and give the resulting system of first-order differential equations in the form

$$\begin{aligned} \dot{x} &= f(x, \lambda) \\ \dot{\lambda} &= g(x, \lambda) \end{aligned}$$

with λ being the current value shadow price.

(b) Transform this system into a second-order differential equation in x and then its solution $x^*(t)$ and the resulting functions $\lambda(t)$ and $u^*(x)$ still including the constants (do **not** compute them).

(c) Why is the pair $(x^*(t), u^*(t))$ indeed a solution?