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Examination in 'Methods for Economists'

(7 February 2014)

Working time: 120 minutes

The derivation of the results must be given clearly. The statement of the result only is not sufficient.

Tools:

- pocket calculator
- printed collection of formulas
- two individually prepared double-sided sheets of paper with arbitrary material
- textbook 'Mathematics of Economics and Business'

It is not allowed to use mobile phones.

Distribution of points obtainable for the problems:

problem	1	2	3	4	5	sum
points	9	11	10	12	8	50

Problems:

1. (a) Given is the function $f : \mathbb{R}^2 \to \mathbb{R}$ with

$$f(x, y) = (y - 1)e^x \ .$$

Check whether the necessary condition for f to be quasi-concave is satisfied. What is the largest possible domain $D^* \subseteq \mathbb{R}^2$ such that the sufficient condition for f to be quasi-concave is satisfied?

   (b) Let $M^* = \{\mathbf{x} = (x_1, x_2)^T \mid \mathbf{a}^T\mathbf{x} \geq \mathbf{b},\ x_1 \geq 0\}$. Prove that M^* is a convex set.

2. Consider the utility maximization problem

$$U(x, y) = 2\sqrt{x + 2} + \sqrt{2y + 2} \to \max!$$

s.t.

$$2x + y = a,$$

where $a \in \mathbb{R}$ is a given parameter.

   (a) Determine all solutions $x(a), y(a)$ satisfying the necessary conditions for a local maximum point.

   (b) Check the sufficient condition for the solution(s) obtained.

   (c) Find an explicit expression for the (maximum) value function $U^*(a)$ and determine

$$\frac{dU^*}{da}(11).$$

3. Consider the nonlinear programming problem:

$$F(x, y) = 20 - 2x - 3\ln(y^2 - 1) \to \max!$$

s.t.

$$\begin{aligned} x + 2y &\geq 3 \\ y &\geq 2 \end{aligned}$$

   (a) Setup the Karush-Kuhn-Tucker (KKT) conditions.

(b) Find all solutions (x^*, y^*) of the KKT conditions.

4. (a) Given is the following macroeconomic control problem:

$$\begin{aligned} \dot{K} &= \frac{1}{2}K^3 - C + 1 \\ \dot{C} &= C + K - 7 \end{aligned}$$

Graph the nullclines. For any sector resulting from the nullclines, graph the directions of motion. Determine the equilibrium point (K^*, C^*) with $K^* > 0, C^* > 0$ graphically.

(b) Given is the following system of differential equations:

$$\begin{aligned} \dot{x} &= x + e^{2t}y - e^{2t} \\ \dot{y} &= 8e^{-2t}x + y + 4 \end{aligned}$$

Transform the above system into a second-order differential equation in x and find the general solution (only for this differential equation in x).

5. Consider the control problem

$$\max_{u>0} \int_0^{20} (6V - u^2)\, e^{-0.1t}\, dt; \qquad \dot{V} = -0.3V + \frac{5}{2}\, u,$$

$$V(0) = 150; \qquad V(20) \text{ free},$$

where $V(t)$ denotes the value of a machine and $u(t) > 0$ denotes the repair effort at time $t \in [0, 20]$. By means of the current value Hamiltonian (with $\lambda_0 = 1$) and the optimality conditions resulting from the maximum principle, determine an optimal control $u^*(t)$ for the repair effort.