Fakultät für Mathematik
Institut für Mathematische Optimierung
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Examination in 'Methods for Economists'

(1 February 2016)

Working time: 120 minutes

The derivation of the results must be given clearly. The statement of the result only is not sufficient.

Tools:

- pocket calculator (according to the instructions of FWW)
- two individually prepared sheets of paper with arbitrary material except solved exercises, numerical examples from the lecture and old examination problems
- textbook 'Mathematics of Economics and Business'

It is not allowed to use mobile phones.

Distribution of points obtainable for the problems:

problem	1	2	3	4	5	sum
points	14	10	11	7	8	50

Problems:

1. Consider the problem

$$F(x, y, z) = xyz - 2u \to \max!$$

s.t.

$$x + y + z = u^2 \,,$$

where $u \in \mathbb{R}$ is a given parameter.

   (a) Determine all solutions $x(u), y(u), z(u)$ satisfying the necessary conditions for a local maximum point (use the fact that in a local maximum point, all variables x, y, z must be positive).

   (b) Check the sufficient condition for the solution(s) obtained (with positive values x, y, z).

   (c) Find an explicit expression for the (maximum) value function $F^*(u)$ and determine

$$\frac{dF^*}{du}(3).$$

2. Consider the nonlinear programming problem:

$$F(x, y) = 4\ln(x^2 + 2) + y^2 - 3 \to \min!$$

s.t.

$$\begin{array}{rcrcl} x & & & \geq & 1 \\ x^2 & + & y & \geq & 2 \end{array}$$

   (a) Setup the Karush-Kuhn-Tucker (KKT) conditions.

   (b) Find all solutions (x^*, y^*) of the KKT conditions.

3. (a) For which values $t \in \mathbb{R}$ is the quadratic form

$$Q(x_1, x_2, x_3) = -2x_1^2 - 4x_2^2 - 2x_3^2 + 2tx_1x_2 + 2x_1x_3 + 4x_2x_3$$

negative definite?

(b) Given is the economic model

$$\begin{aligned} \dot{K} &= (K-1)^3 - I + 3 \\ \dot{I} &= 2K + I - 8 \end{aligned}$$

Determine the nullclines and graph them into a coordinate system. For each sector resulting from the nullclines, insert the directions of motions, and determine the equilibrium point (K^*, I^*) with $K^* > 0, I^* > 0$.

4. Determine the general solution of the separable differential equation

$$(t^3 + 8)\,\frac{dx}{dt} = (t^3 + t^2 + 8) \cdot x$$

and the particular solution satisfying the initial condition $x(0) = 4$.

5. Consider the following variational problem:

$$\max \int_0^3 (-\dot{x}^2 - tx\dot{x} - x^2 - tx)\,dt$$

$$x(0) = 0; \quad x(3) = 10\,.$$

(a) Determine a solution $x = x(t)$ of the necessary optimality conditions by means of the Euler equation (do **not** determine the constants).

(b) Give an argument why the solution found in (a) is indeed a solution of the problem.