1. Multivariable Calculus

Definition 1:

A set M is called **convex** if for any two points (vectors) $\mathbf{x^1}, \mathbf{x^2} \in M$, any convex combination

$$\lambda \mathbf{x^1} + (1 - \lambda)\mathbf{x^2}$$

with $0 \leq \lambda \leq 1$ also belongs to set M.

Theorem 1:

Let $M_1, M_2, \ldots, M_n$ be convex sets. Then

$$M = M_1 \cap M_2 \cap \ldots \cap M_n$$

is also convex.

Definition 2:

Let $M \subseteq \mathbb{R}^n$ be a convex set. A function $f : M \longrightarrow \mathbb{R}$ is called convex on M if

$$f(\lambda \mathbf{x^1} + (1-\lambda)\mathbf{x^2}) \leq \lambda f(\mathbf{x^1}) + (1-\lambda) f(\mathbf{x^2}) \quad (6)$$

for all $\mathbf{x^1}, \mathbf{x^2} \in M$ and all $\lambda \in [0, 1]$.

Function f is called **concave** if (6) holds with $\leq$ replaced by $\geq$.

Theorem 2:

Let $f : D_f \to \mathbb{R}, D_f \subseteq \mathbb{R}^n$, be twice continuously differentiable and $M \subseteq D_f$ convex. Then:

(a) f is **convex** *on $M \iff$ the Hessian matrix $H_f(\mathbf{x})$ is positive semi-definite for all $\mathbf{x} \in M$;*

(b) f is **concave** *on $M \iff$ the Hessian matrix $H_f(\mathbf{x})$ is negative semi-definite for all $\mathbf{x} \in M$;*

(c) the Hessian matrix $H_f(\mathbf{x})$ is positive definite for all $\mathbf{x} \in M \implies f$ is **strictly convex** *on M;*

(d) the Hessian matrix $H_f(\mathbf{x})$ is negative definite for all $\mathbf{x} \in M \implies f$ is **strictly concave** *on M;*

Theorem 3:

Let $f : M \to \mathbb{R}, g : M \to \mathbb{R}$ and $M \subseteq \mathbb{R}^n$ be convex. Then:

(a) f, g are convex on M and $a \geq 0, b \geq 0 \implies af + bg$ is convex on M;

(b) f, g are concave on M and $a \geq 0, b \geq 0 \implies af + bg$ is concave on M.

Theorem 4:

Let $f : M \to \mathbb{R}$ with $M \subseteq \mathbb{R}^n$ be convex and let $F : D_F \to \mathbb{R}$ with $R_f \subseteq D_F$. Then:

(a) f is convex and F is convex and increasing $\Longrightarrow (F \circ f)(x) = F(f(x))$ is convex;

(b) f is convex and F is concave and decreasing $\Longrightarrow (F \circ f)(x) = F(f(x))$ is concave;

(c) f is concave and F is concave and increasing $\Longrightarrow (F \circ f)(x) = F(f(x))$ is concave;

(d) f is concave and F is convex and decreasing $\Longrightarrow (F \circ f)(x) = F(f(x))$ is convex;

Theorem 5:

Let $M \subseteq \mathbb{R}^n$ be a convex set and $f : M \to \mathbb{R}$ be continuously differentiable. Then:

(a) f is **convex** *on M* $\iff$

$\forall \mathbf{x^1}, \mathbf{x^2} \in M :$

$$f(\mathbf{x^2}) \geq f(\mathbf{x^1}) + (\mathbf{x^2} - \mathbf{x^1})^T \cdot \nabla f(\mathbf{x^1})$$

(b) f is **strictly convex** *on M* $\iff$

$\forall \mathbf{x^1}, \mathbf{x^2} \in M,\ \mathbf{x^1} \neq \mathbf{x^2} :$

$$f(\mathbf{x^2}) > f(\mathbf{x^1}) + (\mathbf{x^2} - \mathbf{x^1})^T \cdot \nabla f(\mathbf{x^1})$$

Definition 3:

Let $M \subseteq \mathbb{R}^n$ be convex and $f : M \to \mathbb{R}$.

For any $a \in \mathbb{R}$, the set

$$P_a = \{\mathbf{x} \in M \mid f(\mathbf{x}) \geq a\}$$

is called an **upper level set** for f.

Theorem 6:

Let $M \subseteq \mathbb{R}^n$ *be a convex set and* $f : M \to \mathbb{R}$. *Then:*

(a) If f *is concave, then*

$$P_a = \{\mathbf{x} \in M \mid f(\mathbf{x}) \geq a\}$$

is a convex set for any $a \in \mathbb{R}$.

(b) If f *is convex, then*

$$P^a = \{\mathbf{x} \in M \mid f(\mathbf{x}) \leq a\}$$

(lower level set) is a convex set for any $a \in \mathbb{R}$.

(c) f *is convex* $\iff$

$$\overline{M}^f = \{(\mathbf{x}, y) \mid \mathbf{x} \in M \text{ and } y \geq f(\mathbf{x})\}$$

is a convex set.

(d) f *is concave* $\iff$

$$\overline{M}_f = \{(\mathbf{x}, y) \mid \mathbf{x} \in M \text{ and } y \leq f(\mathbf{x})\}$$

is a convex set.

Definition 4:

Let $M \subseteq \mathbb{R}^n$ be a convex set and $f : M \to \mathbb{R}$.

Function f is called **quasi-concave** if the upper level set

$$P_a = \{\mathbf{x} \in M \mid f(\mathbf{x}) \geq a\}$$

is convex for any number $a \in \mathbb{R}$.

Function f is called **quasi-convex** if $-f$ is quasi-concave.

Theorem 7:

Let $M \subseteq \mathbb{R}^n$ be a convex set, $f : M \to \mathbb{R}$ and $F : D_F \to \mathbb{R}$ with $R_f \subseteq D_F$. Then:

(a) If f is quasi-concave (quasi-convex) and F is increasing, then

$$(F \circ f)(x) = F(f(x))$$

is quasi-concave (quasi-convex).

(b) If f is quasi-concave (quasi-convex) and F is decreasing, then

$$(F \circ f)(x) = F(f(x))$$

is quasi-convex (quasi-concave).

Definition 5:

Let $M \subseteq \mathbb{R}^n$ be a convex set and $f : M \to \mathbb{R}$.

Function f is called **strictly quasi-concave** if

$$f(\lambda \mathbf{x^1} + (1-\lambda)\mathbf{x^2}) > \min\{f(\mathbf{x^1}), f(\mathbf{x^2})\}$$

for all $\mathbf{x^1}, \mathbf{x^2} \in M$ with $\mathbf{x^1} \neq \mathbf{x^2}$ and $\lambda \in (0,1)$.

Function f is called **strictly quasi-convex** if $-f$ is strictly quasi-concave.

Theorem 8:

Let $M \subseteq \mathbb{R}^n$ *be a convex set and* $f : M \to \mathbb{R}$ *continuously differentiable. Then:*

f *is* **quasi-concave** *on* M $\Longleftrightarrow$

for all $\mathbf{x^1}, \mathbf{x^2} \in M$:

$$\left\{ \quad f(\mathbf{x^2}) \geq f(\mathbf{x^1}) \Longrightarrow [\nabla f(\mathbf{x^1})]^T \cdot (\mathbf{x^2} - \mathbf{x^1}) \geq 0 \quad \right\}$$

Theorem 9:

Let $f : D_f \to \mathbb{R}, D_f \subseteq \mathbb{R}^2$ be twice continuously differentiable on a convex set M and $B_2(x, y)$ as defined. Then:

(a) A necessary condition for f to be **quasi-concave** *on M is that*

$$B_2(x, y) \geq 0$$

for all $(x, y) \in M$.

(b) A sufficient condition for f to be **strictly quasi-concave** *on M is that*

$$f_x(x, y) \neq 0 \quad \text{and} \quad B_2(x, y) > 0$$

for all $x, y \in M$.

Theorem 10:

Let $f : D_f \to \mathbb{R}, D_f \subseteq \mathbb{R}^n$, *be twice continuously differentiable on a convex set* $M \subseteq \mathbb{R}^n$ *and let*

$$B_r(\mathbf{x}) = \begin{vmatrix} 0 & f_{x_1}(\mathbf{x}) & \dots & f_{x_r}(\mathbf{x}) \\ f_{x_1}(\mathbf{x}) & f_{x_1x_1}(\mathbf{x}) & \dots & f_{x_1x_r}(\mathbf{x}) \\ \dots & \dots & \dots & \dots \\ f_{x_r}(\mathbf{x}) & f_{x_rx_1}(\mathbf{x}) & \dots & f_{x_rx_r}(\mathbf{x}) \end{vmatrix}$$

Then:

(a) A necessary condition for f *to be* **quasi-concave** *is that*

$$(-1)^r \cdot B_r(\mathbf{x}) \geq 0$$

for all $\mathbf{x} \in M$ *and all* $r = 1, 2, \dots, n$.

(b) A sufficient condition for f *to be* **strictly quasi-concave** *is that*

$$(-1)^r \cdot B_r(\mathbf{x}) > 0$$

for all $\mathbf{x} \in M$ *and all* $r = 1, 2, \dots, n$.

2. Nonlinear Programming

Definition 1:

A point $\mathbf{x}^* \in M$ is called a **global minimum point** for f in M if

$$f(\mathbf{x}^*) \leq f(\mathbf{x})$$

for all $\mathbf{x} \in M$.

Theorem 1: (necessary first-order conditions)

Let $f : M \to \mathbb{R}$ be differentiable and $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)$ be an interior point of M. A necessary condition for $\mathbf{x}^*$ to be an extreme point is

$$\nabla f(\mathbf{x}^*) = \mathbf{0},$$

i.e.

$$f_{x_1}(\mathbf{x}^*) = f_{x_2}(\mathbf{x}^*) = \ldots = f_{x_n}(\mathbf{x}^*) = 0.$$

Theorem 2: (sufficient conditions)

Let $f : M \to \mathbb{R}$ *with* $M \subseteq \mathbb{R}^n$ *being a convex set.*

(a) If f *is convex on* M*, then:*

$\mathbf{x}^*$ *is a (global) minimum point for* f *in* $M \iff \mathbf{x}^*$ *is a stationary point for* f*.*

(b) If f *is concave on* M*, then:*

$\mathbf{x}^*$ *is a (global) maximum point for* f *in* $M \iff \mathbf{x}^*$ *is a stationary point for* f*.*

Definition 2:

The set

$$U_\epsilon(\mathbf{x}^*) := \{\mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x} - \mathbf{x}^*| < \epsilon\}$$

is called an (open) **ϵ-neighborhood** $U_\epsilon(\mathbf{x}^*)$ with $\epsilon > 0$.

Definition 3:

A point $\mathbf{x}^* \in M$ is called a **local minimum point** for function f in M if there exists an $\epsilon > 0$ such that

$$f(\mathbf{x}^*) \leq f(\mathbf{x})$$

for all $\mathbf{x} \in M \cap U_\epsilon(\mathbf{x}^*)$.

Theorem 3: (necessary optimality condition)

Let $f : M \to \mathbb{R}$ *be continuously differentiable and* $\mathbf{x}^*$ *be an interior point of* M *and a local minimum or maximum point. Then*

$$\nabla f(\mathbf{x}^*) = \mathbf{0}.$$

Theorem 4: (sufficient optimality condition)

Let $f : M \to \mathbb{R}$ *be twice continuously differentiable and* $\mathbf{x}^*$ *be an interior point.*

(a) If $\nabla f(\mathbf{x}^*) = \mathbf{0}$ *and* $H(\mathbf{x}^*)$ *is positive definite, then* $\mathbf{x}^*$ *is a local minimum point.*

(b) If $\nabla f(\mathbf{x}^*) = \mathbf{0}$ *and* $H(\mathbf{x}^*)$ *is negative definite, then* $\mathbf{x}^*$ *is a local maximum point.*

Theorem 5: (necessary optimality condition; **Lagrange's theorem**)

Let functions f and g_i, $i = 1, 2, \ldots, m$, $m < n$, be continuously differentiable and let $\mathbf{x^0} = (x_1^0, x_2^0, \ldots, x_n^0) \in D_f$ be a local extreme point of function f subject to the constraints

$$g_i(x_1, x_2, \ldots, x_n) = 0,\ i = 1, 2, \ldots, m.$$

Moreover, let $|J(x_1^0, x_2^0, \ldots x_n^0)| \neq 0$.

Then there exists a $\lambda^{\mathbf{0}} = (\lambda_1^0, \lambda_2^0, \ldots, \lambda_m^0)$ such that

$$\nabla\ L(\mathbf{x^0}, \lambda^{\mathbf{0}}) = \mathbf{0}.$$

Theorem 6: (local sufficient conditions)

Let functions f and g_i, $i = 1, 2, \dots, m$, $m < n$, be twice continuously differentiable and $(\mathbf{x^0}; \lambda^{\mathbf{0}})$ with $\mathbf{x^0} \in D_f$ be a solution of the system $\nabla L(\mathbf{x}; \lambda) = \mathbf{0}$. Moreover, let

$$H_L(\mathbf{x};\lambda) = \left(\begin{array}{ccc|ccc} 0 & \cdots & 0 & L_{\lambda_1 x_1}(\mathbf{x};\lambda) & \cdots & L_{\lambda_1 x_n}(\mathbf{x};\lambda) \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & L_{\lambda_m x_1}(\mathbf{x};\lambda) & \cdots & L_{\lambda_m x_n}(\mathbf{x};\lambda) \\ \hline L_{x_1 \lambda_1}(\mathbf{x};\lambda) & \cdots & L_{x_1 \lambda_m}(\mathbf{x};\lambda) & L_{x_1 x_1}(\mathbf{x};\lambda) & \cdots & L_{x_1 x_n}(\mathbf{x};\lambda) \\ \vdots & & \vdots & \vdots & & \vdots \\ L_{x_n \lambda_1}(\mathbf{x};\lambda) & \cdots & L_{x_n \lambda_m}(\mathbf{x};\lambda) & L_{x_n x_1}(\mathbf{x};\lambda) & \cdots & L_{x_n x_n}(\mathbf{x};\lambda) \end{array}\right)$$

be the bordered Hessian and consider its leading principal minors $D_j(\mathbf{x^0}; \lambda^{\mathbf{0}})$ of order $j = 2m+1, 2m+2, \dots, n+m$ at point $(\mathbf{x^0}; \lambda^{\mathbf{0}})$. Then:

(1) If all leading principal minors $D_j(\mathbf{x^0}; \lambda^{\mathbf{0}})$, $2m+1 \le j \le n+m$, have the sign $(-1)^m$, then $\mathbf{x^0} = (x_1^0, x_2^0, \dots, x_n^0)$ is a local minimum point of function f subject to the given constraints.

(2) If all leading principal minors $D_j(\mathbf{x^0}; \lambda^{\mathbf{0}})$, $2m+1 \le j \le n+m$, alternate in sign, the sign of $D_{n+m}(\mathbf{x^0}; \lambda^{\mathbf{0}})$ being that of $(-1)^n$, then $\mathbf{x^0} = (x_1^0, x_2^0, \dots, x_n^0)$ is a local maximum point of function f subject to the given constraints.

(3) If neither the conditions of (1) nor of (2) are satisfied, then $\mathbf{x^0}$ is not a local extreme point of function f subject to the constraints $g_i(\mathbf{x}) = 0, i = 1, 2, \dots, m$. Here the case when one or several leading principal minors have value zero is not considered as a violation of condition (1) or (2).

Theorem 7: (sufficient condition for global optimality)

Let there exist numbers $(\lambda_1^0, \lambda_2^0, \ldots, \lambda_m^0) = \lambda^0$ *and an* $\mathbf{x^0} \in D_f$ *such that* $\nabla L(\mathbf{x^0}; \lambda^{\mathbf{0}}) = \mathbf{0}$. *Then:*

(a) If

$$L(\mathbf{x}) = f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i^0 \, g_i(\mathbf{x})$$

is concave in $\mathbf{x}$*, then* $\mathbf{x^0}$ *is a global maximum point.*

(b) If

$$L(\mathbf{x}) = f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i^0 \, g_i(\mathbf{x})$$

is convex in $\mathbf{x}$*, then* $\mathbf{x^0}$ *is a global minimum point.*

Definition 4:

A point $(\mathbf{x}^*, \lambda^*)$ is called a **saddle point** of the Lagrangian function L, if

$$L(\mathbf{x}^*, \lambda) \leq L(\mathbf{x}^*, \lambda^*) \leq L(\mathbf{x}, \lambda^*) \quad (12)$$

for all $\mathbf{x} \in \mathbb{R}^n, \lambda \in \mathbb{R}^m_+$.

Theorem 8:

If $(\mathbf{x}^*, \lambda^*)$ *with* $\lambda^* \geq \mathbf{0}$ *is a saddle point of* L*, then* $\mathbf{x}^*$ *is an optimal solution of problem (11).*

Theorem 9: (Theorem by **Kuhn** and **Tucker**)

If condition (S) is satisfied, then $\mathbf{x}^*$ *is an optimal solution of the convex problem*

$$f(\mathbf{x}) \to \min!$$

s.t. *(13)*

$$g_i(\mathbf{x}) \leq 0, \quad i = 1, 2, \ldots, m$$

$$f, g_1, \ldots, g_m \quad \textit{convex functions}$$

if and only if *L has a saddle point* $(\mathbf{x}^*, \lambda^*)$ *with* $\lambda^* \geq \mathbf{0}$.

Theorem 10: (KKT conditions)

If condition (S) is satisfied and functions $f, g_1, \ldots, g_m$ *are continuously differentiable and convex, then* $\mathbf{x}^*$ *is an optimal solution of problem (13)* **if and only if** *the following Karush-Kuhn-Tucker (KKT) conditions (14) are satisfied:*

$$\begin{aligned} \nabla f(\mathbf{x}^*) + \sum_{i=1}^{m} \lambda_i^* \, \nabla g_i(\mathbf{x}^*) \; &= \; \mathbf{0} \\ \lambda_i^* \cdot g_i(\mathbf{x}^*) \; &= \; 0 \\ g_i(\mathbf{x}^*) \; &\leq \; 0 \\ \lambda_i^* \; &\geq \; 0 \\ i = 1, 2, \ldots, m & \end{aligned}$$

DUALITY IN LINEAR PROGRAMMING

Theorem 11:

The dual problem of (D) is again problem (P).

Theorem 12:

Let

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$$

be an arbitrary feasible solution of problem P and

$$\mathbf{u} = (u_{n+1}, u_{n+2}, \ldots, u_{n+m})^T$$

be an arbitrary feasible solution of problem (D). Then

$$z_0 = \mathbf{c}^T \cdot \mathbf{x} \leq \mathbf{b}^T \cdot \mathbf{u} = w_0.$$

Theorem 13:

If one of the problems (P) or (D) has an optimal solution, then the other one has also an optimal solution, and both optimal solutions $\mathbf{x}^*$ *and* $\mathbf{u}^*$ *have the same objective function value, i.e.*

$$z_0^{max} = \mathbf{c}^T \cdot \mathbf{x}^* = \mathbf{b}^T \cdot \mathbf{u}^* = w_0^{min}.$$

Theorem 14:

If one of the problems (P) or (D) has a feasible solution but no optimal solution, then the other one does not have a feasible solution at all.

Theorem 15:

The coefficients of the non-basic variables x_j in the objective row of the optimal tableau of problem (P) are equal to the optimal values of the corresponding variables u_j of problem (D), and conversely:

The optimal values of the basic variables of problem (P) are equal to the coefficients of the corresponding dual variables in the objective row of problem (D).

Some comments on quasi-convex programming

Theorem 16:

Consider a problem (20) (or (11)) , where function f *is continuously differentiable and quasi-convex. Assume that there exist numbers* $\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*$ *and a vector* $\mathbf{x}^*$ *such that*

(a) the KKT conditions are satisfied,

(b) $\nabla f(\mathbf{x}^*) \neq \mathbf{0}$, *and*

(c) $\lambda_i^* \cdot g_i(\mathbf{x})$ *is quasi-convex for*

$i = 1, 2, \ldots, m.$

Then $\mathbf{x}^*$ *is optimal for problem (20) (problem (11), respectively).*

Constraint qualifications

Definition 5:

A constraint function g_i with

$$g_i(\mathbf{x}^*) = 0, \qquad \mathbf{x}^* \in M,$$

is said to be **active** at $\mathbf{x}^*$.

Theorem 17: (Arrow-Hurwicz-Uzawa)

Conditions (KKT) are **necessary** *for condition (LM), provided that any one of the following conditions holds:*

(a) The functions $g_i(\mathbf{x}),\ i = 1, 2, \ldots, m$, *are all concave.*

(b) The functions $g_i(\mathbf{x}),\ i = 1, 2, \ldots, m$, *are all linear.*

(c) The functions $g_i(\mathbf{x}),\ i = 1, 2, \ldots, m$, *are all convex and condition (S) is satisfied.*

(d) The constraint set M *is convex and possesses an interior point, and* $\nabla g_j(\mathbf{x}^*) \neq \mathbf{0}$ *for all* $j \in E$, *where* E *is the set of all the active constraints at* $\mathbf{x}^*$.

(e) Rank condition (R) is satisfied.

Theorem 18: (Arrow and Enthoven)

Let function f and g_i, $i = 1, 2, \ldots, m$, be quasi-convex functions. Then conditions (KKT) are **sufficient** *for condition (M), if any one of the following conditions is satisfied:*

(a) $f_{x_i}(\mathbf{x}^*) > 0$ for at least one variable x_i.

(b) $f_{x_i}(\mathbf{x}^*) < 0$ for at least one relevant variable x_i, where x_i is said to be a relevant variable, if there exists a feasible $\overline{\mathbf{x}} \in \mathbb{R}^n_+$ such that $\overline{x_i} > 0$.

(c) $\nabla f(\mathbf{x}^*) \neq \mathbf{0}$ and f is twice continuously differentiable in a neighborhood.

(d) Function f is convex.

4. Differential Equations

Definition 1:

A relationship

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0$$

between the independent variable x, a function $y(x)$ and its derivatives is called an **ordinary differential equation**.

The **order** of the differential equation is determined by the highest order of the derivatives appearing in the differential equation.

Definition 2:

A function $y(x)$ for which relationship

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0$$

holds for all values $x \in D_y$, is called a **solution** of the differential equation.

The set

$$S = \{y(x) \mid F(x, y, y', y'', \ldots, y^{(n)}) = 0 \\ \text{for all } x \in D_y\}$$

is called the **set of solutions** or **general solution** of the differential equation.

Definition 3:

A point a represents an **equilibrium** or **stationary state** for equation (7) if

$$F(a) = 0.$$

Theorem 1:

The homogeneous differential equation (9) has the general solution

$$x_H(t) = C_1\, x_1(t) + C_2\, x_2(t); \quad C_1, C_2 \in \mathbb{R},$$

where $x_1(t)$, $x_2(t)$ are two solutions that are not proportional (i.e. linearly independent).

The non-homogeneous equation (8) has the general solution

$$\begin{aligned} x(t) &= x_H(t) + x_N(t) \\ &= C_1\, x_1(t) + C_2\, x_2(t) + x_N(t), \end{aligned}$$

where $x_N(t)$ is any particular solution of the non-homogeneous equation.

Table: Settings for special forcing terms

Forcing term $q(t)$	**Setting** $x_N(t)$
$p \cdot e^{st}$	(a) $A \cdot e^{st}$ if s is not a root of the characteristic equation (b) $A \cdot t^k \cdot e^{st}$ if s is a root of multiplicity k of the characteristic equation
$p_n t^n + p_{n-1} t^{n-1} + \ldots + p_1 t + p_0$	(a) $A_n t^n + A_{n-1} t^{n-1} + \ldots + A_1 t + A_0$ if $b \neq 0$ in the homogeneous equation (b) $t^k (A_n t^n + A_{n-1} t^{n-1} + \ldots + A_1 t + A_0)$ with $k = 1$ if $b = 0$ (x does not occur) and $k = 2$ if $a = b = 0$ ($x, \dot{x}$ do not occur) in the homogeneous equation,
$p \cdot \cos st + r \cdot \sin st$ (p or r can be equal to zero)	(a) $A \cos st + B \sin st$ if si is not a root of the characteristic equation (b) $t^k \cdot (A \cdot \cos st + B \cdot \sin st)$ if si is a root of multiplicity $k (\leq 2)$ of the characteristic equation

Definition 4:

Equation (11) is called **globally asymptotically stable** if every solution

$$x_H(t) = C_1\, x_1(t) + C_2\, x_2(t)$$

of the associated homogeneous equation tends to 0 as $t \to \infty$ for all values of C_1 and C_2.

Theorem 2:

Equation

$$\ddot{x} + a\dot{x} + bx = q(t)$$

is globally asymptotically stable if and only if $a > 0$ *and* $b > 0$.

Theorem 3:

Suppose that $|A| \neq 0$. For the linear system

$$\begin{aligned} \dot{x} &= a_{11}\,x + a_{12}\,y + q_1 \\ \dot{y} &= a_{21}\,x + a_{22}\,y + q_2, \end{aligned}$$

the equilibrium point (x^*, y^*) is globally asymptotically stable **if and only if**

$$tr(A) = a_{11} + a_{22} < 0$$

and

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0,$$

where $tr(A)$ is the trace of A

(or equivalently, if and only if both eigenvalues of A have negative real parts).

Theorem 4: (Lyapunov)

Let (a, b) *be an equilibrium point of system (17) and*

$$A = J(a, b) = \begin{pmatrix} f_x(a, b) & f_y(a, b) \\ g_x(a, b) & g_y(a, b) \end{pmatrix}.$$

If

$$tr(A) = f_x(a, b) + g_y(a, b) < 0$$

and

$$|A| = f_x(a, b) \cdot g_y(a, b) - g_x(a, b) \cdot f_y(a, b) > 0$$

(i.e. both eigenvalues of A have negative real parts), then (a, b) *is* **locally asymptotically stable**.

Theorem 5: (Olech)

Let (a,b) be an equilibrium point of system (17) and $A(x,y)$ be the Jacobian matrix at point $(x,y) \in \mathbb{R}^2$. Assume that the following three conditions are satisfied:

(a) $tr(A(x,y)) = f_x(x,y) + g_y(x,y) < 0$ *for all* $(x,y) \in \mathbb{R}^2$;

(b) $det\, A(x,y) = f_x(x,y) \cdot g_y(x,y) - f_y(x,y) \cdot g_x(x,y) > 0$ *for all* $(x,y) \in \mathbb{R}^2$;

(c) $f_x(x,y) \cdot g_y(x,y) \neq 0$ *for all* $(x,y) \in \mathbb{R}^2$
or
$f_y(x,y) \cdot g_x(x,y) \neq 0$ *for all* $(x,y) \in \mathbb{R}^2$.

Then (a,b) is globally asymptotically stable.

Theorem 6:

Let (a, b) be an equilibrium point of system (17) and

$$A = J(a, b) = \begin{pmatrix} f_x(a, b) & f_y(a, b) \\ g_x(a, b) & g_y(a, b) \end{pmatrix}.$$

Moreover, let det $A < 0$
(or equivalently, the eigenvectors of A are nonzero real numbers of opposite signs).

Then:
For any given start point t_0, there exist exactly two solution paths

$$(x_1(t), y_1(t)) \quad \textit{and} \quad (x_2(t), y_2(t))$$

defined on $[t_0, \infty)$ that converge towards (a, b) from opposite directions in the phase plane
(i.e. (a, b) is a local saddle point).

4.3 Linear differential equations of order n

Definition 5:
The solutions

$$x_1(t), x_2(t), \ldots, x_m(t), \quad m \le n,$$

of a linear homogeneous differential equation of order n are said to be **linearly independent** if

$$C_1\, x_1(t) + C_2\, x_2(t) + \ldots + C_m\, x_m(t) \;=\; 0$$

for all $t \in D_x$ is only possible for

$$C_1 = C_2 = \ldots = C_m = 0.$$

Otherwise, the solutions are said to be **linearly dependent**.

Theorem 7: *The solutions*

$$x_1(t), x_2(t), \ldots, x_m(t), \quad m \leq n,$$

of a linear homogeneous differential equation of order n *are linearly independent if and only if*

$$W(t) = \begin{vmatrix} x_1(t) & x_2(t) & \ldots & x_m(t) \\ \dot{x}_1(t) & \dot{x}_2(t) & \ldots & \dot{x}_m(t) \\ \vdots & \vdots & & \vdots \\ x_1^{(m-1)}(t) & x_2^{(m-1)}(t) & \ldots & x_m^{(m-1)}(t) \end{vmatrix} \neq 0$$

for $t \in D_x$.

Theorem 8: *A linear homogeneous differential equation of order n has the general solution*

$$x_H(t) = C_1x_1(t) + C_2x_2(t) + \ldots + C_nx_n(t),$$

where $x_1(t), x_2(t), \ldots, x_n(t)$ are n linearly independent solutions and $C_1, C_2, \ldots, C_n \in \mathbb{R}$.

The non-homogeneous equation (18) has the general solution

$$x(t) = x_H(t) + x_N(t),$$

where x_N is a particular solution of equation (18).

5 CALCULUS OF VARIATIONS AND CONTROL THEORY

5.1 Calculus of variations

Theorem 1: *If*

$$F(t, x, \dot{x})$$

is concave in $(x, \dot{x})$*, a feasible* $x^*(t)$ *that satisfies the Euler equation solves the maximization problem (1).*

Theorem 2: (Transversality conditions)

If $x^*(t)$ solves problem (3) with either (a) or (b) as the transversality condition, then $x^*(t)$ must satisfy the Euler equation.

With the terminal condition (a), the transversality condition is

$$\left(\frac{\partial F^*}{\partial \dot{x}}\right)_{t=t_1} = 0. \tag{4}$$

With the terminal condition (b), the transversality condition is

$$\left(\frac{\partial F^*}{\partial \dot{x}}\right)_{t=t_1} \leq 0$$

$$\left(\left(\frac{\partial F^*}{\partial \dot{x}}\right)_{t=t_1} = 0 \quad \textit{if } x^*(t_1) > x_1 \right) \tag{5}$$

5.2 Control Theory

5.2.1 Basic Problems

Theorem 3: (Maximum principle)

Suppose that

$$(x^*(t),\ u^*(t))$$

is an optimal pair for problem (9) - (10).

Then there exists a continuous function $p(t)$ such that, for all $t \in [t_0, t_1]$,

- $u = u^*(t)$ *maximizes*

  $$H(t, x^*(t), u, p(t)) \qquad \text{for } u \in (-\infty, \infty) \qquad (11)$$

- $$\dot{p}(t) = -H_x(t, x^*(t), u^*(t), p(t)), \quad p(t_1) = 0 \qquad (12)$$

Theorem 4:

If the condition

$$H(t, x, u, p(t)) \text{ is concave in } (x, u)$$
$$\text{for each } \ t \in [t_0, t_1] \qquad (13)$$

is added to the conditions of Theorem 3, then we obtain a sufficient optimality condition, i.e.,
if we find a triple

$$(x^*(t),\, u^*(t),\, p(t))$$

that satisfies (10) - (13), then

$$(x^*(t),\ u^*(t))$$

is optimal.

5.2.2 Standard Problems

Theorem 5: (Maximum principle for standard end constraints)

Suppose that $(x^*(t), u^*(t))$ is an optimal pair for problem (16) - (18). Then there exist a continuous function $p(t)$ and a number $p_0 \in \{0, 1\}$ such that for all $t \in [t_0, t_1]$, we have $(p_0, p(t)) \neq (0, 0)$ and, moreover:

- *The control $u = u^*(t)$ maximizes the Hamiltonian $H(t, x^*(t), u, p(t))$ w. r. t. $u \in U$, i.e., $H(t, x^*(t), u, p(t)) \leq H(t, x^*(t), u^*(t), p(t))$ for all $u \in U$;*

- $\dot{p}(t) = -H_x(t, x^*(t), u^*(t), p(t))$; *(19)*

- *Corresponding to each of the terminal conditions (b) and (c) in (18), there is a transversality condition on $p(t_1)$:*

  (b') $p(t_1) \geq 0$ (with $p(t_1) = 0$ if $x^*(t_1) > x_1$);

  (c') $p(t_1) = 0$;

  (In case (a) there is no condition on $p(t_1)$).

Theorem 6: (Mangasarian)

Suppose that

$$(x^*(t),\, u^*(t))$$

is a feasible pair with corresponding costate variable $p(t)$ *such that conditions (I) - (III) in Theorem 5 are satisfied with* $p_0 = 1$. *Suppose further that*

- *the control region* U *is convex and that*
- $H(t, x, u, p(t))$ *is concave in* (x, u) *for every* $t \in [t_0, t_1]$.

Then

$$(x^*(t),\, u^*(t))$$

is an optimal pair.

5.2.3 Current Value Formulations

Theorem 7: (Maximum principle; current value formulation)

Suppose that a feasible pair

$$(x^*(t), u^*(t))$$

solves problem (20) a let H^c *be the current value Hamiltonian. Then there exist a continuous function* $\lambda(t)$ *and a number* λ_0 *(either 0 or 1) such that for all* $t \in [t_0, t_1]$*, we have* $(\lambda_0, \lambda(t)) \neq (0, 0)$ *and:*

(I) $u = u^*(t)$ *maximizes* $H^c(t, x^*(t), u, \lambda(t))$ *for* $u \in U$*;*

(II) $\dot{\lambda}(t) = -r\lambda(t) = -\frac{\partial H^c(t, x^*(t), u^*(t), \lambda(t))}{\partial x}$ *;*

(III) The transversality conditions are:

(a') no condition on $\lambda(t_1)$*;*

(b') $\lambda(t_1) \geq 0$
($\lambda(t_1) = 0$ *if* $x^*(t_1) > x_1$*);*

(c') $\lambda(t_1) = 0$*.*

5.2.4 Scrap Values

Theorem 8: (Sufficient conditions with scrap value)

Suppose that $(x^*(t),\, u^*(t))$ *is a feasible pair for the scrap-value problem (21) and that there exists a continuous function* $p(t)$ *such that for all* $t \in [t_0, t_1]$, *we have:*

- *The control* $u = u^*(t)$ *maximizes the Hamiltonian* $H(t, x^*(t), u, p(t))$ *with respect to* $u \in U$*;*
- $\dot{p}(t) = -H_x(t, x^*(t), u^*(t), p(t))$;
  $p(t_1) = S'(x^*(t_1))$
- $H(t, x, u, p(t))$ *is concave in* (x, u) *and* $S(x)$ *is concave.*

Then

$$(x^*(t),\, u^*(t))$$

solves the problem.

Current value formulation (with scrap value)

Theorem 9: (Current value maximum principle with scrap value)

Suppose that a feasible pair $(x^*(t), u^*(t))$ solves problem (22). Then there exist a continuous function $\lambda(t)$ and a number $\lambda_0 \in \{0, 1\}$ such that for all $t \in [t_0, t_1]$, we have $(\lambda_0, \lambda(t)) \neq (0, 0)$ and:

- *The control $u = u^*(t)$ maximizes*

  $$H^c(t, x^*(t), u, \lambda(t))$$

  with respect to $u \in U$;

- $$\dot{\lambda}(t) - r\lambda(t) = -\frac{\partial H^c(t, x^*(t), u^*(t), \lambda(t))}{\partial x};$$

- *The transversality conditions are:*

  (a') no condition on $\lambda(t_1)$;

  (b') $\lambda(t_1) \geq \lambda_0 \cdot \dfrac{\partial S(x^*(t_1))}{\partial x}$

  (with equality if $x^*(t_1) > x_1$);

  (c') $\lambda(t_1) = \lambda_0 \cdot \dfrac{\partial S(x^*(t_1))}{\partial x}$

Theorem 10:

The conditions in Theorem 9 with $\lambda_0 = 1$ *are sufficient if*

- U *is convex,*
- $H^c(t, x, u, \lambda(t))$ *is concave in* (x, u) *and*
- $S(x)$ *is concave in* x*.*