Solutions to the exercises of the book

A Refresher Course in Mathematics

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1 Some Mathematical Foundations

1.1: $A \cap B = \{11, 13, 17, 19, 23, 29\}$;
$A \setminus B = \{2, 3, 5, 7, 31, 37, 41, 43, 47, 53, 59, \ldots\}$,
$B \setminus A = \{10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28, 30\}$;

1.2: $I_1 \cup I_2 = [-3, 4]$; $I_1 \cap I_2 = (-1, 4)$; $I_1 \setminus I_2 = [-3, -1] \cup \{4\}$; $I_2 \setminus I_1 = \emptyset$;

$I_1 \cup I_3 = [-3, 5)$; $I_1 \cap I_3 = (3, 4]$; $I_1 \setminus I_3 = [-3, 3]$; $I_3 \setminus I_1 = (4, 5)$

$I_2 \cup I_3 = (-1, 5)$; $I_2 \cap I_3 = (3, 4)$; $I_2 \setminus I_3 = (-1, 3]$; $I_3 \setminus I_2 = [4, 5)$.

1.3: $X \cup Y = \{a, b, d, e, f, g, h\}$; $X \cap Y = \{b, e\}$; $X \setminus Y = \{a, d, f\}$;
$(X \cup Y) \cap Z = \{a, h\}$; $(Y \setminus Z) \cup X = \{a, b, d, e, f, g\}$.

1.4: (a) X; (b) Y; (c) $X \cup Y$.

1.5: (a) 120; (b) 211; (c) $48,788,677,504$.

2 Real Numbers and Arithmetic Operations

2.1: (a) $4a^2 - 20ab^2 + 25b^4$; (b) $x^4 - 9y^2$.

2.2: (a) $4a^2(3a + 1)$; (b) $3(x - 2y)(a - 5b)$; (c) $x^{a-1}(x - x^4 - 1)$.

2.3: (a) $x_1 = 1, \quad x_2 = \frac{13}{3}$; (b) $x_1 = \frac{3}{2}, \quad x_2 = -\frac{17}{2}$.

2.4: $\frac{247}{504}$.

2.5: $\frac{a(a-1)(a-2)}{2-a^2}$.

2.6: $\frac{x+y}{x-y}$.

2.7: $\Big($**Note:** In the third fraction of the task, x must be replaced by u and y must be replaced by v, i.e.: $\frac{9v-u}{2uv-6v^2}\Big)$

$-\frac{2}{u}$.

2.8: $-\frac{4}{xy}$.

2.9: $\frac{3y-5x}{5y-3x}$.

2.10: $\frac{512}{27}$.

2.11: (a) $x^{10}y^{15}z^5$; (b) $\frac{9x^2}{1600a^7}$.

2.12: $x^3\, y^{n-2}\, z^4$.

2.13: (a) $27\,\sqrt[3]{9}$; (b) $56+14\sqrt{15}$; (c) $\sqrt[3]{2}$.

2.14: (a) $a^{4/5}=\sqrt[5]{a^4}$; (b) $\frac{1}{5}\sqrt{2}$; (c) $\frac{1+x}{1-x}\cdot\sqrt{1-x}$; (d) $\frac{a+\sqrt{a^3x}}{1-ax}$.

2.15: (a) $\frac{\ln 12}{\ln 1.03}\approx 84.06656$; (b) $\frac{\ln 10}{\ln 1.1}\approx 24.15886$; (c) -4.

2.16: (a) 3; (b) $\frac{7}{2}$.

3 Equations

3.1: (a) $x=\frac{5}{2}$; (b) $x=\frac{293}{53}$; (c) $x=2$;

(d) for $a=1$: $x\neq 0$ arbitrary; for $a\neq 1$: $x=1$;

(e) $x=12$; (f) $x=\frac{a^2+2ab-b^2}{a^2(b+1)}$ for $b\neq -1$.

3.2: $g=\frac{2(v_0\,t-s)}{t^2}$.

3.3: (a) $x=1$, $y=-3$; (b) no solution.

3.4: (a) $x_1=3$, $x_2=-4$; (b) $x_1=-\frac{2b}{a}$ for $a\neq 0$, $x_2=\frac{a^2}{b}$ for $b\neq 0$;

(c) no real solution; (d) $x_1=-2,\ x_2=4$;

(e) $x_1=\sqrt{\frac{b+1}{a-1}},\ x_2=-\sqrt{\frac{b+1}{a-1}}$ for $a>1,\ b\geq -1$ or $a<1,\ b\leq -1$;

x arbitrary for $a=1$ and $b=-1$; no solution for $a=1$ and $b\neq -1$;

(f) $x_1=a,\ x_2=\frac{b}{2}$; (g) $x_1=\sqrt{2},\ x_2=-\sqrt{2}$; (h) $x_1=100,\ x_2=\frac{1}{4}$.

3.5: (a) $x^2 + x - 30 = 0$; (b) $x^2 - 4x - 1 = 0$.

3.6: $\Big($**Note:** In the formulation of the task, the fraction $\frac{a}{x}$ must be replaced by the fraction $\frac{x}{a}$.$\Big)$

$x_{1/2} = -\frac{1}{2a} \pm \frac{\sqrt{1-4a^3}}{2a}$; the solutions are real for $a \leq \sqrt[3]{\frac{1}{4}}$;

3.7: (a) $x_1 = 2, \quad y_1 = 3$ and $x_2 = -2, \quad y_2 = -3$;

(b) $x_1 = 3, \quad y_1 = 5$ and $x_2 = 5, \quad y_2 = 3$.

3.8: (a) $x = 11$; (b) $x = \frac{1}{9}$; (c) $x = 16$; (d) $x = \left(\frac{a+b}{a-b}\right)^2$ for $a \neq b$;

(e) $x = a$ for $a < b$.

3.9: (a) $x = 5$; (b) $x = 9$ ($x = -9$ is not a solution); (c) $x = 1,000$;
(d) $x = 3$; (e) $x = 2$; (f) $x = 0$ ($x = 8$ is not a solution).

3.10: (a) $x = 3$; (b) $x = -4$; (c) $x = 1$; (d) $x = -2$; (e) $x = 0$;
(f) $x = -\frac{5}{7}$; (g) $x_1 = 1, \quad x_2 = 2$; (h) $x = \frac{\lg a}{1 + \lg b}$ for $\lg b \neq -1$.

3.11: 37.67 EUR.

3.12: 175 minutes.

3.13: daily distances: 190 km, 152 km and 114 km.

3.14: $x \approx 1.51$.

3.15: $x \approx 5.22$.

4 Inequalities

4.1: (a) $x \leq 17$; (b) $x \leq \frac{18}{11}$; (c) $x \in \left(-1, \frac{1}{3}\right]$; (d) $x \in (-\infty, -1) \cup [1, \infty)$;

(e) $x \in \left(-\infty, \frac{4}{7}\right) \cup \left(\frac{3}{2}, \infty\right)$.

4.2: (a) $x \in (-\infty, -4] \cup [3, \infty)$; (b) $x \in \left(-\infty, -\frac{3}{2}\right] \cup \left[\frac{10}{8}, \infty\right)$; (c) $x \in [-2, 2) \cup [4, \infty)$;

(d) $x \in (-3, -2) \cup \left(-\frac{1}{2}, \infty\right)$; (e) $x \in (-\infty, 2 - \sqrt{34}) \cup (7, 2 + \sqrt{34})$.

4.3: (a) $x \in (-4, -3) \cup (2, \infty)$; (b) $x \in [0, 2] \cup [3, \infty)$.

4.4: (a) $x \in (-\infty, -6] \cup [-\sqrt{14}, \sqrt{14}] \cup [6, \infty)$; (b) $x \in [-4, 4]$;

(c) $x \in (-\infty, -1) \cup (1, \infty)$; (d) $x \in \left[\frac{2}{3}, 2\right]$;

(e) $x \in (-\infty, -8] \cup \left[2, \frac{8}{3}\right]$; (f) $x \in [-3, 2]$.

4.5: (a) $x \in [-3, 0] \cup [1, 4]$; (b) $x \in (-2, 0)$.

4.6: (a) $x \in \left[2, \frac{33}{16}\right]$; (b) $x < 97$; (c) $x \leq \frac{2}{3}$; (d) $x \in (-\infty, -2-\sqrt{2}) \cup (-2+\sqrt{2}, \infty)$.

5 Trigonometry and Goniometric Equations

5.1: (a) $\alpha_1 = 14.4775^o$, $\alpha_2 = 165.5225^o$; (b) $\alpha_1 = 123.367^o$, $\alpha_2 = 236.633^o$;

(c) $\alpha_1 = 63.4349^o$, $\alpha_2 = 116.5651^o$, $\alpha_3 = 243.4349$, $\alpha_4 = 296.5651$;

(d) $\alpha_1 = 30^o$, $\alpha_2 = 150^o$, $\alpha_3 = 210^o$, $\alpha_4 = 330^o$.

5.2: $\beta = 72^0$, $a = 6.18$ cm, $b = 19.02$ cm.

5.3: $\alpha = 49,97^o$, $\beta = 40.03^o$, $c = 3.265$ m.

5.4: $\alpha = 30.37^o$, $\beta = 42.39^o$, $\gamma = 107.24^o$.

5.5: $A = 78.42$ cm^2.

5.6: (a) $x_1 = 30^o$, $x_2 = 150^o$;

(b) $x_1 = 42.58^o$, $x_2 = 137.42^o$, $x_3 = 227.65^o$, $x_4 = 312.35^o$;

(c) $x_1 = 90^o$, $x_2 = 270^o$, $x_3 = 187.18^o$, $x_4 = 352.82^o$;

(d) $x_1 = 71.57^o$, $x_2 = 251.57^o$, $x_3 = 45^o$, $x_4 = 225^o$.

6 Analytic Geometry in the Plane

6.1: (a) normal form: $y = -\frac{3}{7}x + \frac{2}{7}$, intercept form $\frac{x}{\frac{2}{3}} + \frac{y}{\frac{2}{7}} = 1$;

(b) normal form: $y = \frac{2}{5}x + 1$, intercept form: $\frac{x}{-\frac{5}{2}} + \frac{y}{1} = 1$;

(c) normal form: $y = -\frac{3}{11}x$, intercept form does not exist.

6.2: (a) $y = \sqrt{3}x + (1 - 3\sqrt{3})$; (b) $y = x$; (c) $y = -x + 2$.

6.3: (a) $y = -\frac{7}{2}x - \frac{13}{2}$; (b) $y = \frac{2}{3}x + \frac{19}{3}$.

6.4: $(x-1)^2 + (y-1)^2 = 25$.

6.5: parabola open from the right, apex: $(x_0, y_0) = \left(\frac{80}{24}, 0\right)$.

6.6: (a) parabola open from the left, apex: $(x_0, y_0) = (28, -2)$;

(b) parabola open from above, apex: $(x_0, y_0) = \left(0, \frac{10}{8}\right)$;

(c) parabola open from below, apex: $(x_0, y_0) = \left(-3, \frac{1}{5}\right)$;

6.7: the other half axis is $b = 10$, equation of the ellipse is $\frac{x^2}{25} + \frac{y^2}{100} = 1$

(note that $b = 5$ leads to a contradiction).

6.8: equation of the hyperbola is $\frac{(x-1)^2}{16} - \frac{(y-1)^2}{49} = 1$.

7 Sequences and Partial Sums

7.1: (a) $a_1 = 6$, $a_2 = 4$, $a_3 = \frac{10}{3}$, $a_4 = 3$, $a_5 = \frac{14}{5}$, $a_6 = \frac{8}{3}$;

(b) $b_1 = 0$, $b_2 = -\frac{1}{6}$, $b_3 = \frac{2}{7}$, $b_4 = -\frac{3}{8}$, $b_5 = \frac{4}{9}$, $b_6 = -\frac{1}{2}$;

(c) $c_1 = 3$, $c_2 = 9$, $c_3 = 19$, $c_4 = 33$, $c_5 = 51$, $c_6 = 73$.

7.2: a_n cannot be determined since $\{a_n\}$ is neither an arithmetic nor a geometric sequence.

7.3: 815.

7.4: $d = 3$, $a_1 = 0$, $a_n = 3(n-1)$.

7.5: There is no unique solution to the problem. We obtain the equality

$$n = \frac{650}{d} + 1.$$

Since n must be integer, we may have

$$d \in \{1, 2, 5, 10, 13, 25, 26, 50, 65, 130, 325, 650\}$$

(i.e., there are 12 solutions). Moreover,

$$s_n = \left(\frac{650}{d} + 1\right) \cdot 1402 \cdot \frac{1}{2}.$$

For instance, for $d = 10$, we obtain $n = 66$ and $s_{66} = 46,266$.

7.6: $\{a_n\}$ is not monotonic, the largest term is $a_3 = \frac{9}{16}$, limit $L = 0$.

7.7: $a_1 = 3$, a_{16} is the first term with an absolute value less than 0.01.

7.8: 174 terms are less than 700.

7.9: $a_1 = 18$, $q = \frac{1}{3}$.

7.10: (a) sequence $\{a_n\}$ is strictly decreasing;

(b) sequence $\{b_n\}$ is strictly decreasing

(note that the obtained quadratic equation in n does not change the sign for $n \geq 1$);

(c) sequence $\{c_n\}$ is strictly decreasing.

7.11: (a) sequence $\{a_n\}$ is strictly decreasing and bounded, limit $L = -6$;

(b) sequence $\{b_n\}$ is not monotonic but bounded, limit $L = 0$;

(c) sequence $\{c_n\}$ is decreasing and bounded, limit $L = 0$.

7.12: (a) $\frac{2}{e}$;

(b) for $a = 0$, the limit L does exist: $L = \frac{1}{3}$,

for $a \neq 0$, the limit L does not exist ($L = \infty$ for $a > 0$ and $L = -\infty$ for $a < 0$);

(c) for $c_1 = 1$ and $c_1 = 4$: $L = 0$.

7.13: (a) $s_1 = \frac{7}{2}$, $s_2 = \frac{15}{2}$, $s_3 = 12$, $s_4 = 17$, $s_5 = \frac{45}{2}$, $s_6 = \frac{57}{2}$;

(b) $s_1 = 0.3333$, $s_2 = 1.1458$, $s_3 = 2.6125$, $s_4 = 4.8417$, $s_5 = 7.8988$, $s_6 = 11.8259$;

(c) $s_1 = 0$, $s_2 = \frac{1}{3}$, $s_3 = -\frac{1}{6}$, $s_4 = \frac{13}{30} = 0.4333$, $s_5 = -\frac{7}{30} = -0.2333$,

$s_6 = \frac{101}{210} = 0.4810$.

7.14: (a) 97.5; (b) is not an arithmetic sequence, but for $b_3 = +1$, we get $s_{10} = -290$;

(c) -560.

7.15: (a) 0.2977; (b) 0.6; (c) $-12,285$.

7.16: (a) $14,450$; (b) 2.9986; (c) 63.9844.

8 Functions

8.1: (a) $D_f = \mathbb{R}, \quad R_f = \mathbb{R}$; (b) $D_f = \mathbb{R}, \quad R_f = \mathbb{R}_{\leq 2}$; (c) $D_f = \mathbb{R}, \quad R_f = \left[-\frac{25}{8}, \infty\right)$;

(d) $D_f = \mathbb{R}_{\geq -3}, \quad R_f = \mathbb{R}_{\geq 0}$; (e) $D_f = \mathbb{R}_{>0}, \quad R_f = \mathbb{R}$; (f) $D_f = \mathbb{R}, \quad R_f = \mathbb{R}_{\geq 0}$;

(g) $D_f = (\infty, -3] \cup [3, \infty), \quad R_f = \mathbb{R}_{\geq 0}$; (h) $D_f = \mathbb{R}_{>0}, \quad R_f = \mathbb{R}$;

(i) $D_f = \mathbb{R}, \quad R_f = \mathbb{R}_{\geq 0}$.

8.3: (a) $f(x) = 3x^3 + 2x^2 - 7x - 4 + \dfrac{16x + 8}{x^2 + 2}$;

(b) $f(x) = x^2 - x - 3 + \dfrac{4x^2 + 10x + 4}{x^3 + 3x + 1}$.

8.4: (a) $D_f = \mathbb{R}, \quad R_f = [-2, 2]$, zeroes: $x_k = \frac{\pi}{2} k - \frac{\pi}{4}, k \in \mathbb{Z}$; period: π;

(b) $D_f = \mathbb{R}, \quad R_f = \left[-\frac{1}{2}, \frac{1}{2}\right]$; zeroes: $x_k = 3\pi + 2k\pi, k \in \mathbb{Z}$, period: 4π;

(c) $D_f = \{x \in \mathbb{R} \mid x \neq \left(\frac{\pi}{2} - \sqrt{3}\right) + k\pi,\ k \in \mathbb{Z}\}, \quad R_f = \mathbb{R}$,

zeroes: $x_k = k\pi - \sqrt{3}, k \in \mathbb{Z}$, period: π;

(d) $D_f = \{x \in \mathbb{R} \mid x \neq k\pi - 1, k \in \mathbb{Z}\}, \quad R_f = \mathbb{R}$ (for $s \neq 0$), zeroes: $x_k = \left(\frac{\pi}{2} - 1\right) + k\pi,\ k \in \mathbb{Z}$,

period: π.

8.5: (a) $(f \circ g)(x) = e^x + 3, \qquad (g \circ f)(x) = e^{x+3}$;

(b) $f \circ g$ not defined, $(g \circ f)(x) = \sqrt{(x-1)(x^2+1)}$;

(c) $f(\circ g)(x) = (\ln x + 1)^2, \quad (g \circ f)(x) = \ln(x+1)^2$;

(d) $(f \circ g)(x) = 2x^2 + 2x - 5, \qquad (g \circ f)(x) = 4x^2 + 6x + 1$.

8.6: (a) $y = f^{-1}(x) = \frac{1}{2}(x + 7)$; (b) $y = f^{-1}(x) = \sqrt{x - 3}$; (c) $y = f^{-1}(x) = (x+1)^4$;

(d) $y = f^{-1}(x) = \frac{1}{2}\ln(x+1)$; (e) $y = f^{-1}(x) = 16\left(\dfrac{x+1}{x-1}\right)^2$; (f) f^{-1} not defined.

9 Differentiation

9.1: (a) $L_r = L_l = L = s$ (limit exists); (b) $L_r = L_l = 2$ (limit exists);

(c) $L_l = L_r = L = 0$ (limit exists); (d) $L_l = 5 \neq L_r = 4$ (limit does not exist) .

9.2: (a) $L = -4$; (b) $L_l = \infty, L_r = -\infty$ (limit does not exist);

(c) $L_l = -\infty, L_r = \infty$ (limit does not exist).

9.3: (a) f discontinuous at $x_0 = 4$, pole; (b) f continuous at $x_0 = -3$;

(c) f discontinuous at $x_0 = 1$, jump.

9.4: (a) $f'(x) = 2x - 5 + 3\sin x$; (b) $f'(x) = (3x^2 - 1)\cos x - (x^3 - x)\sin x$;

(c) $f'(x) = \dfrac{1 - \cos x + x\sin x}{(2 + \cos x)^2}$; (d) $f'(x) = 4(2x^3 - 3x + \ln x)^3 \cdot \left(6x^2 - 3 + \dfrac{1}{x}\right)$;

(e) $f'(x) = \cos(x^2 + 4x + 1)^3 \cdot 3(x^2 + 4x + 1)^2 \cdot (2x + 4)$;

(f) $f'(x) = 3\cos^2(x^2 + 4x + 1) \cdot [-\sin(x^2 + 4x + 1)] \cdot (2x + 4)$;

(g) $f'(x) = \dfrac{e^x \cdot \cos e^x}{2\sqrt{\sin e^x}}$; (h) $f'(x) = \dfrac{4x}{2x^2 - 1}$; (i) $f'(x) = \dfrac{1 - 2x - x^2}{(x^2 + 1)^2}$.

9.5: (a) $f'(x) = \dfrac{\ln 3x \cdot (2 - \ln 3x)}{x}$; (b) $f'(x) = \cos x - 2\sin x$; (c) $f'(x) = -\dfrac{1}{\sin x}$;

(d) $f'(x) = -\dfrac{3}{4}(x^{-3/4} - 2x^{-1/2} + x^{-1/4})$; (e) $f'(x) = 2xe^{x^2}(1 + x^2)$;

(f) $f'(x) = \dfrac{\left(e^x + \frac{1}{x}\right)\sin x - (e^x + \ln x)\cos x}{\sin^2 x}$; (g) $f'(x) = 3\sqrt{x^2 + 4x} \cdot (x + 2)$.

9.6: (a) $f'''(x) = \sin 2x(8x^2 - 12) - 24x\cos 2x$; (b) $f'''(x) = \dfrac{4}{x^3}$; (c) $f'''(x) = -\dfrac{72(5 - 3x)}{(x + 1)^5}$;

(d) $f'''(x) = 4e^{2x}(1 + 2x)$.

9.7: (a) $x_1 = 0$ local minimum point, $x_2 = \dfrac{1}{4}$ local maximum point,

$x_3 = 2$ local minimum point;

(b) $x = -1$ local maximum point;

(c) $x = 0$ local maximum point;

(d) $x_1 = 0$ local maximum point, $x_2 = 4$ local minimum point;

(e) $x_1 = \sqrt{3}$ local maximum point, $x_2 = -\sqrt{3}$ local minimum point;

(f) $x = \dfrac{3}{2}$ local maximum point;

(g) $x_1 = 2$ local minimum point with $f(2) = 0$;

$x_2 = e^{-2} + 1$ local maximum point with $f(e^{-2} + 1) = 4e^{-2}$;

(h) $x = -1$ local maximum point.

9.8: (a) $\dfrac{3}{2}$; (b) ∞ (i.e., limit does not exist); (c) $\dfrac{20}{5}$; (d) ∞; (e) ∞;

(f) 1; (g) 1; (h) 0.

9.9: (a) $D_f = \mathbb{R}$, zeroes: $x_1 = -2$, $x_2 = x_3 = x_4 = 2$; no discontinuities;

extreme point: $x_1 = -1$ local minimum point;

monotonicity: strictly decreasing for $x \in (-\infty - 1]$, strictly increasing for $x \in [-1, \infty)$;

inflection points: $x_1 = 0$, $x_2 = 2$;

convexity/concavity: strictly convex for $x \in (-\infty, 0]$ and $x \in [2, \infty)$, strictly concave for $x \in [0, 2]$;

$\lim\limits_{x \to \pm\infty} f(x) = \infty$;

(b) $D_f = \{x \in \mathbb{R} \mid x \neq 1\}$, zeroes: $x_1 = x_2 = x_3 = 0$, discontinuity: $x = 1$ pole;

extreme point: $x_1 = 3$ local minimum point;

monotonicity: strictly increasing for $x \in (-\infty, 1)$ and $[3, \infty)$, strictly decreasing for $x \in (1, 3]$;

inflection point: $x = 0$;

convexity/concavity: strictly convex for $x \in (0, 1)$ and $x \in (1, \infty)$,

strictly concave for $x \in (-\infty, 0)$;

$\lim\limits_{x \to -\infty} f(x) = -\infty$, $\lim\limits_{x \to \infty} f(x) = \infty$;

(c) $D_f = \{x \in \mathbb{R} \mid x \neq 2\}$, zero: $x = \frac{4}{3}$, discontinuity: $x = 2$ pole;

extreme point: $x = 1$ local maximum point,

monotonicity: strictly increasing for $x \leq 1$, strictly decreasing for $x \in [1, 2)$ and $x \in (2, \infty)$,

inflection point: $x = \frac{2}{3}$,

convexity/concavity: strictly convex for $x \leq \frac{2}{3}$ and $x > 2$, strictly concave for $x \in \left[\frac{2}{3}, 2\right)$,

$\lim\limits_{x \to \pm\infty} f(x) = 0$;

(d) $D_f = \mathbb{R}$, no zeroes, no discontinuities,

extreme point: $x = 1$ local maximum point,

monotonicity: strictly decreasing for $x \in (-\infty, 1]$, strictly increasing for $x \in [1, \infty)$;

no inflection points;

convexity/concavity: strictly convex for $x \in (-\infty, \infty)$;

$\lim\limits_{x \to \pm\infty} f(x) = \infty$;

(e) $D_f = \mathbb{R}$, zeroes: $x_1 = x_2 = 0$, extreme points: $x_1 = 0$ local minimum point, $x_2 = 2$ local maximum point;

monotonicity: strictly decreasing for $x \in (-\infty, 0]$ and $x \in [2, \infty)$, strictly increasing for $x \in [0, 2]$;

inflection points: $x_1 = 2 - \sqrt{2}$, $x_2 = 2 + \sqrt{2}$;

convexity/concavity: strictly convex for $x \in (-\infty, 2 - \sqrt{2}]$ and $x \in [2 + \sqrt{2}, \infty)$,

strictly concave for $x \in [2 - \sqrt{2}, 2 + \sqrt{2}]$;

$\lim_{x \to -\infty} f(x) = \infty$; $\lim_{x \to \infty} f(x) = 0$;

(f) $D_f = \{x \in \mathbb{R} \mid x \leq 2\}$, zeroes: $x = 0, x_2 = 2$, no discontinuities,

extreme points: $x_1 = 0$ local minimum point, $x_2 = \frac{4}{3}$ local maximum point,

monotonicity: strictly decreasing for $x \in (-\infty, 0]$ and $x \in \left[\frac{4}{3}, 2\right]$,

strictly increasing for $x \in \left[0, \frac{4}{3}\right]$;

inflection point: $x = 2$;

convexity/concavity: strictly concave for $x \in (-\infty, 0]$ and $x \in [0, 2]$,

$\lim_{x \to -\infty} f(x) = \infty$;

(**comment**: We defined roots in general only for non-negative numbers. If one would allow to calculate the root $x = \sqrt[n]{a}$ with an odd root exponent n also for negative numbers a as a solution of the equation $x^n = a$, one could extend the domain to $D_f = \mathbb{R}$ and gets the additional results: convex for $x \in [2, \infty)$; $\lim_{x \to \infty} f(x) = -\infty$);

(g) $D_f = \mathbb{R}$, zero: $x = 0$, no discontinuity;

extreme point: $x = \frac{\ln 4}{3} \approx 0.46$ local maximum point;

inflection point: $x = \frac{\ln 16}{3} \approx 0.92$;

$\lim_{x \to -\infty} f(x) = -\infty$, $\lim_{x \to \infty} f(x) = 0$;

(h) $D_f = \mathbb{R}$, zero: $x = 10$;

extreme points: $x_1 = 10(1 - \sqrt{2})$ local minimum point,

$x_2 = 10(1 + \sqrt{2})$ local maximum point,

monotonicity: strictly increasing for $x \in [10(1 - \sqrt{2}), 10(1 + \sqrt{2}]$,

strictly decreasing for $x \in (-\infty, 10(1 - \sqrt{2}]$ and $x \in [10(1 + \sqrt{2}, \infty)$;

inflection points: $x_1 = -10$, $x_2 = 10(2 - \sqrt{3})$, $x_3 = 10(2 + \sqrt{3})$

(**hint:** one zero of a cubic equation must be numerically found (or guessed), then perform polynomial division to get a quadratic equation and apply the solution formula);

convexity/concavity: strictly concave for $x \in (-\infty, x_1]$ and $x \in [x_2, x_3]$,

strictly convex for $x \in [x_1, x_2]$ and $x \in [x_3, \infty)$;

$\lim_{x \to \pm\infty} f(x) = 0$;

(i) $D_f = \{x \in \mathbb{R} \mid x \neq 0\}$, zero: $x = -\frac{1}{2}$,

extreme point: $x = \frac{1}{4}\sqrt[3]{4} \approx 0.39685$,

monotonicity: strictly increasing for $x \geq \frac{1}{4}\sqrt[3]{4}$, strictly decreasing for $x \leq \frac{1}{4}\sqrt[3]{4}$,

convexity/concavity: strictly convex for $x \leq -\frac{1}{2}$ and $x > 0$, strictly concave for $-\frac{1}{2} \leq x < 0$,

$\lim_{x \to \pm\infty} f(x) = \infty$.

9.10: $R = \frac{100}{\pi}$, $L = 100$ (with $A \approx 6,366 \text{ m}^2$).

9.11: radius $R = \sqrt[3]{\frac{V}{2\pi}}$, height $H = \sqrt[3]{\frac{4V}{\pi}}$

(i.e., the height is equal to the diameter $2R$).

9.12: $x \approx -0.4263$.

9.13: $x \approx 2.19582$.

10 Integration

10.1: (a) $x^3 + \frac{1}{2}\ln|x| + C$; (b) $\frac{5}{24}\sqrt[5]{x^{24}} - \frac{15}{17}\sqrt[15]{x^{17}} + C$; (c) $\frac{2^x}{\ln 2} - 0.4x^{-2.5} + C$.

10.2: (a) $\frac{1}{6}\ln^2 x + C$; (b) $-e^{\cos x} + C$; (c) $\frac{2}{3}\ln|3x-1| + C$;

(d) $-\frac{1}{3}e^{-(4+3x)} + C$; (e) $\frac{2}{15}\sqrt{(2+5x)^3} + C$; (f) $\frac{1}{3}\sqrt{1+x^2}\cdot(x^2-2) + C$;

(g) $\frac{1}{2}(1 + x^2 - \ln(1+x^2)) + C$; (h) $-\frac{1}{3}\sqrt{(2-x^2)^3} + C$; (i) $-\frac{1}{\sin x} - \sin x + C$;

(j) $\frac{1}{2}\sin^2 x + C$; (k) $2\ln|e^x + 1| - x + C$.

10.3: (a) $\frac{1}{4}e^{4x}\left(x-\frac{1}{4}\right)+C$; (b) $\frac{1}{2}e^{x}(\sin x-\cos x)+C$; (c) $x\tan x+\ln|\cos x|+C$;

(d) $\frac{1}{2}(x-\sin x\,\cos x)+C$; (e) $\frac{x^2}{2}\left(\ln x-\frac{1}{2}\right)+C$; (f) $\cos x(2-x^2)+2x\sin x+C$.

10.4: (a) 54; (b) 1; (c) $\frac{2}{3}$; (d) $\sqrt{20}-2$; (e) $\frac{1}{2}(9\ln 9-8)$; (f) $\frac{2}{3}$;

(g) $4-\frac{3}{2}\pi$.

10.5: (a)

$$\int_0^{2\pi}\cos x\,dx=0 \qquad\qquad \text{area: } 2$$

(b) 54; (c) $10\frac{2}{3}$.

10.6: (a) 0.5659; (b) 0.5589; (c) 0.5371.

11 Vectors

11.1: (a) sums:

$$\mathbf{a}+\mathbf{b}=\begin{pmatrix}2\\2\\9\end{pmatrix},\quad \mathbf{a}+\mathbf{c}=\begin{pmatrix}8\\-3\\-4\end{pmatrix},\quad \mathbf{b}+\mathbf{c}=\begin{pmatrix}2\\5\\1\end{pmatrix},$$

differences

$$\mathbf{a}-\mathbf{b}=\begin{pmatrix}6\\-8\\-5\end{pmatrix},\quad \mathbf{a}-\mathbf{c}=\begin{pmatrix}0\\-3\\8\end{pmatrix},\quad \mathbf{b}-\mathbf{c}=\begin{pmatrix}-6\\5\\13\end{pmatrix},$$

(for the opposite differences, the sign of each component changes);

(b)

$$6\mathbf{a}-3\mathbf{b}+2\mathbf{c}=\begin{pmatrix}38\\-33\\-21\end{pmatrix},\quad 5\mathbf{b}-3\mathbf{c}-\ =\begin{pmatrix}-26\\28\\51\end{pmatrix};$$

(c) angles:

$$\angle(\mathbf{a},\mathbf{b})=100.91^0,\qquad \angle(\mathbf{a},\mathbf{c})=84.10^0,\quad \angle(\mathbf{b},\mathbf{c})=141.73^0\,.$$

11.2: $|\mathbf{a}|=4,\quad \frac{1}{4}\begin{pmatrix}3\\\sqrt{6}\\1\end{pmatrix}.$

11.3: $z=4$.

11.4: $\lambda=-1$.

11.5: $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} 7 \\ -2 \\ 17 \end{pmatrix}$;

11.6: $A = \frac{1}{2}\sqrt{552} \approx 11.7473.$

12 Combinatorics, Probability Theory and Statistics

12.1: $P(4) \cdot P(3) = 4! \cdot 3! = 144.$

12.2: $P(6) = 6! = 720.$

12.3: $P(9; 2, 4, 1, 2) = 3,780.$

12.4: $\binom{10}{5} \cdot \binom{25}{3} \cdot \binom{5}{2} = 5,796,000.$

12.5: $\overline{V}(3, 8) = 3^8 = 6,561.$

12.6: (a) $C(4, 12) = \binom{12}{4} = 495.$ (b) $V(4, 12) = 11,880;$

12.7: (a) $\frac{1}{4}$; (b) $\frac{1}{2}$; (c) $\frac{1}{8}$; (d) $\frac{5}{8}$.

12.8: $B = A_1 \cup A_2 \cup A_3$, $C = \overline{A_1 \cap A_2 \cap A_3}$, $D = \overline{A_1} \cap \overline{A_2} \cap \overline{A_3}$.

12.9: $E(X) = 1.07$, $\sigma^2(X) = 1.7451.$

12.10: $P(0 \le X < 2) = 0.6321.$

12.11: (a) 0.8042; (b) 0.9308; (c) 0.2054.

12.12: (a) 0.2639; (b) 0.0574; (c) 0.0016.

12.13: (a) 0.97725; (b) $0,000032$; (c) 0.9744; (d) 0.000064.

12.14: 0.9545.

12.15: For $\alpha = 0.05$ we get: $26 \in \mathbf{A} = \{26, 27, 28, 29, 30\}$.

12.16: $10 \in \mathbf{A} = \{7, 8, \ldots, 17\}$.