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Exercises 'Mathematical Economics'

Series 2

1. Determine which of the following sets are convex by drawing each of them in the xy-plane:
   (a) $M_1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 > 8\}$;
   (b) $M_2 = \{(x,y) \in \mathbb{R}^2_+ \mid xy \geq 1\}$;
   (c) $M_3 = \{(x,y) \in \mathbb{R}^2 \mid \sqrt{x} + \sqrt{y} \leq 2\}$.

2. Consider the set of solutions of a system of linear inequalities:
$$M = \Big\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\Big\},$$
   where A is a matrix of order $m \times n$, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^m$. Prove that M is a convex set.

3. (a) Check the following functions f and g for convexity/concavity:
   (a) $f(x,y) = x + y - e^x - e^{x+y}$;
   (b) $g(x,y,z) = (x + 2y + 3z)^2$.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given with
$$f(x,y) = ax^2 + 2bxy + cy^2 + px + qy + r \qquad (a,b,c,p,q,r \in \mathbb{R}).$$
   (a) Show that f is strictly concave if $ac - b^2 > 0$ and $a < 0$, whereas it is strictly convex if $ac - b^2 > 0$ and $a > 0$.
   (b) Find necessary and sufficient conditions for function f to be concave/convex.

5. For which values of $a \in \mathbb{R}$ is the following function $f : \mathbb{R}^2 \to \mathbb{R}$ concave/convex:
$$f(x,y) = -6x^2 + (2a+4)xy - y^2 + 4ay.$$

6. Determine whether the following functions are (quasi-)convex or (quasi-)concave for $(x,y) \in \mathbb{R}^2_+ \setminus (0,0)$:
   (a) $f(x,y) = 100x^{1/3}y^{1/4}$;
   (b) $g(x,y) = x^2y^3$;
   (c) $h(x,y) = 250x^{0.02}y^{0.98}$. → **Homework** ✍

7. (a) Check whether the following functions $f : D_f \to \mathbb{R}$ are homogeneous. If so, of what degree?

$$f(x_1, x_2) = 5x_1^3 - 2x_1x_2 + 7x_2^3;$$
$$f(x_1, x_2) = 2x_1 + x_2 + 3\sqrt{x_1x_2};$$
$$f(x_1, x_2, x_3) = \frac{x_1x_2^2}{x_3} + 2x_1x_3.$$

(b) Give a proof of the '$\Longrightarrow$'-part of Euler's theorem:

$$\text{A function} f \text{ is homogeneous of degree } k \Longleftrightarrow$$

$$f_{x_1}(x_1, \ldots, x_n) \cdot x_1 + \ldots + f_{x_n}(x_1, \ldots, x_n) \cdot x_n = k \cdot f(x_1, \ldots, x_n)$$

(Hint: Start with the definition of a homogeneous function and interpret it as a function of α, thereby treating $x_1, \ldots, x_n$ as given parameters. Then take the derivative with respect to α on both sides.)

(c) Let $f : \mathbb{R}^n_+ \to \mathbb{R}$. Prove that function f cannot be *strictly* concave if f is homogeneous of degree one.
(Hint: Strict concavity requires

$$f(\mathbf{x}/2 + \mathbf{y}/2) > f(\mathbf{x})/2 + f(\mathbf{y})/2 \qquad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+$ with $\mathbf{x} \neq \mathbf{y}$. Let $\mathbf{y} = 2\mathbf{x} \in \mathbb{R}^n_+$ provided that $\mathbf{x} \in \mathbb{R}^n_+$. Then, assuming that f is homogeneous of degree one, obtain a contradiction.)

8. (a) Given

$$F(x, y) = x^2 + 3xy + 2yz + y^2 + z^2 - 11 = 0.$$

Does F implicitly define a function $z = f(x, y)$ around the point $(x^0, y^0, z^0) = (1, 2, 0)$. If so, determine f_x and f_y by the implicit-function theorem and evaluate them at the given point.

(b) Consider the following equilibrium condition for the goods-market:

$$\begin{aligned} Y &= C + I + G \\ C &= C_0 + c(Y - T) \\ T &= T_0 + tY \end{aligned}$$

Apply the implicit-function theorem and determine the partial derivatives Y_G, C_G and T_G (do **not** reduce the given system to one single equation before!).

9. Let the following system of equations be given: $\to$ **Homework** ✻

$$\begin{aligned} axy^3 + 4aby - 3bc &= 25 \\ 7bx^2 + 2xy + c &= 48 \end{aligned}$$

Determine the partial derivatives x_a, x_b and y_c. Which condition must be fulfilled for these derivatives to exist?