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Exercises 'Mathematical Economics'

Series 5

1. Consider the following nonlinear programming problem:

$$\begin{array}{rcl} f(x_1,x_2) & = & (x_1-4)^2+(x_2-3)^2 \to \max! \\ \text{subject to} & & x_1+x_2 \geq 5 \\ & & -3x_1-x_2 \leq -9 \\ & & x_1,\ x_2 \geq 0 \end{array}$$

and assume that $x_1 > 0$ and $x_2 > 0$.

   (a) Set up the KKT-conditions. Proceed in the following way: → **Homework** ✍
   – Show that an optimal point is not compatible with both constraints being active (i.e., satisfied with equality) simultaneously.
   – Assume that the second constraint is active whereas the first one is not and obtain a contradiction.
   – Now assume the opposite situation and solve for x_1, x_2 and λ_1.
   – Check whether a further solution exists for $\lambda_1 = \lambda_2 = 0$.

   (b) Check whether the KKT-conditions are necessary and/or sufficient for a local or global maximum point.

   (c) Draw a picture with the constraints and the level curves of the objective function.

2. Consider the following problem dealing with the determination of optimal labor supply by a given individual:

$$\begin{array}{rcl} U(c,\ell) & \to & \max! \\ \text{subject to} & & p\,c \leq w(T-\ell)+V \\ & & \ell \leq T \\ & & c,\ \ell \geq 0 \end{array}$$

with c denoting consumption, ℓ leisure, T the maximum amount of time at the individual's disposal, p the price of the consumption good, w the nominal wage rate (per hour of work) and V other (nominal) income.

   (a) Set up the KKT-conditions and discuss the three typical constellations that can occur:

   (i) $\ell = 0$, (ii) $\ell = T$ and (iii) $0 < \ell < T$.

   Give also a graphical representation for these cases.

(b) Under which conditions are the KKT-conditions necessary and/or sufficient for a maximum here?

3. Consider the following nonlinear programming problem: → **Homework** ✍

$$\begin{aligned} f(x_1, x_2, x_3) &= -x_1 x_2 x_3 \to \min! \\ \text{subject to} \quad & -2x_1^2 - 2x_2^2 - x_3 + 46 \geq 0 \\ & 10 - x_3 \geq 0 \\ & x_1 + x_2 + x_3 - 1 \geq 0 \\ & x_1,\ x_2,\ x_3 \geq 0 \end{aligned}$$

Show that $x_1^* = 3, x_2^* = 3$ and $x_3^* = 10$ is a solution of the KKT-conditions and determine also the values of λ_1^*, λ_2^* and λ_3^*.

4. (a) Prove Theorem 8.

   (b) Prove the complementary slackness condition $\lambda^* \cdot g(\mathbf{x}^*) = 0$, where λ^* is the vector of the Lagrangian multipliers at the optimum and $g(\mathbf{x}^*)$ is the vector of the constraint functions evaluated at the optimum.

5. Consider again the linear programming problem given as problem 5 in Series 4.

   (a) Give a graphical representation of the constraints as well as the level curves of the objective function.

   (b) Solve the problem by the simplex algorithm. → **Homework** ✍

   (c) If you compare this solution procedure with the application of the KKT-conditions, where do you see the most important difference(s)?

6. Consider the following optimization problem:

$$\begin{aligned} f(x_1, x_2) &= 15 - 7x_1 + 7x_2^3 - 21x_2^2 + 14x_2 \to \max! \\ \text{subject to} \quad g(x_1, x_2) &= 1.5x_1 + x_2 - 3 = 0 \\ & x_1,\ x_2 \geq 0 \end{aligned}$$

   (a) Solve the problem first by the Lagrange multiplier method. Doing this, disregard the non-negativity constraints and check them only after you have obtained the solution(s). → **Homework** ✍

   (b) Now solve the problem by means of the KKT-conditions. → **Homework** ✍

   (c) What is the difference to the solution(s) obtained in (a) and how can it be explained?