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**Exercises 'Mathematical Economics'**

Series 6

1. Consider the following minimization problem:

$$f(x,y) = 3x^2 + 2y^2 \to \min!$$

s.t.

$$g(x,y) = \frac{1}{2}\,x + \frac{1}{3}\,y - 10 = t, \quad t \in \mathbb{R}; \quad x, y \in \mathbb{R}.$$

(a) Solve the problem in the usual way by means of the Lagrangian function and the corresponding necessary and sufficient optimality conditions. After having obtained the solution $(x^*(t), y^*(t), \lambda^*(t))$, determine

$$\frac{dx^*}{dt}(t=0), \quad \frac{dy^*}{dt}(t=0) \quad \text{and} \quad \frac{d\lambda^*}{dt}(t=0).$$

(b) Let $F(t) := f(x^*(t), y^*(t))$. Determine $F'(t=0)$ by a direct calculation and by means of the envelope theorem.

2. Assume that a firm has a certain quantity $\overline{y}$ of a good $y$ at its disposal, which can be sold on two separated markets. The demand on the first market is given by

$$y_1(p_1) = -ap_1 + b$$

and that on the second market by

$$y_2(p_2) = -cp_2 + d$$

$(a, b, c, d > 0)$, where it is assumed that $b + d > 2\overline{y}$. Let $\mathbf{r} = (a, b, c, d, \overline{y})^T$ be the vector of the parameters. The firm's problem is now to choose the two prices $p_1$ and $p_2$ in such a way that total revenue $\overline{R}$ is maximized, i.e.:

$$R(p_1, p_2) = -\overline{R}(p_1, p_2) = -p_1 \cdot y_1(p_1) - p_2 \cdot y_2(p_2) \to \min!$$

s.t.

$$y_1(p_1) + y_2(p_2) \leq \overline{y}, \qquad p_1, p_2 \geq 0.$$

(**Remark:** In principle, one should also explicitly postulate the two non-negativity constraints for $y_1(p_1)$ and $y_2(p_2)$, i.e. $-ap_1 + b \geq 0$ and $-cp_2 + d \geq 0$). However, if the conditions

$$bc - ad - 2c\overline{y} < 0 \qquad (1)$$

and

$$bc - ad + 2a\overline{y} > 0 \tag{2}$$

are satisfied, then $y_1(p_1^*) > 0$ and $y_2(p_2^*) > 0$ are satisfied so that one can drop the corresponding non-negativity constraints - hereafter an asterisk denotes the values for an optimal solution).

(a) Set up the KKT-conditions for this problem, assume directly that $p_1, p_2, \lambda > 0$ and determine an optimal solution.

(b) Now prove the two parts of the envelope theorem for this concrete problem, thereby making use of the information obtained from part (a). Take $b$ as the parameter to be varied. Thus, show
- first that

$$\frac{\partial L^*(\mathbf{r})}{\partial b} = \frac{\partial L(p_1, p_2, \lambda, \mathbf{r})}{\partial b}\Big|_{p_1^*(\mathbf{r}), p_2^*(\mathbf{r}), \lambda^*(\mathbf{r})}$$

- second that

$$\frac{\partial R^*(\mathbf{r})}{\partial b} = \frac{\partial L(p_1, p_2, \lambda, \mathbf{r})}{\partial b}\Big|_{p_1^*(\mathbf{r}), p_2^*(\mathbf{r}), \lambda^*(\mathbf{r})}$$

(c) Now apply the second part above of the envelope theorem directly for the determination of

$$\frac{\partial R^*(\mathbf{r})}{\partial \overline{y}} \quad \text{and} \quad \frac{\partial R^*(\mathbf{r})}{\partial c}.$$

(d) Now verify that

$$\frac{\partial R^*(\mathbf{r})}{\partial \overline{y}} < 0, \quad \frac{\partial R^*(\mathbf{r})}{\partial b} < 0 \quad \text{and} \quad \frac{\partial R^*(\mathbf{r})}{\partial c} > 0.$$

Try to give an economic explanation for these results.

3. → **Homework** ✍

Two resources $x$ and $y$ are to be allocated to two agents, each of them having a utility function of the form

$$U(x_i, y_i) = x_i\, y_i, \quad i = 1, 2.$$

Let $\alpha \in (0, 1)$ denote the weight of the welfare of agent 1 in the social planner's utility function and $1 - \alpha$ the corresponding weight of agent 2. Assume that the maximum amounts $\overline{x}$ and $\overline{y}$ of the two goods are given. Thus, the social planner's optimization problem can be formulated as follows:

$$\begin{aligned} F(x_1, x_2, y_1, y_2) &= \alpha x_1 y_1 + (1-\alpha) x_2 y_2 \to \max! \\ \text{subject to} \quad & x_1 + x_2 \le \overline{x} \\ & y_1 + y_2 \le \overline{y} \\ & x_1,\ x_2,\ y_1,\ y_2 \ge 0 \end{aligned}$$

(a) Since the optimum must be characterized by a full exploitation of the two resources (for obvious reasons), the two inequality constraints can be replaced by equalities. Now neglect in a first step the non-negativity constraints and solve the problem in the usual way using the necessary optimality conditions for the Lagrangian function. After having obtained a solution, check whether the non-negativity constraints are fulfilled.

(b) Now solve the same problem by means of the KKT-conditions and compare the solution with the previous one. How can the difference be explained?

4. Have a look at the following utility maximization problem:

$$\begin{aligned} U(x_1, x_2) &= x_1^{2/3}\, x_2^{1/3} \to \max! \\ \text{subject to} \quad & p_1 x_1 + p_2 x_2 \leq 12 \qquad \text{(with } p_1 = 1 \text{ and } p_2 = 4\text{)} \\ & x_1,\ x_2 \geq 0 \qquad \text{and additionally } x_1 \geq C. \end{aligned}$$

The last constraint can be interpreted, e.g., as the quantity of fuel oil needed to keep the temperature in one's flat above a certain minimum level.

(a) Let $C = 4$ and solve the problem by means of the KKT-conditions.
→ **Homework** ✍

(b) Try to give an economic interpretation of the Lagrangian multipliers.

(c) Now assume $C = 8$. Try to explain why the Lagrangian multiplier corresponding to the constraint $x_1 \geq 8$ is zero here despite the fact that this constraint is active at the optimum.

(d) Finally, consider the case $C = 15$.