

Exercises ‘Mathematical Economics’

Series 8

1. Consider the following second-order differential equation:

$$\ddot{x} = -6\dot{x} - 5x + 15.$$

- (a) Transform the above equation into a system of two first-order equations.
- (b) Solve the given second-order differential equation by solving the corresponding first-order system obtained in (a).
- (c) Check that your solution determined in (b) does indeed fulfill the above second-order differential equation.
- (d) Determine the particular solution for the following initial values:

$$x(0) = -2 \quad \text{and} \quad x(1) = 3.68859.$$

2. → **Homework**

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- (a) Let Λ be a diagonal matrix, the diagonal elements of which consist of the eigenvalues of a matrix A , which are assumed to be real and distinct, and P be a matrix, the columns of which consist of the corresponding eigenvectors, which are linearly independent. Prove that

$$e^{P\Lambda t P^{-1}} = P e^{\Lambda t} P^{-1}.$$

- (b) Show (for the 2-dimensional case) that

$$e^{\Lambda t} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}.$$

3. Consider the following systems of differential equations:

- (a) $\dot{x} = x + y + 4;$ $x(0) = 3,$
 $\dot{y} = 4x + y + 19;$ $y(0) = 13;$ → **Homework**
- (b) $\dot{x} = 3x + 6y - 15;$ $x(0) = 8;$
 $\dot{y} = -6x + 3y + 15;$ $y(0) = 3.$

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Determine in both cases:

- i) the solution of the corresponding homogeneous system;
- ii) a particular solution of the non-homogeneous system

(**Hint:** Compute simply an equilibrium state which is characterized by $\dot{x} = 0$ and $\dot{y} = 0$);

- iii) the general solution of the non-homogeneous system and
 - iv) the particular solution of the initial value problem.
4. Consider again both systems given in Exercise 3 and do the following:
- (a) Draw the nullclines into the phase plane and determine the directions of motion above and below them.
 - (b) Draw (in a qualitative way) some typical trajectories into this picture.
5. Consider the following small macroeconomic model:

$$\pi(t) = 0.375 [U(t) - 2.2]^2 - 1.5 + \pi^e(t) \quad (\text{expectations-augmented Phillips-curve}) \quad (1)$$

$$\dot{\pi}^e(t) = \alpha [\pi(t) - \pi^e(t)], \quad \alpha > 0 \quad (\text{adaptive expectations}) \quad (2)$$

$$\dot{U} = \beta [\pi(t) - \bar{m}], \quad \beta > 0 \quad (\text{macroeconomic context, e.g. IS-LM}) \quad (3)$$

where:

$\pi(t)$: actual rate of inflation, $\pi \in \mathbb{R}$;

$\pi^e(t)$: expected rate of inflation, $\pi^e \in \mathbb{R}$;

$U(t)$: actual rate of unemployment, $U \in [0, 1]$;

\bar{m} : constant growth rate of nominal money supply.

- (a) Insert equation (1) into equations (2) and (3), thereby reducing the model to a two-dimensional nonlinear system of differential equations in the two dynamical variables π^e and U .
 - (b) Draw the nullclines of the system obtained in (a) into the phase plane and determine the directions of motion above and below them. Then draw (in a qualitative way) some trajectories into this picture.
6. → **Homework** ✎

In an economic model, $K = K(t)$ denotes capital, $C = C(t)$ consumption, while α, A , and r are positive constants with $\alpha < 1$. Assume that

$$\dot{K} = AK^\alpha - C, \quad \dot{C} = C(\alpha AK^{\alpha-1} - r).$$

Perform a phase plane analysis of this system when $A = 2, \alpha = 0.5$, and $r = 0.05$.