

Exercises ‘Mathematical Economics’

Series 9

1. Solve the problem

$$\int_0^1 (-t\dot{x} - \dot{x}^2) dt \rightarrow \max!$$

subject to $x(0) = 1$ and

- (a) $x(1)$ free;
- (b) $x(1) \geq 1$.

2. Solve the control problem

$$\max_{u \in \mathbb{R}_+} \int_0^1 [-x(t) - u(t)^2] dt; \quad \dot{x}(t) = -u(t), \quad x(0) = 0, \quad x(1) \text{ free.}$$

3. → **Homework** (see Kamien/Schwartz ‘*Dynamic Optimization*’) ◀

A firm has an order of B units of a commodity to be delivered at time T . Let $x(t)$ be the stock at time t and assume that the cost per unit of time of storing $x(t)$ units is $ax(t)$. The increase in $x(t)$, which equals production per unit of time, is $u(t) = \dot{x}(t)$. Assume that the total cost of production per unit of time is equal to $b(u(t))^2$. Here a and b are positive constants. So the firm’s problem is

$$\min_{\{u(t) \geq 0\}} \int_0^T [ax(t) + bu(t)^2] dt; \quad \dot{x}(t) = u(t), \quad x(0) = 0, \quad x(T) = B$$

- Give the necessary conditions for an optimal solution.
- Find the only possible solution to the problem and explain why it really is a solution.
 (Hint: Distinguish between the cases $B \geq aT^2/4b$ and $B < aT^2/4b$.)
- Now assume that the time horizon can be chosen freely by the firm so that the costs are now minimized with respect to $u(t)$ and T . Determine the optimal value for T .

4. Assume that a social planner wants to optimize the path of per-capita-consumption $c(t)$ of the population over time:

$$U = \int_0^{100} 15 \ln[c(t)] \cdot e^{-0.06t} dt$$

subject to

$$\begin{aligned} \dot{k}(t) &= 3.2 \cdot [k(t)]^{0.25} - \delta k(t) - c(t), & \delta &= 0.04 \\ k(0) &= 10, & k(100) &= 0 \end{aligned}$$

with k denoting the per-capita-capital stock and δ the rate of depreciation.

- (a) Set up the **current value** Hamiltonian function for this problem and derive the necessary conditions for an optimal path.
- (b) Derive from the results obtained in (a) a dynamical system in the variables k and c and determine its equilibrium state.
- (c) Draw the nullclines into the phase plane and determine the directions of motion above and below them. Draw some typical trajectories into the phase plane as well.
- (d) Now linearize the system obtained in (b) at the (economically relevant) equilibrium state.
- (e) Compute the explicit solution of the linearized system obtained in (d). Which consumption level $c(0)$ has optimally to be chosen at $t = 0$, given the above initial value $k(0) = 10$?