Faculty of Mathematics
Institute of Mathematical Optimization
Prof. Dr. F. Werner

Exercises 'Methods for Economists'

Series 1

1. Investigate the definiteness of the following quadratic forms $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$:
   (a) $Q(x_1, x_2, x_3) = 5x_1^2 + 2x_1x_3 + 2x_2^2 + 2x_2x_3 + 4x_3^2$;
   (b) $Q(x_1, x_2) = -(x_1 - x_2)^2$;
   (c) $Q(x_1, x_2, x_3) = -3x_1^2 + 2x_1x_2 - x_2^2 + 4x_2x_3 - 8x_3^2$.
2. Determine the gradients of functions f and g at the given points:
   (a) $f(x, y) = xy + y^2$ at $(x^0, y^0) = (2, 1)$;
   (b) $g(x, y, z) = xe^{xy} - z^2$ at $(x^0, y^0, z^0) = (0, 0, 1)$.
3. Determine the directional derivatives of functions f and g at the given points in the given direction:
   (a) $f(x, y) = 2x + y - 1$ at $(x_0, y_0) = (2, 1)$ in the direction $\mathbf{r^1} = (1, 1)^T$;
   (b) $g(x, y, z) = xe^{xy} - z^2 - xy$ at $(x_0, y_0, z_0) = (0, 1, 1)$ in the direction $\mathbf{r^2} = (1, 1, 1)^T$.
4. → **Homework** ✍
   Given is the function $f : D_f \to \mathbb{R}$ with
   $$f(x, y, z) = xy \ln(x^2 + y^2 + z^2).$$
   (a) Find the directional derivative of f at the point $(x_0, y_0, z_0) = (1, 1, 1)$ in the direction given by the vector from point $(3, 2, 1)$ to point $(-1, 1, 2)$.
   (b) Determine the direction of maximal increase from point $(x_0, y_0, z_0) = (1, 1, 1)$.
5. Find the quadratic approximations for the following functions at the point $(0, 0)$:
   (a) $f(x, y) = e^{x+y}(xy - 1)$;
   (b) $g(x, y) = e^{xe^y}$; → **Homework** ✍
   (c) $h(x, y) = \ln(1 + x^2 + y^2)$.
6. Assume that point $(x_0, y_0; u_0, v_0)$ satisfies the two equations
   $$F(x, y; u, v) = x^2 - y^2 + uv - v^2 + 3 = 0$$
   $$G(x, y; u, v) = x + y^2 + u^2 + uv - 2 = 0$$
   Give sufficient conditions for this system to be represented by two equations
   $$u = f(x, y), \qquad v = g(x, y)$$

in a neighborhood of this point. Show that this condition is satisfied for

$$(x_0, y_0; u_0, v_0) = (2, 1; -1, 2).$$

(a) Compute $f_x(2, 1)$ and $g_x(2, 1)$.

(b) Compute $f_y(2, 1)$ and $g_y(2, 1)$. → **Homework** ✍