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Exercises 'Methods for Economists'

Series 4

1. Solve the following problem by means of the KKT-conditions:

$$\begin{array}{rcl} f(x_1,x_2) & = & x_1^2 + x_2 \to \min! \\ \text{subject to} & & x_1^2 + x_2^2 \quad \le 9 \\ & & x_1 + x_2 \quad \le 1 \end{array}$$

   Is the solution obtained globally minimal?

2. Solve the following problem by means of the KKT-conditions: → **Homework** ✍

$$\begin{array}{rcl} f(x_1,x_2) & = & (x_1-1)^2 + e^{x_2^2} - 1 \to \min! \\ \text{subject to} & & x_1^2 + x_2^2 \ \le \ 1 \end{array}$$

   Why is the obtained solution a global minimum point?

3. Consider the following nonlinear programming problem:

$$\begin{array}{rcl} f(x_1,x_2) & = & -x_1 - x_2 - 8 \to \min! \\ \text{subject to} & & 2x_1^2 + x_2^2 \quad \le 6 \\ & & -x_1 + 3x_2 \ \ge 1 \\ & & x_1, \ x_2 \ \ge 0 \end{array}$$

   (a) Set up the KKT-conditions and show that $x_1 = 0$ or $x_2 = 0$ is not compatible with them.

   (b) Compute the concrete solution (Hint: $\lambda_2^* = 0$).

   (c) Show by means of the saddle-point theorem that the solution obtained in (b) is indeed a global minimum point for the problem under consideration.

4. Consider the problem:

$$\begin{array}{rcl} f(x_1,x_2) & = & \min\{x_1, 2x_2\} \to \max! \\ \text{subject to} & & x_1 + x_2 \le 3 \\ & & x_1, \ x_2 \ \ge 0 \end{array}$$

   (a) Give a graphical representation of the problem and identify the solution for x_1 and x_2.

(b) Show by means of the saddle-point theorem that the point obtained in (a) is compatible with it and determine the value of λ^* (Hint: Treat the two half-spaces $x_1 \geq 2x_2$ and $x_1 \leq 2x_2$ separately).

5. Solve the following linear programming problem by means of the KKT-conditions:
$\rightarrow$ **Homework** ✍

$$\begin{aligned} f(x_1, x_2) &= 4x_1 + 3x_2 \rightarrow \max! \\ \text{subject to} \quad & x_1 + 5x_2 \leq 45 \\ & x_1 + 2x_2 \leq 21 \\ & 3x_1 + x_2 \leq 33 \\ & x_1, \ x_2 \geq 0 \end{aligned}$$

(Hint: Show first that only $x_1 > 0$ and $x_2 > 0$ can be a solution. Then analyze the different possible constellations for the λ_i values, i.e. $\lambda_1, \lambda_2, \lambda_3 > 0$; $\lambda_1 = 0, \lambda_2, \lambda_3 > 0$; etc.)

6. Consider the following nonlinear programming problem:

$$\begin{aligned} f(x_1, x_2) &= (x_1 - 5)^2 + 4(x_2 - 1)^2 - 14 \rightarrow \min! \\ \text{subject to} \quad & x_1^2 - 4x_1 - x_2 \leq -4 \\ & 2x_1 + x_2 \leq 7 \\ & x_1, \ x_2 \geq 0 \end{aligned}$$

(a) Check whether the conditions are fulfilled which ensure that the KKT-conditions are sufficient and necessary for a global constrained minimum point.

(b) Give a graphical representation of
– the two constraints,
– the resulting set of feasible solutions,
– the level curves of the objective function
and try to identify the solution.

(c) Now verify that the solution found in (b) satisfies the KKT-conditions.
$\rightarrow$ **Homework** ✍