Faculty of Mathematics
Institute of Mathematical Optimization
Prof. Dr. F. Werner

Exercises „Mathematical Economics“

Series 1

1. Investigate the definiteness of the following quadratic forms $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ resp. matrix A:
   (a) $Q(x_1, x_2, x_3) = 5x_1^2 + 2x_1x_3 + 2x_2^2 + 2x_2x_3 + 4x_3^2$;
   (b) $Q(x_1, x_2) = -(x_1 - x_2)^2$;
   (c) $Q(x_1, x_2, x_3) = -3x_1^2 + 2x_1x_2 - x_2^2 + 4x_2x_3 - 8x_3^2$.
2. Determine the gradients of functions f and g at the given points:
   (a) $f(x, y) = xy + y^2$ at $(x^0, y^0) = (2, 1)$;
   (b) $g(x, y, z) = xe^{xy} - z^2$ at $(x^0, y^0, z^0) = (0, 0, 1)$.
3. Determine the directional derivatives of functions f and g at the given points in the given direction:
   (a) $f(x, y) = 2x + y - 1$ at $(x^0, y^0) = (2, 1)$ in the direction $\mathbf{r^1} = (1, 1)^T$;
   (b) $g(x, y, z) = xe^{xy} - z^2 - xy$ at $(x^0, y^0, z^0) = (0, 1, 1)$ in the direction $\mathbf{r^2} = (1, 1, 1)^T$.
4. Given is the function $f : D_f \to \mathbb{R}$ with
$$f(x, y, z) = xy \ln(x^2 + y^2 + z^2).$$
   (a) Find the directional derivative of f at the point $(x^0, y^0, z^0) = (1, 1, 1)$ in the directien given by the vector from point $(3, 2, 1)$ to point $(-1, 1, 2)$.
   (b) Determine the direction of maximal increase from point $(x^0, y^0, z^0) = (1, 1, 1)$
5. Find the quadratic approximations for the following functions at the point $(0, 0)$:
   (a) $f(x, y) = e^{x+y}(xy - 1)$;
   (b) $g(x, y) = e^{xe^y}$;
   (c) $h(x, y) = \ln(1 + x^2 + y^2)$.
6. Assume that point $(x^0, y^0; u^0, v^0)$ satisfies the two equations
$$\begin{aligned} F(x, y; u, v) &= x^2 - y^2 + uv - v^2 + 3 = 0 \\ G(x, y; u, v) &= x + y^2 + u^2 + uv - 2 = 0 \end{aligned}$$

Give sufficient conditions for this system to be represented by two equations

$$u = f(x, y), \quad v = g(x, y)$$

in a neighborhood of this point. Show that this condition is satisfied for $(x^0, y^0; u^0, v^0) = (2, 1; -1, 2)$. Compute $f_x(2, 1), f_y(2, 1), g_x(2, 1)$ and $g_y(2, 1)$.

7. Determine which of the following sets are convex by drawing each of them in the xy-plane:

   (a) $M_1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 > 8\}$;

   (b) $M_2 = \{(x, y) \in \mathbb{R}^2_+ \mid xy \geq 1\}$;

   (c) $M_3 = \{(x, y) \in \mathbb{R}^2 \mid \sqrt{x} + \sqrt{y} \leq 2\}$.

8. Consider the set of solutions of a system of linear inequalities:

$$M = \Big\{ \mathbf{x} \mid A\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq \mathbf{0} \Big\},$$

   where A is a matrix of order $m \times n$, $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^m$. Prove that M is a convex set.

9. Check the following functions f and g for convexity/concavity:

   (a) $f(x, y) = x + y - e^x - e^{x+y}$;

   (b) $g(x, y, z) = (x + 2y + 3z)^2$.

10. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given with

$$f(x, y) = ax^2 + 2bxy + cy^2 + px + qy + r \qquad (a, b, c, p, q, r \in \mathbb{R}).$$

   (a) Show that f is strictly concave if $ac - b^2 > 0$ and $a < 0$, whereas it is strictly convex if $ac - b^2 > 0$ and $a > 0$.

   (b) Find necessary and sufficient conditions for function f to be concave/convex.

11. For which values of $a \in \mathbb{R}$ is the following function $f : \mathbb{R}^2 \to \mathbb{R}$ concave/convex:

$$f(x, y) = -6x^2 + (2a + 4)xy - y^2 + 4ay.$$

12. Determine whether the following functions are (quasi-)convex or (quasi-)concave for $(x, y) \in \mathbb{R}^2 \setminus (0, 0)$:

   (a) $f(x, y) = 100x^{1/3}y^{1/4}$;

   (b) $g(x, y) = x^2y^3$;

   (c) $h(x, y) = 250x^{0.02}y^{0.98}$.

13. Determine whether the following functions are quasi-concave:

   (a) $f(x) = 5x + 7$;

   (b) $g(x, y) = ye^x \quad$ with $y > 0$;

   (c) $h(x, y) = -x^2y^3$.