Faculty of Mathematics
Institute of Mathematical Optimization
Prof. Dr. F. Werner

Exercises „Mathematical Economics“

Series 2

1. (a) Check whether the following functions $f : D_f \to \mathbb{R}$ are homogeneous. If so, of what degree?

$$f(x_1, x_2) = 5x_1^3 - 2x_1x_2 + 7x_2^3;$$
$$f(x_1, x_2) = 2x_1 + x_2 + 3\sqrt{x_1x_2};$$
$$f(x_1, x_2, x_3) = \frac{x_1x_2^2}{x_3} + 2x_1x_3.$$

   (b) Give a proof of the '$\Longrightarrow$'-part of Euler's theorem:

$$\text{A function} f \text{ is homogeneous of degree } k \Longleftrightarrow$$

$$f_{x_1}(x_1, \ldots, x_n) \cdot x_1 + \ldots + f_{x_n}(x_1, \ldots, x_n) \cdot x_n = k \cdot f(x_1, \ldots, x_n)$$

   (Hint: Start with the definition of a homogeneous function and interpret it as a function of α, thereby treating $x_1, \ldots, x_n$ as given parameters. Then take the derivative with respect to α on both sides.)

   (c) Let $f : \mathbb{R}^n_+ \to \mathbb{R}$. Prove that function f cannot be *strictly* concave if f is homogeneous of degree one.
   (Hint: Strict concavity requires

$$f(\mathbf{x}/2 + \mathbf{y}/2) > f(\mathbf{x})/2 + f(\mathbf{y})/2 \qquad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+$$

   for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+$ with $\mathbf{x} \neq \mathbf{y}$. Let $\mathbf{y} = 2\mathbf{x} \in \mathbb{R}^n_+$ provided that $\mathbf{x} \in \mathbb{R}^n_+$. Then, assuming that f is homogeneous of degree one, obtain a contradiction.)

2. (a) Given

$$F(x, y) = x^2 + 3xy + 2yz + y^2 + z^2 - 11 = 0.$$

   Does F implicitly define a function $z = f(x, y)$ around the point $(x^0, y^0, z^0) = (1, 2, 0)$. If so, determine f_x and f_y by the implicit-function theorem and evaluate them at the given point.

   (b) Let the following system of equations be given:

$$\begin{aligned} axy^3 + 4aby - 3bc &= 25 \\ 7bx^2 + 2xy + c &= 48 \end{aligned}$$

   Determine the partial derivatives x_a, x_b and y_c. Which condition must be fulfilled for these derivatives to exist?

(c) Consider the following equilibrium condition for the goods-market:

$$\begin{aligned} Y &= C + I + G \\ C &= C_0 + c(Y - T) \\ T &= T_0 + tY \end{aligned}$$

Apply the implicit-function theorem and determine the partial derivatives Y_G, C_G and T_G (do **not** reduce the given system to one single equation before!).

3. Consider the function $g : D_g \to \mathbb{R}$ with $D_g = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$ given by

$$g(x, y) = x^3 + y^3 - 3x - 2y.$$

Show that function g is convex and determine its global minimum value.

4. Let function $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = x^3 + y^3 - 3xy.$$

Determine all local extreme points of function f.

5. Let $f : \mathbb{R}^3 \to \mathbb{R}$ and consider the problem:

$$\begin{aligned} f(x_1, x_2, x_3) &= 4x_1 + 5x_2 + 3x_3 \to \max! \\ \text{subject to} \quad & 2x_1^2 + x_2^2 + 3x_3^2 = 36 \end{aligned}$$

Solve this problem by using the Lagrange multiplier method and checking the necessary and sufficient conditions.

6. Let $f : \mathbb{R}^3 \to \mathbb{R}$ and consider the problem:

$$\begin{aligned} f(x_1, x_2, x_3) &= 5x_1 + 3x_2 - x_3 \to \max! \\ \text{subject to} \quad & x_1^2 + x_2^2 + x_3^2 = 72 \\ & 4x_1 - x_2 = 0 \end{aligned}$$

Apply the Lagrange multiplier method (i.e. do **not** reduce the problem to one with only two variables and one constraint).

7. Consider the following cost minimization problem. Let

$$f(x, y) = x^\alpha y^\beta \qquad (\alpha, \beta > 0,\ \alpha + \beta = 1)$$

denote the production function and

$$c(x, y) = ax + by$$

the cost associated with the use of the two production factors x and y. Finally, let w denote the required output level.

(a) Set up the corresponding optimization problem and the Lagrangian function.

(b) Solve the problem in the usual way.

(c) Give an economic interpretation of the Lagrangian multiplier involved here.

8. Assume the following loss function for a central bank:

$$\mathcal{L}(U, \pi) = \frac{U^2}{2} + \theta\frac{\pi^2}{2}, \quad \theta = 0.1,$$

which shall be minimized subject to the following (short-run) Phillips-curve:

$$\pi = (-\beta_w)(U - \overline{U}) + \pi^e$$

with π denoting the actual rate of inflation, π^e the expected one, U the unemployment rate and $\overline{U}$ the 'natural level' of the latter. Assume that $\overline{U} = 7\% = 0.07$ and $\beta_w = 2$.

- Determine the short-run optimum with regard to U and π for $\pi^e = 0$.
- Give a (qualitative) graphical representation of your results.