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Exercises „Mathematical Economics“

Series 3

1. Solve the following problem by means of the KKT conditions:

$$\begin{array}{rcl} f(x_1, x_2) & = & x_1^2 + x_2 \to \min! \\ \text{subject to} & & x_1^2 + x_2^2 \quad \le 9 \\ & & x_1 + x_2 \quad \le 1 \end{array}$$

   Is the solution obtained globally minimal?

2. Solve the following problem by means of the KKT conditions:

$$\begin{array}{rcl} f(x_1, x_2) & = & (x_1 - 1)^2 + e^{x_2^2} - 1 \to \min! \\ \text{subject to} & & x_1^2 + x_2^2 \ \le \ 1 \end{array}$$

   Why is the obtained solution a global minimum point?

3. Two resources x and y are to be allocated to two agents, each of them having a utility function of the form

$$U(x_i, y_i) = x_i \, y_i, \quad i = 1, 2.$$

   Let $\alpha \in (0, 1)$ denote the weight of the welfare of agent 1 in the social planner's utility function and $1 - \alpha$ the corresponding weight of agent 2. Assume that the maximum amounts $\overline{x}$ and $\overline{y}$ of the two goods are given. Thus, the social planner's optimization problem can be formulated as follows:

$$\begin{array}{rcl} F(x_1, x_2, y_1, y_2) & = & \alpha x_1 y_1 + (1 - \alpha) x_2 y_2 \to \max! \\ \text{subject to} & & x_1 + x_2 \ \le \overline{x} \\ & & y_1 + y_2 \ \le \overline{y} \\ & & x_1, \ x_2, \ y_1, \ y_2 \ \ge 0 \end{array}$$

   (a) Since the optimum must be characterized by a full exploitation of the two resources (for obvious reasons), the two inequality constraints can be replaced by equalities. Now neglect in a first step the non-negativity constraints and solve the problem in the usual way using the first-order conditions for the Lagrangian function. After having obtained a solution, check whether the non-negativity constraints are fulfilled.

   (b) Now solve the same problem by means of the KKT conditions and compare the solution with the previous one. How can the difference be explained?

4. Consider the following nonlinear programming problem:

$$\begin{aligned} f(x_1, x_2) &= -x_1 - x_2 - 8 \to \min! \\ \text{subject to} \quad & 2x_1^2 + x_2^2 \le 6 \\ & -x_1 + 3x_2 \ge 1 \\ & x_1,\ x_2 \ge 0 \end{aligned}$$

(a) Set up the KKT conditions and show that $x_1 = 0$ or $x_2 = 0$ is not compatible with them.

(b) Compute the concrete solution (Hint: $\lambda_2^* = 0$).

(c) Show by means of the saddle-point theorem that the solution obtained in (b) is indeed a global minimum point for the problem under consideration.

5. Consider the problem:

$$\begin{aligned} f(x_1, x_2) &= \min\{x_1, 2x_2\} \to \max! \\ \text{subject to} \quad & x_1 + x_2 \le 3 \\ & x_1,\ x_2 \ge 0 \end{aligned}$$

(a) Give a graphical representation of the problem and identify the solution for x_1 and x_2.

(b) Show by means of the saddle-point theorem that the point obtained in (a) is compatible with it and determine the value of λ^* (Hint: Treat the two half-spaces $x_1 \ge 2x_2$ and $x_1 \le 2x_2$ separately).

6. Solve the following linear programming problem by means of the KKT conditions:

$$\begin{aligned} f(x_1, x_2) &= 4x_1 + 3x_2 \to \max! \\ \text{subject to} \quad & x_1 + 5x_2 \le 45 \\ & x_1 + 2x_2 \le 21 \\ & 3x_1 + x_2 \le 33 \\ & x_1,\ x_2 \ge 0 \end{aligned}$$

(Hint: Show first that only $x_1 > 0$ and $x_2 > 0$ can be a solution. Then analyze the different possible constellations for the λ_i values, i.e. $\lambda_1, \lambda_2, \lambda_3 > 0$; $\lambda_1 = 0, \lambda_2, \lambda_3 > 0$; etc.)

7. Consider the following nonlinear programming problem:

$$\begin{aligned} f(x_1, x_2) &= (x_1 - 5)^2 + 4(x_2 - 1)^2 - 14 \to \min! \\ \text{subject to} \quad & x_1^2 - 4x_1 - x_2 \le -4 \\ & 2x_1 + x_2 \le 7 \\ & x_1,\ x_2 \ge 0 \end{aligned}$$

(a) Check whether the conditions are fulfilled which ensure that the KKT conditions are sufficient and necessary for a global constrained minimum point.

(b) Give a graphical representation of
  – the two constraints,
  – the resulting set of feasible solutions,
  – the level curves of the objective function
  and try to identify the solution.

(c) Now verify that the solution found in (b) satisfies the KKT conditions.

8. Consider the following nonlinear programming problem:

$$\begin{aligned} f(x_1, x_2) \quad &= \quad (x_1-4)^2 + (x_2-3)^2 \to \max! \\ \text{subject to} \qquad & \quad x_1 + x_2 \quad \geq \; 5 \\ & \quad -3x_1 - x_2 \; \leq \; -9 \\ & \quad x_1,\; x_2 \; \geq 0 \end{aligned}$$

and assume that $x_1 > 0$ and $x_2 > 0$.

(a) Set up the KKT conditions. Then proceed in the following way:
  – Show that an optimal point is not compatible with both constraints being active simultaneously.
  – Assume that the second constraint is active whereas the first one is not and obtain a contradiction.
  – Now assume the opposite situation and solve for x_1, x_2 and λ_1.
  – Check whether a further solution exists for $\lambda_1 = \lambda_2 = 0$.

(b) Check whether the KKT conditions are necessary and/or sufficient for a local or global maximum point.

(c) Draw a picture with the constraints and the level curves of the objective function.

9. Consider the following problem dealing with the determination of optimal labor supply by a given individual:

$$\begin{aligned} U(c, \ell) \quad &\to \quad \max! \\ \text{subject to} \qquad & \quad p\,c \leq w(T-\ell) + V \\ & \quad \ell \leq T \\ & \quad c,\; \ell \; \geq 0 \end{aligned}$$

with c denoting consumption, ℓ leisure, T the maximum amount of time at the individual's disposal, p the price of the consumption good, w the nominal wage rate (per hour of work) and V other (nominal) income.

(a) Set up the KKT conditions and discuss the three typical constellations that can occur:

  (i) $\ell = 0$, (ii) $\ell = T$ and (iii) $0 < \ell < T$.

  Give also a graphical representation for these cases.

(b) Under which conditions are the KKT conditions necessary and/or sufficient for a maximum here?

10. Have a look at the following utility maximization problem:

$$\begin{aligned} U(x_1, x_2) &= x_1^{2/3}\, x_2^{1/3} \to \max! \\ \text{subject to} \quad & p_1 x_1 + p_2 x_2 \le 12 \qquad \text{(with } p_1 = 1 \text{ and } p_2 = 4) \\ & x_1,\ x_2\ \ge 0 \qquad \text{and additionally } x_1 \ge C. \end{aligned}$$

The last constraint can be interpreted, e.g., as the quantity of fuel oil needed to keep the temperature in one's flat above a certain minimum level.

(a) Let $C = 4$ and solve the problem by means of the KKT conditions. Try to give an economic interpretation of the Lagrangian multipliers.

(b) Now assume $C = 8$. Try to explain why the Lagrangian multiplier corresponding to the constraint $x_1 \ge 8$ is zero here despite the fact that this constraint is active at the optimum.

(c) Finally, consider the case $C = 15$.