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Exercises „Mathematical Economics“

Series 4

1. Consider the following nonlinear programming problem:

$$\begin{aligned} f(x_1, x_2, x_3) &= -x_1x_2x_3 \to \min! \\ \text{subject to} \quad & -2x_1^2 - 2x_2^2 - x_3 + 46 \geq 0 \\ & 10 - x_3 \geq 0 \\ & x_1 + x_2 + x_3 - 1 \geq 0 \\ & x_1,\ x_2,\ x_3 \geq 0 \end{aligned}$$

   (a) Show that $x_1^* = 3, x_2^* = 3$ and $x_3^* = 10$ is a solution of the KKT conditions and determine also the values of λ_1^*, λ_2^* and λ_3^*.

   (b) Now check **all** conditions listed in the Arrow-Hurwicz-Uzawa-Theorem.

   (c) Now do the same for the Arrow-Enthoven-Theorem, where you can consider its basic requirements concerning the quasi-convexity of the objective function and the convexity of the constraints as satisfied.

2. (a) Prove the following part of Theorem 10:

   If functions f and $g_i,\ i = 1, 2, \ldots, m$, are all convex functions defined on $\mathbb{R}^n$, then the KKT conditions imply condition (M) for a minimum point, i.e. a point $\mathbf{x}^* \in \mathbb{R}^n$ satisfying the KKT conditions is indeed a global (constrained) minimum of the underlying optimization problem.

   (b) Prove Theorem 8.

   Prove also that $\lambda^* \cdot g(\mathbf{x}^*) = 0$, where λ^* is the vector of the Lagrangian multipliers at the optimum and $g(\mathbf{x}^*)$ is the vector of the constraint functions evaluated at the optimum.

3. Consider again the linear programming problem given as problem 6 in Series 3.

   (a) Give a graphical representation of the constraints as well as the level curves of the objective function.

   (b) Solve the problem by the simplex algorithm.

   (c) If you compare this solution procedure with the application of the KKT conditions, where do you see the most important difference(s)?

4. (a) Give a proof of the following two theorems on duality, thereby making use of the application of the saddle-point theorem.

- **Duality Theorem 1:** see Theorem 13 in the lecture.
- **Duality Theorem 2:**

  (a) If the optimal value of a decision variable in the primal problem is nonzero (i.e. $x_i > 0$), then the corresponding slack variable u_i in the i-th constraint must be equal to zero at the optimum.

  (b) If a slack variable x_{n+j} in the primal problem is nonzero at the optimum, then the optimal value of the corresponding decision variable in the dual problem must be zero (i.e. $u_{n+j} = 0$).

(Hint: For the proof of the second theorem, you should think about the relationship between inactive constraints and the values of the corresponding Lagrange and slack/surplus variables. On the other hand, strictly positive Lagrange variables also have some implications for the corresponding constraints and their slack/surplus variables. Finally, remember the relationships between the decision and Lagrange variables of the primal and dual problems.)

(b) Solve problem 6 in Series 3 graphically. Compute also the values of the slack variables by means of the information you get from the graph.

(c) Formulate the dual problem.

(d) Solve the dual problem solely by means of the two duality theorems mentioned above in part (a).

5. Consider the following optimization problem:

$$\begin{aligned} f(x_1, x_2) &= 15 - 7x_1 + 7x_2^3 - 21x_2^2 + 14x_2 \rightarrow \max! \\ \text{subject to} \quad g(x_1, x_2) &= 1.5x_1 + x_2 - 3 = 0 \\ & x_1,\ x_2 \geq 0 \end{aligned}$$

(a) Solve the problem first by the Lagrange multiplier method. Doing this, disregard the non-negativity constraints and check them only after you have obtained the solution(s).

(b) Now solve the problem by means of the KKT conditions. What is the difference to the solution(s) obtained in (a) and how can it be explained?

(c) What conclusion about the solution(s) can be drawn in (b) and which theorem can be applied in this context?