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Exercises „Mathematical Economics“

Series 5

1. Consider the following minimization problem:

$$f(x,y) = 3x^2 + 2y^2 \to \min!$$

s.t.

$$g(x,y) = \frac{1}{2}\,x + \frac{1}{3}\,y - 10 = t, \quad t \in \mathbb{R}; \quad x, y \in \mathbb{R}.$$

   (a) Solve the problem in the usual way by means of the Lagrangian function and the corresponding first-order and second-order conditions. After having obtained the solution $(x^*(t), y^*(t), \lambda^*(t))$, determine

$$\frac{dx^*}{dt}(t=0), \quad \frac{dy^*}{dt}(t=0) \quad \text{and} \quad \frac{d\lambda^*}{dt}(t=0).$$

   (b) Now determine the three partial derivatives mentioned in (a) in an alternative way, thereby making use of the ‘fundamental equation of comparative statics’.

   (c) Let $F(t) := f(x^*(t), y^*(t))$. Determine $F'(t=0)$ by a direct calculation and with the aid of the envelope theorem.

2. Assume that a firm has a certain quantity $\overline{y}$ of a good y at its disposal, which can be sold on two separated markets. The demand on the first market is given by

$$y_1(p_1) = -ap_1 + b$$

and that on the second market by

$$y_2(p_2) = -cp_2 + d$$

$(a, b, c, d > 0)$, where it is assumed that $b + d > 2\overline{y}$. The firm's problem is now to choose the two prices p_1 and p_2 in such a way that total revenue R^* is maximized, i.e.:

$$R(p_1, p_2) = -R^*(p_1, p_2) = -p_1 \cdot y_1(p_1) - p_2 \cdot y_2(p_2) \to \min!$$

s.t.

$$y_1(p_1) + y_2(p_2) \leq \overline{y}, \qquad p_1, p_2 \geq 0.$$

(**Remark:** In principle, one should also explicitly postulate the two non-negativity constraints for $y_1(p_1)$ and $y_2(p_2)$, i.e. $-ap_1 + b \geq 0$ and $-cp_2 + d \geq 0$). However, if the conditions

$$bc - ad - 2c\overline{y} < 0 \tag{1}$$

and

$$bc - ad + 2a\overline{y} > 0 \tag{2}$$

are satisfied, then $y_1(p_1^*) > 0$ and $y_2(p_2^*) > 0$ are satisfied so that one can drop the corresponding non-negativity constraints - hereafter an asterisk denotes the values for an optimal solution).

Now apply the 'fundamental equation of comparative statics' (in matrix-vector form) to this problem and determine the partial derivatives of λ^*, p_1^* and p_2^* with respect to the following parameters:

- $\overline{y}$ (i.e. the consequences of a change in the firm's capacity for the optimal values of λ, p_1 and p_2 are considered here);
- b (here the partial derivatives to be determined mirror the effects of a shift of the demand curve on market 1);
- (c) c (now the consequences due to a change of the inclination of the demand curve on market 2 are considered).

3. Consider again the optimization problem in the second exercise.

   (a) Set up the KKT conditions for this problem and assume directly that $p_1, p_2, \lambda > 0$ (but do not compute the solutions).

   (b) Now prove the two parts of the envelope theorem for this concrete problem, thereby making use of the information obtained from part (a). Take b as the parameter to be varied. Thus, show
   - first that
   $$\frac{\partial L^*}{\partial b}(p_1^*(b), p_2^*(b), \lambda^*(b)) = \frac{\partial L}{\partial b}(p_1, p_2, \lambda)\Big|_{p_1^*, p_2^*, \lambda^*}$$
   - second that
   $$\frac{\partial R}{\partial b}(p_1^*(b), p_2^*(b)) = \frac{\partial L}{\partial b}(p_1, p_2, \lambda)\Big|_{p_1^*, p_2^*, \lambda^*}$$

   Try to follow the logic in the book of Takayama, i.e. apply the general procedure outlined there to the concrete problem given here.

   (c) Now apply the second part above of the envelope theorem directly for the determination of
   $$\frac{\partial R}{\partial \overline{y}}(p_1^*(\overline{y}), p_2^*(\overline{y}), \lambda^*(\overline{y})) \quad \text{and} \quad \frac{\partial R}{\partial c}(p_1^*(c), p_2^*(c), \lambda^*(c)).$$

   (d) Now compute the solutions for p_1^*, p_2^* and λ^* from the KKT conditions derived in (a) and verify that
   $$\frac{\partial R}{\partial \overline{y}} < 0, \quad \frac{\partial R}{\partial b} < 0 \quad \text{and} \quad \frac{\partial R}{\partial c} > 0.$$

[Do not forget conditions (1) and (2)!] Try to give an economic explanation for these results.

4. (a) Consider the following general cost-minimization problem:

$$C(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} w_i x_i \to \min!$$

s.t.

$$f(x_1, \ldots, x_n) = y$$

and derive Samuelson's reciprocity relation:

$$\frac{\partial x_i}{\partial w_j} = \frac{\partial x_j}{\partial w_i}, \; i, j = 1, \ldots, n \quad \text{and} \quad \frac{\partial x_i}{\partial y} = \frac{\partial \lambda}{\partial w_i}, \; i = 1, \ldots, n.$$

(b) Assume a cost function

$$C(w, y) = w_1^{\alpha} w_2^{1-\alpha} y^{\beta}, \qquad 0 < \alpha < 1, \quad \beta > 0$$

and show that the underlying production function is

$$f(x_1, x_2) = \left(\frac{x_1^{\alpha} x_2^{1-\alpha}}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}} \right)^{1/\beta}.$$

(**Hint:** Apply Shephard's lemma to the cost function).