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Exercises „Mathematical Economics“

Series 6

1. Solve the following initial value problems:
   (a) $(1+t^3)\cdot\dot{x} = t^2\cdot x, \qquad x(0)=2;$
   (b) $\dot{x} = 6t^2\cdot x^2, \qquad x(3) = -\frac{1}{38}.$

2. (a) Assume that the marginal prospensity to consume in an economy is given by $C'(Y) = 0.8$. Furthermore, for the current income $Y_0 = 1000$ the consumption level is $C(Y_0) = 900$. Which consumption level will be reached if income rises to $Y_1 = 1200$.
   (b) Now assume that aggregate demand Y^d of an economy is composed of a consumption function $C(Y) = c\cdot Y,\ 0 < c < 1$, an autonomous investment $\overline{I}$ and a given level of government expenditure $\overline{G}$, i.e. $Y^d = C(Y) + \overline{I} + \overline{G}$. Assume furthermore that output reacts on the difference between demand and supply in the following way:
   $$\dot{Y}(t) = \mu\,[Y^d(t) - Y(t)] \ = \mu\,[C(Y) + \overline{I} + \overline{G}) - Y(t)], \quad \mu > 0.$$
   Now proceed in the following way:
      i. Determine the equilibrium state of this differential equation.
      ii. Find the general solution of the corresponding homogeneous equation.
      iii. Show that the sum of the solutions obtained in (i) and (ii) solves the original differential equation.

3. Find the general solution of the following differential equation:
$$\dot{x} - \frac{t}{t^2-1}\cdot x = t, \qquad (t > 1).$$

4. Determine the equilibrium states and their stability properties of the following differential equation:
$$\dot{x} = [7-x]^2\cdot[x^2 - 6x + 5].$$

5. Solve the following initial value problems:
   (a) $\ddot{x} + 2\dot{x} + x = t^2, \qquad x(0) = 0,\ \dot{x}(0) = 1;$
   (b) $\ddot{x} + 4x = 4t + 1, \qquad x(\pi/2) = 0, \dot{x}(\pi/2) = 0.$

6. A car driver observes a traffic sign that requires him to stop. In the following, let

- t denote the time in seconds *after the beginning of the braking process* and
- $s(t)$ denote the distance in meters between the point, *where the drivers observes the sign* and the actual position of the car at time t.

Assume that the driver brakes with a negative acceleration of $a = -4$ m/sec^2.

(a) Assume that the driver reacts instantaneously and that after 5 seconds he has covered a distance of 100 m from his original position, i.e. $s(5) = 100$. Determine the time at which the car stops as well as the corresponding braking distance.

(b) Now assume that the driver has a certain reaction time, i.e. he does not immediately begin to brake when he observes the sign (note that now $s(0)$ is no longer equal to zero). Three seconds after the beginning of the braking process he has covered a distance of 132 m from this original position (i.e. *where he observed the sign*) and after two further seconds this distance has increased to 160 m. Once again compute the time at which the car stops and the whole distance covered.

(c) Now assume that the car has a speed of 25 m/sec (which corresponds to 90 km/h) when the driver begins to brake (i.e. $v(0) = 25$). After having passed 118 m from the point, *where the traffic sign was observed*, the speed has declined to 9 m/s. Determine again the same value as in (a) and (b). How long is the driver's reaction time here? (Assume that the initial speed does not change between the observation of the traffic sign and the beginning of the braking process).

(**Hint:** Recall from physics that acceleration is the second derivative of distance with respect to time and that (instantaneous) speed is the first time derivative of distance, i.e. $a(t) = \dot{s}(t)$ and $(v(t) = \dot{s}(t)$. The acceleration is assumed here to be constant for all t, see above).