Faculty of Mathematics
Institute of Mathematical Optimization
Prof. Dr. F. Werner

Exercises „Mathematical Economics“

Series 7

1. Consider the following second-order differential equation:

$$\ddot{x} = -6\dot{x} - 5x + 15.$$

   (a) Transform the above equation into a system of two first-order equations.

   (b) Solve the given second-order differential equation by solving the corresponding first-order system obtained in (a).

   (c) Check that your solution determined in (b) does indeed fulfill the above second-order differential equation.

   (d) Determine the particular solution for the following initial values:

$$x(0) = -2 \qquad \text{and} \qquad x(1) = 3.68859.$$

2. (a) Let Λ be a diagonal matrix, the diagonal elements of which consist of the eigenvalues of a matrix A, which are assumed to be real and distinct, and P be a matrix, the columns of which consist of the corresponding eigenvectors, which are linearly independent. Prove that

$$e^{P\Lambda t P^{-1}} = Pe^{\Lambda t}P^{-1}.$$

   (b) Show (for the 2-dimensional case) that

$$e^{\Lambda t} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}.$$

3. Consider the following systems of differential equations:

   (a) $\dot{x} = x + y + 4; \qquad x(0) = 3,$
   $\dot{y} = 4x + y + 19; \qquad y(0) = 13;$

   (b) $\dot{x} = 3x + 6y - 15; \qquad x(0) = 8;$
   $\dot{y} = -6x + 3y + 15; \qquad y(0) = 3.$

   Determine in both cases:

   i) the solution of the corresponding homogeneous system;

   ii) a particular solution of the non-homogeneous system
   (**Hint:** Compute simply an equilibrium state which is characterized by $\dot{x} = 0$ and $\dot{y} = 0$);

iii) the general solution of the non-homogeneous system and

iv) the particular solution of the initial value problem.

4. Consider again both systems given in Exercise 3 and do the following:

   (a) Determine the type of dynamics (node/spiral/saddle point; stable/unstable).

   (b) Draw the nullclines into the phase plane and determine the directions of motion above and below them.

   (c) Draw (in a qualitative way) some typical trajectories into this picture.

5. Consider the following small macroeconomic model:

$$\begin{aligned} \pi(t) &= 0.375\,[U(t) - 2.2]^2 - 1.5 + \pi^e(t) \quad \text{(expectations-augmented Phillips-curve)} \quad (1) \\ \dot{\pi}^e(t) &= \alpha\,[\pi(t) - \pi^e(t)], \quad \alpha > 0 \quad \text{(adaptive expectations)} \quad (2) \\ \dot{U} &= \beta\,[\pi(t) - \bar{m}], \quad \beta > 0 \quad \text{(macroeconomic context, e.g. IS-LM)} \quad (3) \end{aligned}$$

   where:
   $\pi(t)$: actual rate of inflation, $\pi \in \mathbb{R}$;
   $\pi^e(t)$: expected rate of inflation, $\pi^e \in \mathbb{R}$;
   $U(t)$: actual rate of unemployment, $U \in [0, 1]$;
   $\bar{m}$: constant growth rate of nominal money supply.

   (a) Insert equation (1) into equations (2) and (3), thereby reducing the model to a two-dimensional nonlinear system of differential equations in the two dynamical variables π^e and U.

   (b) Draw the nullclines of the system obtained in (a) into the phase plane and determine the directions of motion above and below them. Then draw (in a qualitative way) one or more (depending on the type of dynamics) trajectories into this picture.

6. In an economic model, $K = K(t)$ denotes capital, $C = C(t)$ consumption, while α, A, and r are positive constants with $\alpha < 1$. Assume that

$$\dot{K} = AK^\alpha - C, \qquad \dot{C} = C(\alpha A K^{\alpha-1} - r).$$

   Perform a phase plane analysis of this system when $A = 2, \alpha = 0.5$, and $r = 0.05$.

7. Consider the system

$$\dot{x} = -x + \frac{1}{2}\,y^2, \qquad \dot{y} = 2x - 2y$$

   and investigate by means of Lyapunov's theorem whether the equilibrium state (0,0) is locally asymptotically stable.

8. Find the general solution of the differential equation

$$\frac{d^4x}{dt^4} - 3\frac{d^3x}{dt^3} + \frac{d^2x}{dt^2} + 4x = 2t - 1$$

   (**Hint:** Use that $\lambda^4 - 3\lambda^3 + \lambda^2 + 4 = (\lambda^2 + \lambda + 1)(\lambda - 2)^2$.)