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Exercises „Mathematical Economics“

Series 8

1. Solve the problem
$$\int_0^1 (-t\dot{x} - \dot{x}^2)dt \to \max!$$
subject to $x(0) = 1$ and
   (a) $x(1)$ free;
   (b) $x(1) \geq 1$.

2. Solve the control problem
$$\max_{u \in \mathbb{R}_+} \int_0^1 \left[-x(t) - u(t)^2\right] \, dt; \quad \dot{x}(t) = -u(t), \quad x(0) = 0, \quad x(1) \text{ free.}$$

3. A firm has an oder of B units of a commodity to be delivered at time T. Let $x(t)$ be the stock at time t and assume that the cost per unit of time of storing $x(t)$ units is $ax(t)$. The increase in $x(t)$, which equals production per unit of time, is $u(t) = \dot{x}(t)$. Assume that the total cost of production per unit of time is equal to $b(u(t))^2$. Here a and b are positive constants. So the firm's problem is
$$\min_{\{u(t) \geq 0\}} \int_0^T \left[ax(t) + bu(t)^2\right] \, dt; \quad \dot{x}(t) = u(t), \quad x(0) = 0, \quad x(T) = B$$
   - Give the necessary conditions for an optimal solution.
   - Find the only possible solution to the problem and explain why it really is a solution.
     (Hint: Distinguish between the cases $B \geq aT^2/4b$ and $B < aT^2/4b$.)
   - Now assume that the time horizon can be chosen freely by the firm so that the costs are now minimized with respect to $u(t)$ and T. Determine the optimal value for T.

4. Assume that a social planner wants to optimize the path of per-capita-consumption $c(t)$ of the population over time:
$$U = \int_0^{100} 15 \, \ln[c(t)] \cdot e^{-0.06t} dt$$
subject to
$$\begin{aligned} \dot{k}(t) &= 3.2 \cdot [k(t)]^{0.25} - \delta k(t) - c(t), \qquad \delta = 0.04 \\ k(0) &= 10, \qquad k(100) = 0 \end{aligned}$$
with k denoting the per-capita-capital stock and δ the rate of depreciation.

(a) Set up the **current value** Hamiltonian function for this problem and derive the necessary conditions for an optimal path.

(b) Derive from the results obtained in (a) a dynamical system in the variables k and c and determine its equilibrium state.

(c) Draw the nullclines into the phase plane and determine the directions of motion above and below them. Assuming that the (economically relevant) equilibrium state is a saddle point, draw some typical trajectories into the phase plane, too.

(d) Now linearize the system obtained in (b) at the (economically relevant) equilibrium state and verify that the dynamics are indeed of the saddle point type.

(e) Compute the explicit solution of the linearized system obtained in (d). Which consumption level $c(0)$ has optimally to be chosen at $t = 0$, given the above initial value $k(0) = 10$?

5. Find the solution to the control problem

$$\max \left\{ \int_0^1 (1 - tu - u^2)dt + 2x(1) + 3 \right\}, \quad \dot{x} = u, \quad x(0) = 1, \quad u \in \mathbb{R}.$$